# On (v, k, λ) graphs and designs without involutions

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# ON $(v, k, \lambda)$ GRAPHS AND DESIGNS WITHOUT INVOLUTIONS (*)

The adjacency matrix of a $(v, k, \lambda)$ graph is the incidence matrix of a $(v, k, \lambda)$ design. The subject of the paper is $(v, k, \lambda)$ graphs for which the automorphism group of the graph differs from the automorphism group of the corresponding design. In a foregoing paper it has been shown that there is just one example of this phenomenon if $v \leq 36$ and the automorphism group of the graph is trivial. We shall extend these results by determining all such $(v, k, \lambda)$ graphs in case $k - \lambda \leq 9$ (i.e. $v < 63$) and the graph has no involution.

## 1 Introduction

A $(v, k, \lambda)$ graph $G$ is a graph whose adjacency matrix $A$ satisfies

$$A^2 = (k - \lambda)I + \lambda J,$$

where $I$ and $J$ are the identity and the all-one matrix, respectively. Since $A^2 = AA^T$, the above equation also defines $A$ to be the incidence matrix of a symmetric $(v, k, \lambda)$ design, denoted by $D(G)$. It can happen, however, that non isomorphic $(v, k, \lambda)$ graphs produce isomorphic designs. For example, the two non isomorphic $(16, 6, 2)$ graphs produce the same design. In [3] it is shown that this is not the case if $\mathrm{aut}(G)$ (the full automorphism group of $G$) has odd order (i.e. $G$ has no involution). It is clear that $\mathrm{aut}(G)$ is a subgroup of $\mathrm{aut}(D(G))$. Sometimes $\mathrm{aut}(G)$ is a proper subgroup of $\mathrm{aut}(D(G))$, as is again illustrated by the two $(16, 6, 2)$ graphs. Thus the following question poses itself.

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QUESTION (1) – *Is* $\mathrm{aut}(G) = \mathrm{aut}(D(G))$ *if* $\mathrm{aut}(G)$ *has odd order?*

This question is the subject of the present paper. The answer will turn out to be negative, but the conditions for $G$ to be a counter example are strong enough to enable us to prove that the answer is affirmative for $v < 63$, except for just three graphs: one $(36, 15, 6)$ graph and two $(45, 12, 3)$ graphs. These three graphs are exhibited in section 4.

A $(v, k, \lambda)$ graph is strongly regular. By use of elementary properties of strongly regular graphs (see for instance [5]), we deduce the existence of integers $s$ and $a$ $(s > 1, a \geq 1)$, such that

$$k - \lambda = s^2, \quad \lambda = sa, \quad k = s(s + a), \quad v = s(s + a - 1)(s + a + 1)/a.$$

By use of these formulas we find eight feasible parameter sets for $s \leq 3$:

| nr. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $s$ | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| $a$ | 1 | 2 | 3 | 1 | 2 | 3 | 4 | 6 | 12 |
| $v$ | 16 | 15 | 16 | 45 | 36 | 35 | 36 | 40 | 56 |
| $k$ | 6 | 8 | 10 | 12 | 15 | 18 | 21 | 27 | 45 |
| $\lambda$ | 2 | 4 | 6 | 3 | 6 | 9 | 12 | 18 | 36 |

It is easily seen that for fixed $s$, $v$ is minimal if $a = s$, so the smallest $v$ not in the table equals 63. Graphs with parameter numbers 2, 3 and 9 are known and unique. They have (large) even order automorphism groups. There exist just two $(16, 6, 2)$ graphs, both having an even order automorphism group, so only parameter sets 4 to 8 are relevant to investigate Question (1). The parameter sets 5, 6 and 7 have been treated in [1] in case $\mathrm{aut}(G)$ is trivial. The results in [1], however, remain valid if $\mathrm{aut}(G)$ has odd order. Indeed, the structure theorem (Theorem (4)) remains valid (as we shall see in the next section), and the graphs in [1] that satisfy the desired structure all have involutions, except, of course, for the $(36, 15, 6)$ graph $G$ found in section 4.3. For this graph $\mathrm{aut}(G)$ is trivial and $\mathrm{aut}(D(G)) = c_3$ (the cyclic group of order 3), and thus it provides the first counterexample to Question (1) (see Section 4). In the coming sections we deal with the parameter sets numbered 4 and 8.

