Strongly regular graphs induced by polarities of symmetric designs

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1. Introduction

Throughout, we shall assume that A is the (0, 1) incidence matrix of a symmetric 2-(v, k, λ) design D with a polarity with v_1 $(\neq 0)$ absolute points and $v_2 = v - v_1$ $(\neq 0)$ non-absolute points. So A takes the form:

$$A = \begin{bmatrix} A_1 & C \\ C^T & A_2 \end{bmatrix},$$

where A_1 and A_2 are symmetric matrices, A_1 has 'ones' on the diagonal and A_2 has zeros on the diagonal. Thus $A_1 - I$ and A_2 are adjacency matrices of graphs Γ_1 and Γ_2, say. We call Γ_1 and Γ_2 the graphs *induced* by the polarity of D. The above decomposition of A is called the *polar decomposition*. A polar decomposition is *regular* if Γ_1 and Γ_2 are regular and *strongly regular* if Γ_1 and Γ_2 are strongly regular, complete or empty. The incidence structure of absolute points and non-absolute blocks, given by the matrix C, is called the *polar structure* of the design D.

We assume that D is non-trivial; that is, $0 < \lambda < k - 1 < v - 2$. Polar decompositions of trivial designs are obvious and not very interesting.

In the present paper, we study strongly regular polar decompositions. It is, in a certain sense, an extension to a paper on strongly regular graphs with strongly regular decompositions by Haemers and Higman (1989). The subject is motivated by the following example. Let D be the design of points and hyperplanes of $PG(n, 4)$ $(n \geq 2)$. Then the graphs induced by the unitary polarity are strongly regular (see for instance Hubaut 1975). The parameters for this example are:

$$v = \frac{4^{n+1} - 1}{3},\ k = \frac{4^n - 1}{3},\ \lambda = \frac{4^{n-1} - 1}{3},\ v_1 = \frac{2^{2n+1} - (-2)^n - 1}{3}.$$

We shall prove that all parameters of a strongly regular polar decomposition can be expressed in terms of only one parameter a. If

a is a power of -2 we find the values above, but unfortunately, other values of a for which no examples are known, remain feasible.

For strongly regular Γ_i (i= 1, 2) the parameters are denoted by v_i, k_i', λ_i and μ_i, and the eigenvalues of A_i by k_i, ρ_i and σ_i, such that $|\rho_i| \geq |\sigma_i|$. Note that $k_1 = k_1' + 1$ and $k_2 = k_2'$. The multiplicity of σ_i is denoted by φ_i. If Γ_i is complete or empty we put $\varphi_i = v_i - 1$ (ρ_i disappears).

2. Preliminaries

The following three Lemmas are the analogues of 2.2, 2.4 and 2.5 of Haemers and Higman (1989). Since the proofs are the same, we omit them here.

Lemma 1. *Suppose Γ_1 is regular of degree $k_1' = k_1 - 1$. Then*

$$\left|\frac{k_1 v - v_1 k}{v - v_1}\right| \leq \sqrt{(k - \lambda)}.$$

Equality holds if and only if the polar decomposition is regular.

Note that for the polar decomposition to be regular $k - \lambda$ (the order of D) must be square.

For a regular polar decomposition we define k_2 to be the degree of Γ_2 and

$$a = \frac{k_1 v - v_1 k}{v - v_1}. \qquad (1)$$

Then we easily have

$$a = k_1 + k_2 - k = \pm\sqrt{(k - \lambda)}\,, \ |a| \geq 2. \qquad (2)$$

So, a, $-a$ and k are the eigenvalues of A with multiplicities φ (say), v - 1- φ and 1 respectively.

Lemma 2. *Suppose A_1 has eigenvalues k_1, σ_1 and ρ_1 ($|\sigma_1| \leq |\rho_1|$) with multiplicities 1, φ_1 and $v_1 - 1 - \varphi_1$ ($1 \leq \varphi_1 \leq v_1 - 1$), respectively. Let the polar decomposition be regular. Then A_2 has eigenvalues k_2, $-\sigma_1$, $-\rho_1$, a, $-a$ with multiplicities 1, φ_1, $v_1 - 1 - \varphi_1$, $\varphi - v_1$ and $v_2 - \varphi$, respectively. Also, $|\sigma_1| \leq |\rho_1| \leq |a|$ and $|\sigma_1| \neq |a|$.*

Lemma 3. *With the hypotheses of Lemma 2, the polar decomposition is strongly regular if and only if one of the following occurs:*
(i) $v_1 = v_2 = \varphi$;
(ii) $\rho_1 = a$, $v_1 = \varphi$;
(iii) $\rho_1 = -a$, $v_2 = \varphi$.

