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A NEW PARTIAL GEOMETRY CONSTRUCTED FROM THE HOFFMAN-SINGLETON GRAPH

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We give the construction of a partial geometry with parameters $s = 4$, $t = 17$, $\alpha = 2$. We also obtain two new strongly regular graphs.

A (finite) *partial geometry* with parameters s, t and α is a $1-(v, s+1, t+1)$ design (for which we speak of *lines* rather than blocks), satisfying the following two conditions.

(i) Any two distinct lines have at most one point in common;

(ii) for any non-incident point-line pair (x, L) the number of lines containing x and intersecting L equals α.

A partial geometry is called *proper* if $1 < \alpha < \min\{s, t\}$ (this means that the geometry is not equivalent to a combinatorial object for which another name is more common). Partial geometries were introduced by Bose [2]. At that time no example of a proper one was known. In the meantime some construction methods for proper partial geometries have been found, see [15], [13], [10], [3], [5], [14], [7]. Only one of the known ones has $\alpha = 2$, viz. the sporadic one of van Lint and Schrijver [10]. Here we construct a second proper partial geometry with $\alpha = 2$, which is (up till now) sporadic too.

The *point graph* of a partial geometry is the graph whose vertices are the points, two vertices being adjacent whenever the two corresponding points lie on one line. We need to quote some results. The first one is well-known (see [2]) and easily verified.

RESULT 1 : *The point graph of a partial geometry with parameters* s, t, *and* α *is strongly regular with parameters* $((v, k, \lambda, \mu) = ((s+1)(st+\alpha)/\alpha,\ s(t+1),\ s-1+t(\alpha-1),\ \alpha(t+1)))$. □ (*)

A strongly regular graph is called *pseudo-geometric* if there exist integers s, t and α such that its parameters satisfy (*). The next two results are well-known (see [12] or [2]). For a proof of Result 4 we refer to [6]. For convenience, we call an eigenvalue *restricted* if it has an eigenvector different from the all-one vector.

RESULT 2 : *A graph is strongly regular iff its adjacency matrix has just two distinct restricted eigenvalues.* □

RESULT 3 : *For a strongly regular graph with parameters* v, k, λ, μ *and restricted eigenvalues* r_+, r_- $(r_+ > r_-)$ *we have*

$$\mu = k - r_+ r_- = \lambda - r_+ - r_- = (k - r_+)(k - r_-)/v . \ \square$$

RESULT 4 : *Let* G *be a regular graph of degree* k *with* v *vertices. Let* G_1 *be an induced regular subgraph of* G *with* v_1 *vertices and degree* k_1. *Let* r_+ *be the largest, and* r_- *be the smallest restricted eigenvalue of the adjacency matrix of* G. *Then*

(i) $r_+ \geq (vk_1 - v_1k)/(v - v_1) \geq r_-$,

(ii) *if equality holds on one of the sides then any vertex outside* G_1 *is adjacent to exactly* $v_1(k - k_1)/(v - v_1)$ *vertices of* G_1. □

The following lemma is known, but hard to find in the literature.

LEMMA 1 : *Let* G *be a pseudo-geometric graph corresponding to the parameters* s, t *and* α. *Let* $\sum$ *be a set of* $(s+1)$*-cliques (cliques with* $s+1$ *vertices) of* G. *If any two adjacent vertices of* G *are in exactly one clique of* $\sum$, *then the design with the* *vertices of* G *as points and the cliques of* $\sum$ *as lines is a partial geometry with parameters* s, t *and* α, *having* G *as its point graph.*

PROOF : Only axiom (ii) is not trivially fulfilled. To prove this axiom we apply Result 4. Take G_1 to be a clique of $\sum$, then $k_1 = v_1 - 1 = s$. By use of Result 3 we have $r_+ = s - \alpha$. Now substitution in the left hand inequality shows that equality holds. Hence by (ii) any vertex outside G_1 is adjacent to $v_1(k - k_1)/(v - v_1) = \alpha$ vertices of G_1. This proves the lemma. □

We use a construction of the Hoffman-Singleton graph which is based on the following well-known result (see [4]).