## 2 Basic results

In this section we develop the major tools for our investigation. First we quote Lemma (2) from [1]:

LEMMA (2) – *Let $A$ be the adjacency matrix of a $(v, k, \lambda)$ graph $G$. Let $P \neq I$ be a permutation matrix of odd order such that $PAP = A$. Then*

*i. the orbits of $P$ on $G$ are cocliques,*

*ii. the number of orbits is at most $v - 2k + 2\lambda - \alpha + 1$, where $\alpha$ is the size of the largest orbit.*

LEMMA (3) – *Let $G_1$ be an induced subgraph on $v_1$ vertices of a $(v, k, \lambda)$ graph $G$ $(0 < v_1 < v)$. Let $k_1$ denote the average degree of $G_1$ and let $s = \sqrt{k - \lambda}$. Then*

$$s(v - v_1) \geq vk_1 - kv_1 \geq -s(v - v_1),$$

*if equality holds on either side then $G_1$ is regular, and so is the graph induced by the remaining vertices of $G$.*

PROOF — The eigenvalues of $G$ are $k$, $s$ and $-s$. Thus the result follows from Theorem 2.1.2 of [2]. □

If $G_1$ is a coclique (i.e. $k_1 = 0$), the right hand side of the above inequality becomes: $v_1 \leq (k\sqrt{k - \lambda} - k + \lambda)/\lambda$. This is Hoffman's coclique bound for a $(v, k, \lambda)$ graph.

We recall the following definition. An $n \times n$ matrix $M$ is **anti-cyclic** (or back circulant) whenever

$$M_{i,j+1 (\mathrm{mod}\, n)} = M_{i+1 (\mathrm{mod}\, n),j}, \quad \text{for } i, j = 1, \ldots, n.$$

THEOREM (4) – *Let $A$ be the adjacency matrix of a $(v, k, \lambda)$ graph $G$. Suppose $\mathrm{aut}(G) \neq \mathrm{aut}(D(G))$, and $\mathrm{aut}(G)$ has odd order. Then there exist a prime $p > 2$, an integer $m \leq v/p$, and a partitioning*

$$A = \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m} \\ \vdots & \vdots & & \vdots \\ A_{m,0} & A_{m,1} & \cdots & A_{m,m} \end{pmatrix},$$

*such that*

*i. $p \leq (k\sqrt{k - \lambda} - k + \lambda)/\lambda$,*

*ii. $(p - 1)(m - 1) \geq 2(k - \lambda)$,*

*iii. $A_{i,i} = 0$ for $i = 1, \ldots, m$,*

*iv. $A_{i,j} = A_{j,i}$ is anti cyclic of size $p \times p$, for $i, j = 1, \ldots, m$,*

*v. $A_{i,0} = A_{0,i}^T$ consists of $p$ identical rows, for $i = 1, \ldots, m$,*

*vi. $m(k - \lambda)$ is even if $p = 3$.*

PROOF — We have $QAR = A$ for permutation matrices $Q$ and $R (Q \neq R^T)$. The symmetry of $A$ implies $QAR = R^T AQ^T$, hence $RQARQ = A$. Put $S = RQ$, then $S$ has odd order $n$ (say). Let $p > 2$ be a prime dividing $n$. Then $P = S^{n/p}$ has order $p$ and $PAP = A$. Hence the orbits of $P$ on $G$ have size 1 or $p$. Let $m$ be the number of orbits of size $p$. We partition $A$ according to the set of $v - mp$ fixed points and the $m$ orbits of size $p$. Then Lemma (3) implies iii and ii.
Hoffman's bound gives i. We may write

$$P = \text{ diag } (1, \ldots, 1, P_1, \ldots, P_1) ,$$

where the diagonal blocks $P_1$ are cyclic permutation matrices of size $p$.

