DUAL SEIDEL SWITCHING

by

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Dedicated to J.J. Seidel on the occasion of his retirement.

INTRODUCTION

From 1973 to 1980 I have studied and worked under the stimulating guidance of J.J. Seidel. Already in the beginning of this period Seidel posed the problem treated in the present note, and it has been on my mind ever since. I was never able to deal with the problem in a satisfactory way, but there are some miscellaneous results that have never been published. These results are closely related to Seidel's work. Therefore, it seemed a good idea to publish them here.

The problem treated here has to do with strongly regular graphs with $\lambda = \mu$. These graphs give rise to symmetric block designs (see for instance [2]). The question is when do non-isomorphic graphs produce isomorphic designs. The main result implies that this is impossible if the automorphism group of one of the graphs is trivial. It is also shown that the question is in a certain sense dual to the following one.
When are non-isomorphic strongly regular graphs equivalent under Seidel-switching? This led to the title of this note.

The reader is assumed to be familar with strongly regular graphs and some related topics. A good general reference is Seidel's survey [7]. The present note uses the notation from this survey.

THE ISOMORPHISM PROBLEM

Let A be the adjacency matrix of a strongly regular graph (v,k,λ,μ). It is well known and easily verified that A is the incidence matrix of a symmetric block design whenever $\lambda = \mu$. These strongly regular graphs are often called (v,k,λ) graphs (cf. [2]). Similarly, if $\lambda + 2 = \mu$, then $A + I$ represents a symmetric block design.

Here we deal with the question: when do non-isomorphic strongly regular graphs give rise to the same symmetric block design?

Let A (or $A - I$) and B (or $(B - I)$ be the adjacency matrices of two such graphs. Since the corresponding block designs are isomorphic, there exist permutation matrices Q and R such that

$$B = QAR \ .$$

Hence, $RBR^t = RQA$. Put $P = RQ$, Then $PA \sim B$ ("$\sim$" indicates that the corresponding graphs are isomorphic). This implies that the rows of A can be permuted in such a way that another strongly regular graph appears. Our first result states that this phenomenon is a privilege of (v,k,λ) graphs.

RESULT 1. Let A be the adjacency matrix of a non-trivial strongly regular graph G. Let $P \neq I$ be a permutation matrix such that PA or $PA - I$ is again the adjacency matrix of a strongly regular graph. Then G is a (v,k,λ) graph.

Proof. Because G is strongly regular we have

$$A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J.$$

A strongly regular graph with matrix PA has to have the same parameters as G: (v,k,λ,μ), and a graph with matrix $PA - I$ has parameters $(v,k-1,\lambda-2,\mu)$. In both cases PA satisfies

$$(PA)^2 = (\lambda - \mu)PA + (k - \mu)I + \mu J.$$

By use of

$$(PA)^2 = (PA)^t(PA) = A^2$$

it now follows that $PA = A$, or $\lambda = \mu$. □

The two non-isomorphic (16,6,2) graphs give rise to the same block design. So they produce an example for the case $A \neq PA$. See [5], p. 10 for details.

An example where A and $PA - I$ are both adjacency matrices of strongly regular grpahs (that are obviously non-isomorphic) can be constructed as follows. Consider the following (Hadamard) matrix:

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

Let H^m denote the m-th Kronecker product of H with itself, that is

$$H^m = H \otimes H \otimes \ldots \otimes H \quad \text{(m times)} .$$

Then H^m is a regular symmetric Hadamard matrix with constant diagonal (see [4]) and $A = -\frac{1}{2}(H^m - J)$ is the adjacency matrix of a (v,k,λ) graph. Define

$$P = (J_2 - I_2) \otimes I_{2m}$$

(indices indicate the size of the matrix). Then PA is symmetric and has all-one diagonal. Therefore $PA - I$ represents a strongly regular graph with parameters $(v,k-1,\lambda-2,\lambda)$.

In the sequel we suppose that A represents a (v,k,λ) graph, and that PA or $PA - I$ is the adjacency matrix of a strongly regular graph. Since PA is symmetric we have

$$(*) \qquad PA = AP^t \ , \quad AP = P^tA, \quad PAP = P^tAP^t = A \ .$$

LEMMA 1. For any integer m, P^mA represents a strongly regular graph and

$$P^mA \sim \begin{cases} A \ , & \text{if m is even,} \\ PA \ , & \text{if m is odd.} \end{cases}$$

Proof. Induction on m. For $m = 0$, or 1 the result is obvious. Let $m > 1$, then by use of (*)

$$P^mA = PP^{m-2}(PA) = P(P^{m-2}A)P^t \sim P^{m-2}A. \qquad \square$$

LEMMA 2. Let n be the order of P. If n is odd then $PA \sim A$.
If n is even then $P^{\frac{1}{2}n}$ is an involution of the graphs represented by A and PA.