RESULT 5 : *There exists a* 1 - 1 *correspondence between the* 35 *lines of* PG(3, 2) *and the* 35 *triples of a* 7*-set, such that lines intersect iff the corresponding triples have exactly one element in common.* □

Now the Hoffman-Singleton graph H is constructed as follows: The vertices are the 15 points together with the 35 lines of PG(3, 2) . Points are mutually non-adjacent. A point is adjacent to a line whenever the point lies on that line. Two lines are adjacent whenever the corresponding two triples are disjoint.

LEMMA 2 : H *is strongly regular with parameters* (50, 7, 0, 1) .

PROOF : The only step in the verification that may not be straightforward is to see that for a non-adjacent point-line pair there is exactly one line adjacent to both. This however follows easily after one has verified that a triple of the 7-set together with the four triples disjoint from it correspond to a spread in PG(3, 2) . □

Hoffman and Singleton [8] first constructed and proved uniqueness of a strongly regular graph with the above parameters. Next we describe a known construction of another strongly regular graph G (see [9]). The vertices of G are the 175 edges of H. Two vertices of G are adjacent whenever the corresponding edges of H have distance two (i.e. the two edges are disjoint and there exists an edge connecting the two).

LEMMA 3 : *G is strongly regular with parameters* (175, 72, 20, 36), *i.e. G is pseudo-geometric corresponding to the parameters* $s = 4$, $t = 17$, $\alpha = 2$.

PROOF: Let A, B and C be the adjacency matrices of G, H and the line graph of H respectively (observe that these three graphs are regular). Let N be the incidence matrix of H. Then it is easily seen that

$$N N^t = B + 7I, \quad N^t N = C + 2I,$$

$$A = C^2 - 5C - 12I.$$

B has restricted eigenvalues -3 and 2 (apply Result 3). Because NN^t and N^tN have the same nonzero eigenvalues, it follows that the restricted eigenvalues of C are -2, 2 and 7. Hence A has restricted eigenvalues 2 and -18. So by Result 2 the graph G is strongly regular. The parameters now readily follow by use of Result 3. □

The Hoffman-Singleton graph H contains many Petersen graphs as induced subgraphs. We define such a Petersen subgraph of H to be *special* if it consists of seven lines of PG(3, 2) corresponding to seven triples of the 7-set of the following form:

$$\begin{matrix} (1 & 1 & 1 & 0 & 0 & 0 & 0) \\ (1 & 0 & 0 & 1 & 1 & 0 & 0) \\ (0 & 1 & 0 & 1 & 1 & 0 & 0) \\ (0 & 0 & 1 & 1 & 1 & 0 & 0) \\ (1 & 0 & 0 & 0 & 0 & 1 & 1) \\ (0 & 1 & 0 & 0 & 0 & 1 & 1) \\ (0 & 0 & 1 & 0 & 0 & 1 & 1) \end{matrix},$$

together with the three points of PG(3, 2) that lie on the first of these seven lines. One readily verifies that these ten vertices of H indeed induce a Petersen graph.

LEMMA 4 : *Each pentagon of* H *is contained in exactly one special Petersen graph.*

PROOF : First we must realize that our description of H gives rise to two types of pentagons. The first type consists of three lines of PG(3, 2), whose triples have the form:

$$\begin{matrix} (1 & 1 & 1 & 0 & 0 & 0 & 0) \\ (1 & 0 & 0 & 1 & 1 & 0 & 0) \\ (0 & 1 & 0 & 0 & 0 & 1 & 1) \end{matrix},$$

together with two points of PG(3, 2), viz. the point in the intersection of the first and the second line, and the point in the intersection of the first and the third line. It is not hard to see that there is a unique way of completing this pentagon to a special Petersen graph. The second type of pentagon consists of four lines, whose triples have the form

$$\begin{matrix} (1 & 0 & 0 & 1 & 1 & 0 & 0) \\ (0 & 1 & 0 & 0 & 0 & 1 & 1) \\ (0 & 0 & 1 & 1 & 1 & 0 & 0) \\ (1 & 0 & 0 & 0 & 0 & 1 & 1) \end{matrix},$$

together with the unique vertex (which has to be a point of PG(3, 2)) adjacent to the first and the fourth line (here Lemma 2 is used). It is easy to see that there is a unique special

Petersen graph P containing the four lines. However, inside P there is also a unique vertex adjacent to the first and fourth line, so P contains the whole pentagon. □

A 1-*factor* of a Petersen graph is a set of five mutually disjoint edges.