Furthermore, $PAP = A$ implies

$$P_1 A_{i,j} P_1 = A_{i,j} , \quad P_1 A_{i,0} = A_{i,0} \quad \text{for } i, j = 1, \ldots, m ,$$

which proves iv and v.
Any two rows of $A$ differ in exactly $2(k - \lambda)$ positions. In case $p = 3$ this implies that block row $i (1 \leq i \leq m)$ contains precisely $k - \lambda$ anti-cyclic $p \times p$ blocks with row sum 1 or 2. So in total we have $m(k - \lambda)$ such blocks. This number must be even, since $A$ is symmetric and $A_{i,i} = 0$. This proves vi. □

A direct consequence of Theorem (4) v. is the following observation.

LEMMA (5) – *With the hypothesis of Theorem (4), the graph $G_0$ with adjacency matrix $A_{0,0}$ satisfies:*

*i. Any vertex has degree congruent to $k \bmod p$.*

*ii. The number of common neigbours of any pair of distinct vertices is congruent to $\lambda \bmod p$.*

Throughout the paper we shall use the notation of Theorem (4). Some further notation is needed. We define matrices $N$ and $A_1$ by

$$A = \begin{pmatrix} A_{0,0} & N \\ N^T & A_1 \end{pmatrix} .$$

The orbit matrix $B(A)$ of $A$ is the matrix where in the orbit partitioning of $A$ each block is replaced by its row sum. Corresponding to the above partitioning we write

$$B(A) = \begin{pmatrix} B(A_{0,0}) & B(N) \\ B(N^T) & B(A_1) \end{pmatrix} .$$

It is easily observed that $B(A_{0,0}) = A_{0,0}$ and $N = B(N) \otimes J_{1\times p}$ ($\otimes$ denotes the tensor product). Finally we define

$$K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$r_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

## 3 (40,27,18) and (45,12,3) graphs

In this section we shall deal with $(40, 27, 18)$ and $(45, 12, 3)$ graphs that satisfy the conditions of Theorem (4).

PROPOSITION (6) – *The answer to Question (1) is affirmative for* $(40, 27, 18)$ *graphs.*

PROOF — Theorem (4) yields that $p = 3$ and $m = 10$, or $m = 12$. In both cases we claim that $A_{0,i} (i = 1, \ldots, m)$ has at most one all-zero row (note that $A_{0,i}$ has only all-one and all-zero rows). Indeed, suppose there were two such rows, then since $A_{i,i} = 0$, we find a $5 \times 5$ submatrix with at most two ones, which violates Lemma (3). Now suppose $m = 10$. Then the matrix $N$ has at least $9 \times 30 = 270$ ones. Since $k = 27$ this implies $A_{0,0} = 0$, which is impossible by Hoffman's bound. So for $m = 10$ there is no $(40, 27, 18)$ graph that satisfies the conditions of Theorem (4).
Let $m = 12$. By the same reasoning as above it follows that $A_{0,0} = 0$ and $N = (J_4 - I_4) \otimes J_{1\times 9}$. Thus for each diagonal block $A_{i,i} (i = 1, \ldots, 12)$ there is a vertex of $A_{0,0}$ with which it forms a coclique of size 4. This meets the Hoffman bound, so any other vertex is adjacent to precisely 3 vertices of the coclique. Using this we find the orbit matrix $B(A_1)$ in a straightforward way:

$$B(A_1) = 2(I_4 \otimes J_3) - 3I_{12} + J_{12}.$$

This implies that the matrix $J - A_1 - I_{12} \otimes J_3$ is the adjacency matrix of a $(36, 15, 6)$ graph having a block structure described in Section 4.2 of [1]. There exist only two such $(36, 15, 6)$ graphs, both having involutions that act on the $3 \times 3$ blocks, and thus are also involutions for the corresponding $(40, 27, 18)$ graph. This completes the proof. □

PROPOSITION (7) – *There exists no* $(45, 12, 3)$ *graph that satisfies the conditions of Theorem (4) when* $p > 3$ *and* $mp < 45$.