Proof. The first line follows directly from Lemma 1. To prove the second line we apply (*):

$$A = P^{n}A = P^{\frac{1}{2}n}A(P^{\frac{1}{2}n})^{t} ,$$

and similarly

$$PA = P^{n+1}A = P^{\frac{1}{2}n}(PA)(P^{\frac{1}{2}n})^{t} . \qquad \square$$

THEOREM 1. If the automorphism group of the graph represented by A or PA has odd order then PA ~ A.

Proof. If the order of the automorphism group is odd, the graph has no involution. So the result follows from Lemma 2. □

We illustrate the use of Theorem 1 by the following example. Bussemaker, Mathon and Seidel constructed many (v,k,λ) graphs [1]. They obtained 16448 (36,15,6) graphs. They also constructed 105 strongly regular graphs with parameters (36,14,4,6) and 1853 graphs with parameters (35,16,6,8). Out of these graphs 15417, 28 and 1576 respectively, have trivial automorphism groups. Thus by Theorem 1 the first two families produce at least 15417+28+1 = 15446 non-isomorphic symmetric (36,15,6) designs. The third family produces at least 1577 symmetric (35,17,8) designs.

We finish this section with some minor results that may be of some use to those who come across the problem of the present note.

RESULT 2. Suppose PA has zero diagonal. Then P is an even permutation.

Proof. A and PA have the same eigenvalues, and hence the same determinant. So det P = 1. □

RESULT 3. Suppose PA has zero diagonal. The orbits of P correspond to co-cliques of A and PA.

Proof. By Lemma 1, $P^m A$ has zero diagonal for any integer m. However, every entry of a principal submatrix of A that corresponds to an orbit of P can be moved to the diagonal by P. □

RESULT 4. Without loss of generality we may assume that all orbit sizes of P are powers of 2.

Proof. Suppose that n, the order of P has an odd divisor. Let m be the largest odd divisor of n. Then P^m has no odd orbit sizes and $P^m A \sim PA$ by Lemma 1. □

DUAL SEIDEL SWITCHING

Permuting the rows (and not the columns) of the incidence matrix of a graph will be called dual Seidel switching. This definition will be justified by the forthcoming result. First we want to point at some similarities between Seidel switching (see [6] or [7]) and dual Seidel switching. Both concepts give a method to derive other strongly regular graphs from a given one. The parameters of the new graph are the same or slightly different (the multiplicities differ by 1 in case of Seidel switching). To make this derivation work, the strongly regular graphs have to be very special (regular 2-graphs (see [6]) and (v,k,λ) graphs, respectively). These observations led to the conclusion that these two types of special strongly regular graphs should be dual to each other in the sense of Delsarte [3].

RESULT 5. If a strongly regular graph in the switching class of a regular 2-graph has a dual, then this dual is a (v,k,λ) graph, and vice versa.

Proof. If a strongly regular graph has eigenvalues k, r and s with multiplicities 1, f and g respectively, then the eigenvalues of its dual are (see [7])

$$f,\ fr/k \quad \text{and} \quad -f(r+1)/(n-k-1)\ .$$

Such a graph is a (v,k,λ) graph whenever

$$fr/k = f(r+1)/(n-k-1)\ ,$$

or, equivalently

$$k = r(n-1)/(2r+1)\ .$$

Substitution of this formula in the following identity for strongly regular graphs (see [7])

$$(n-k-1)(k+rs) = -k(r+1)(s+1)$$

leads to

$$n-1+(2r+1)(2s+1) = 0,$$

which is the condition for a strongly regular graph to be in the switching class of a regular 2-graph (see [6]). □

This duality can also be seen in terms of 3-class association schemes since both regular 2-graphs and symmetric block designs form 3-class association schemes. It may be that such a setup produces more insight in the present duality case. We haven't worked this out.

It has turned out that Seidel switching can produce a lot of non-isomorphic strongly regular graphs, often with trivial automorphism groups (see for instance [1]). Dual Seidel switching, however, cannot produce graphs with trivial automorphism groups because of Theorem 1. It is likely that the automorphism groups of most (v,k,λ) graphs are trivial. Therefore dual Seidel switching may not be expected to be as fruitful as Seidel switching.

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