LEMMA 5 : *The Petersen graph satisfies:*

(i) any two edges of a 1-*factor have distance two;*

(ii) any two edges at distance two lie in a unique 1-*factor.*

PROOF : By straightforward verification. □

Now we define the geometry S as follows. The points are the vertices of G, i.e. the edges of H. The lines are the 1-factors of the special Petersen graphs in H.

THEOREM : *S is a partial geometry with parameters* $s = 4$, $t = 17$, $\alpha = 2$.

PROOF : By (i) of Lemma 5 the lines of S are 5-cliques of G. By Lemma 3 G is pseudo-geometric corresponding to the above parameters. Hence, by Lemma 1, it suffices to prove that any edge of G lies on a unique line of S. Take two adjacent vertices of G, i.e. two edges x and y of H at distance two from each other. Then, by Lemma 2, x and y lie in a unique pentagon of H. By Lemma 4 this pentagon lies in a unique special Petersen graph. By (ii) of Lemma 5, inside this special Petersen graph, x and y lie in a unique 1-factor. This completes the proof. □

The *dual* geometry of S (the roles of points and lines are interchanged) is a partial geometry with parameters $s = 17$, $t = 4$, $\alpha = 2$. By Result 1 the *line graph* of S, which is the point graph of its dual, is strongly regular with parameters $(630, 85, 20, 10)$. This strongly regular graph seems to be new. But the present setting leads to another new strongly regular graph. To show this we need another lemma.

LEMMA 6 : *Let* G_1 *be an induced regular subgraph of* G *on* v_1 *vertices of degree* k_1 . *Let* A_1 *and*

$$A = \begin{bmatrix} A_1 & M \\ M^t & A_2 \end{bmatrix}$$

be the adjacency matrices of G_1 *and* G . *Define*

$$A' = \begin{bmatrix} o & j^t & o \\ j & A_1 & J-M \\ o & J-M^t & A_2 \end{bmatrix}$$

where j *is the all-one vector, and* J *the all-one matrix. If* (v_1, k_1) *equals* $(70, 18)$ *or* $(90, 38)$ *then* G' *, the graph with adjacency matrix* A' *, is strongly regular with parameters* $(176, 70, 18, 34)$ *or* $(176, 90, 38, 54)$ *respectively.*

PROOF : Result 4 implies that not only A and A_1 , but also M and A_2 have constant row and column sums. Now the result follows, either by straightforward verification, or more elegantly, from the theory of regular two-graphs, see [11] . □

It is not difficult to see that the 70 edges of H that do not contain a point of $PG(3, 2)$ induce a regular subgraph of G of degree 18 . So, by Lemma 6, we have a strongly regular graph with parameters $(176, 70, 18, 34)$. This graph is known, see [9]. In order to prove existence of the other subgraph of Lemma 6 we need the description of the Hoffman-Singleton graph given in [1], where the vertex set is partitioned into two subsets of size 25 , both inducing 5 disjoint pentagons. Two pentagons, one from each subset, form a Petersen graph and the 1-factor in there connecting the two pentagons forms a 5-clique in G . This gives 25 disjoint 5-cliques in G . By Result 4, any vertex outside such a 5-clique is adjacent to exactly two vertices of that 5-clique. Thus any set of 18 disjoint 5-cliques of G gives a regular induced subgraph with 90 vertices of degree 38. Now, since $18 \leq 25$, we have constructed, by Lemma 6, a strongly regular graph with parameters $(176, 90, 38, 54)$. This graph seems to be new. The amusing thing about this graph is that it is pseudo-geometric corresponding to the parameters $s = 5$, $t = 17$, $\alpha = 3$. We tried to construct this partial geometry in a way similar to that used for S, but did not succeed. The reason for this failure is that the set of 18 disjoint 5-cliques that we used does not look natural. If one finds a more natural set of 18 disjoint 5-cliques in G, then there is a good chance of success.

We conclude with a remark about the automorphism group of S. The automorphism group of H that fixes the partition into 15 points and 35 lines is the alternating group A_7. Clearly this group is also an automorphism group of S. It seems interesting that the action of A_7 on S is transitive on the 630 lines, but not transitive on the 175 points of S. We do not know whether A_7 is the full automorphism group of S.

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