PROOF — Theorem (4) gives six possible block structures: $p = 7, m = 4, 5, 6$ and $p = 5, m = 6, 7, 8$. The number $v_0$ of fixed points (i.e. size of $A_{0,0}$) equals 17,10,3,15,10 and 5, respectively. By Lemma (5), two distinct vertices of $G_0$ have exactly 3 common neighbours. Fix a vertex $x$ of $G_0$. Then the neighbours of $x$ induce a 3-regular graph, so the degree $k_0$ of $x$ is even and $k_0 \geq 4$. On the other hand Lemma (5) gives $k_0 = 12 \pmod p$, hence $k = 12$. If $v_0 < 13$ this is obviously impossible, and for $v_0 = 17$ or 15 it cannot be true since then two distinct vertices have more than 3 common neighbours. □

By Theorem (4) only four possibilities for $(45, 12, 3)$ graphs are left to be investigated: $(p, m) = (5, 9), (3, 10), (3, 12)$ and $(3, 14)$.

PROPOSITION (8) – *There exists no* $(45, 12, 3)$ *graph that satisfies the conditions of Theorem (4) with* $p = 5$ *and* $m = 9$.

PROOF — Clearly $A_{i,j} \neq J$, since $\lambda = 3$. By Hoffman's bound $A_{i,j} \neq 0$ if $i \neq j$. For any block row let $a_k$ $(k = 1, \ldots, 4)$ denote the number of matrices $A_{i,j}$ with row sum k. Then

$$\sum_{k=1}^{4} a_k = 8\,, \quad \sum_{k=1}^{4} k a_k = 12\,, \quad \sum_{k=1}^{4} 5\binom{k}{2} a_k = 3\binom{5}{2}\,.$$

From this it follows that $a_3 + 3a_4 = 2$. Hence $a_4 = 0, a_3 = 2, a_2 = 0, a_1 = 6$. Let $C_1$ and $C_2$ be the two anti-cyclic matrices with row sum 3. Then from

$$\sum_{j=1}^{9} A_{i,j}^2 = 3J + 9I$$

it follows that $C_1^2 + C_2^2 = 3J + 3I$. So, up to a cyclic shift

$$C_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

(or vice versa). So in total there are nine matrices of type $C_1$. This is a contradiction because the symmetry of $A$ requires the number to be even. □

PROPOSITION (9) – *There exists no* $(45, 12, 3)$ *graph with the structure of Theorem (4) if* $p = 3$ *and* $m = 10$.

PROOF — A column of $A_{0,i}$ ($i = 1, \ldots, 10$) has at most three ones, since $\lambda = 3$, and equality implies that $A_{i,j}$ is a permutation matrix for $i, j = 1, \ldots, 10, i \neq j$. Therefore the matrix $N$ has at most 90 ones, hence the average row sum $k_0$ of $A_{0,0}$ satisfies $k_0 \geq 6$. On the other hand Lemma (3) implies $k_0 \leq 6$. So $k_0 = 6$, and each $A_{i,j}$ ($i, j = 1, \ldots, 10, i \neq j$) is a permutation matrix. From

$$\sum_{j=0}^{10} A_{1,j}A_{j,2} = 3J\,, \quad A_{1,1} = A_{2,2} = 0\,, \quad A_{1,0}A_{0,2} \in < J >\,,$$

it follows that

$$\sum_{j=3}^{10} A_{1,j}A_{j,2} \in < J > \,.$$

This is clearly impossible for permutation matrices $A_{1,j}A_{j,2}$. □

PROPOSITION (10) – *There exists exactly one (45, 12, 3) graph without involution that satisfies the block structure of Theorem (4) with $p = 3$ and $m = 12$.*

PROOF — We first prove that without loss of generality either $A_{0,0} = 0$ or

$$(*) \qquad A_{0,0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes J_3\,.$$

By Lemma (5) a vertex $x$ of $G_0$ has degree 0, 3 or 6, and any pair of vertices of $G_0$ has 0 or 3 common neighbours. Suppose $x$ has degree 6, and let $a$ denote the number of vertices having 3 neighbours in common with $x$. Then by counting common neighbours of pairs of vertices adjacent to $x$ we obtain

$$3a = 2\binom{6}{2}\,.$$

So $a = 10$, which is impossible. Hence any vertex of $G_0$ has degree 0 or 3. Now is follows easily that $A_{0,0}$ must be one of the matrices above.

Next we shall show that both types of $A_{0,0}$ lead to the following structure

$$(**) \quad A = \left(\begin{array}{c|ccc} P_6 \otimes J_3 & \begin{matrix} R_0 \\ C_0 \end{matrix} & \begin{matrix} R_1 \\ C_1 \end{matrix} & \begin{matrix} R_2 \\ C_2 \end{matrix} \\ \hline R_0^T C_0^T & 0 & C_3 & C_4 \\ R_1^T C_1^T & C_3^T & 0 & C_5 \\ R_2^T C_2^T & C_4^T & C_5^T & 0 \end{array}\right)\,, \qquad R_i = \begin{pmatrix} r_1 & r_2 & r_3 \\ r_{1+i} & r_{2+i} & r_{3+i} \\ r_{1-i} & r_{2-i} & r_{3-i} \end{pmatrix}\,, \quad C_i = \begin{pmatrix} K_{i,1} & K_{i,2} & K_{i,3} \\ K_{i,4} & K_{i,5} & K_{i,6} \\ K_{i,7} & K_{i,8} & K_{i,9} \end{pmatrix}\,,$$

where $r_j = r_k$ if $j = k$ mod 3, and $K_{i,j} = K, L$ or $M$. Hoffman's coclique bound equals 9 and, by Lemma (3), a vertex outside a 9-coclique $C$ is adjacent to exactly 3 vertices of $C$. Suppose $A_{0,0} = 0$. Then $N$ has constant column sum 3. Hence $B(N)$ is the incidence matrix of a $2-(9,3,1)$ design $D$, i.e. the affine plane of order 3. We have $A_{i,j} = 0$ if $i$ and $j$ correspond to lines in the same parallel class of $D$, since otherwise we can easily find a coclique $C$ and a vertex with more than 3 neighbours in $C$. Now it is straightforward to show that $A$ has the above structure. Next suppose $A_{0,0}$ is as given in (*). Then it is easily seen that without loss of generality the upper left $9 \times 18$ submatrix of $A$ has is one of the following:

$$\begin{pmatrix} 0 & J & 0 & 0 & 0 & 0 \\ J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & J & 0 & 0 & 0 & 0 \\ J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J-r_1 & J-r_2 & J-r_3 \end{pmatrix}.$$

Similar to above it follows that the first submatrix leads to the desired structure. So suppose we have the second submatrix. Then we readily find the following upper left $11 \times 11$ submatrix of $B(A)$.

$$C = \begin{pmatrix} 0 & J & 0 & 0 \\ J & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & F & I \end{pmatrix}, \qquad E^T = F = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \end{pmatrix}.$$

The blocks in the orbit partitioning of $A$ have constant row sums, therefore the eigenvalues of $B(A)$ are also eigenvalues of $A$ (see for instance [2], Theorem 1.2.3). By use of trace $B(A) = 0$ we find that $B(A)$ has eigenvalues $12, 3$ and $-3$ with multiplicities $1, 8$ and $12$ respectively. Hence rank $B(A) + 3I = 9$. So rank $C + 3I \leq 9$. But it is easily checked that rank $C + 3I = 10$. This contradiction proves that $A$ indeed has the stated form. In [4] Mathon and Spence find all $(45, 12, 3)$ graphs with that particular form. Only one of these graphs has no involution. This proves the proposition. □

We remark that for $A_{0,0} = 0$ one can also prove the above result in a manner similar to the case $(p, m) = (3, 12)$ for $(40, 27, 18)$ graphs, by transforming $A_1$ into a $(36, 15, 6)$ graph treated in [1].

Proposition (11) – *There exists precisely one $(45, 12, 3)$ graph with no involution that satisfies the conditions of Theorem (4) for $p = 3, m = 14$.*

PROOF — Similar to the previous case, $A$ must have the structure given in (4), except for the $R_i$'s, that are defined differently by:

$$R_i = \begin{pmatrix} r_1 & r_2 & r_3 \\ K_{i,1} & K_{i,2} & K_{i,3} \\ K_{i,4} & K_{i,5} & K_{i,6} \end{pmatrix}.$$

Unlike the previous case, however, there is no reasonably short argument to prove this. The result followed from an examination of all the possibilities and was confirmed by a computer search. Also for the present structure Mathon and Spence found all (45, 12, 3) graphs. Again there is precisely one such graph with no involution. □

## 4 Recapitulation

THEOREM (12) – *For $k - \lambda \leq 9$ the only $(v, k, \lambda)$ graphs with no involution for which* $\mathrm{aut}(G) \neq \mathrm{aut}(D(G))$ *are the following ones:*

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & J & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
 & 0 & 0 & 0 & J & 0 & 1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \\
 & & 0 & J & 0 & 0 & 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \\
 & & & 0 & 0 & 0 & K & K & K & K & K & K & K & K & K \\
 & & & & 0 & 0 & K & K & K & L & L & L & M & M & M \\
 & & & & & 0 & K & K & K & M & M & M & L & L & L \\
 & & & & & & 0 & 0 & 0 & K & L & M & K & L & M \\
 & & & & & & & 0 & 0 & L & M & K & M & K & L \\
 & & & & & & & & 0 & M & K & L & L & M & K \\
 & & & & & & & & & 0 & 0 & 0 & L & K & M \\
 & & & & & & & & & & 0 & 0 & K & M & L \\
 & & & & & & & & & & & 0 & M & L & K \\
 & & & & & & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & & & & 0
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & J & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \\
 & 0 & 0 & 0 & J & 0 & K & L & M & K & L & M & K & L & M \\
 & & 0 & 0 & 0 & 0 & K & M & L & K & M & L & K & M & L \\
 & & & 0 & 0 & J & K & K & K & L & L & L & K & K & K \\
 & & & & 0 & 0 & K & K & K & M & M & M & M & M & M \\
 & & & & & 0 & K & K & K & K & K & K & L & L & L \\
 & & & & & & 0 & 0 & 0 & 0 & \mathbf{K} & K & 0 & K & \mathbf{K} \\
 & & & & & & & 0 & 0 & K & 0 & \mathbf{K} & \mathbf{K} & 0 & K \\
 & & & & & & & & 0 & \mathbf{K} & K & 0 & K & \mathbf{K} & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 0 & \mathbf{K} & K \\
 & & & & & & & & & & 0 & 0 & K & 0 & \mathbf{K} \\
 & & & & & & & & & & & 0 & \mathbf{K} & K & 0 \\
 & & & & & & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & & & & 0
\end{pmatrix},$$

$$\begin{pmatrix}
0 & J & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{3} & 0 & \mathbf{2} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{1} \\
 & 0 & 1 & \mathbf{1} & \mathbf{2} & \mathbf{3} & 0 & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\
 & & 0 & \mathbf{K} & K & K & \mathbf{K} & \mathbf{K} & K & K & K & \mathbf{K} \\
 & & & 0 & L & L & \mathbf{K} & K & \mathbf{L} & K & \mathbf{L} & K \\
 & & & & 0 & \mathbf{L} & \mathbf{L} & K & K & K & \mathbf{K} & \mathbf{L} \\
 & & & & & 0 & \mathbf{K} & \mathbf{L} & \mathbf{K} & L & K & K \\
 & & & & & & 0 & \mathbf{L} & L & K & \mathbf{K} & L \\
 & & & & & & & 0 & L & \mathbf{L} & K & K \\
 & & & & & & & & 0 & K & L & K \\
 & & & & & & & & & 0 & \mathbf{L} & L \\
 & & & & & & & & & & 0 & L \\
 & & & & & & & & & & & 0
\end{pmatrix},$$

*where we wrote 1,2,3 instead of* $r_1, r_2, r_3$*, and used the fat symbols to indicate the complement (i.e.* $K = J - K$*, etc.). The respective automorphism groups of the two* $(45, 12, 3)$ *graphs are:* $c_3 \times c_3$*, and* $c_3$*. The automorphism group of the* $(36, 15, 6)$ *graph is trivial. In each case the automorphism group of the corresponding design is the direct product of* $c_3$ *with the automorphism group of the graph.*

Finally we remark that by modifications of the structures of Proposition (10) and (11), Mathon and Spence [4] found 45 $(45, 12, 3)$ graphs, and over 2000 $(45, 12, 3)$ designs. One of the graphs has a trivial automorphism group, hence by Theorem (12) the corresponding design also has a trivial automorphism group. Thus with the results of [1] we have a $(v, k, \lambda)$ graph and design with a trivial automorphism group for each of the parameter sets 4 to 8 of the table in Section 1.

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