It is easy to see that if (ii) occurs for a matrix A, then (i) occurs for the complementary case; that is, for the design with incidence matrix (J denotes the all-one matrix),

$$J - \begin{bmatrix} A_2 & C^T \\ C & A_1 \end{bmatrix}.$$

Note that Lemma 3 also applies if Γ_1 is complete or empty (the conditions $\rho_1 = \pm a$ are meaningless in this case). Next we show that case (i) does not occur.

Lemma 4. *A strongly regular polar decomposition with $v_1 = v_2 = \varphi$ does not exist.*

Proof. Assume $v \geq 2k$ (otherwise consider the complement). Trace $A = v_1 = \varphi = k + \varphi a - (v - \varphi - 1)a$ implies $k = v_1 - a$. Hence, by (1) and (2), $k_1 = k_2 = v_1/2 = v_2/2$. Therefore, since Γ_2 is strongly regular, $k_2(k_2 - \lambda_2 - 1) = \mu_2(v_2 - k_2 - 1) = \mu_2(k_2 - 1)$. This implies $\mu_2 \equiv 0 \bmod k_2$, hence Γ_2 is complete bipartite, so $\rho_2 = -k_2$. This is impossible, since then by Lemma 2, $|a| \geq |\rho_2| = k_2 = v_2/2 = v/4$, which is a contradiction to $a^2 < k \leq v/2$ and $|a| \geq 2$. □

3. Main theorem

Theorem 5. *If D has a strongly regular polar decomposition then $a \equiv 4$ mod 6, and, up to taking complements, all other parameters are expressible in terms of a as follows:*

$v = (16a^2 - 1)/3$, $k = (4a^2 - 1)/3$, $\lambda = (a^2 - 1)/3$,
$v_1 = \varphi = (2a + 1)(4a - 1)/3$, $v_2 = 2a(4a - 1)/3$,
$k_1' = k_1 - 1 = 2(a - 1)(a + 2)/3$, $k_2' = k_2 = a(2a - 1)/3$,
$\rho_1 = -\rho_2 = a$, $\sigma_2 = -\sigma_1 = a/2$,
$\varphi_1 = \varphi_2 = 8(a - 1)(2a + 1)/9$,
$\mu_1 = (a - 1)(a + 2)/6$, $\lambda_1 = (a^2 + 4a - 2)/6$,
$\mu_2 = a(a + 2)/6$, $\lambda_2 = a(a - 1)/6$.

Proof. We may restrict ourselves to case (ii) of Lemma 3. By this result we can express all parameters in terms of k and a only. We have $\lambda = k - a^2$ and hence, using $\lambda(v - 1) = k(k - 1)$:

$$v = \frac{k^2 - a^2}{k - a^2}.$$

Trace $A = v_1 = \varphi = k + \varphi a - (v - 1 - \varphi)a$ yields

$$v_1 = \frac{k(k + a)(a - 1)}{(k - a^2)(2a - 1)}, \qquad v_2 = \frac{a(k + a)(k + 1 - 2a)}{(k - a^2)(2a - 1)}.$$

Next, using (1) and (2), we find

$$k_2 = \frac{ak}{2a - 1}, \; v_2 - k_2 - 1 = \frac{k(a + 1)(a - 1)^2}{(k - a^2)(2a - 1)}.$$

Since $k - a^2 = \lambda > 0$ and $|a| \geq 2$ it follows that $0 < k_2 < v_2 - 1$, so Γ_2 is not the empty graph or the complete graph. Moreover, $2a - 1$ divides k. Define $x = k/(2a - 1)$, then $k_2 = ax$ and a and x have the same sign. By use of $\rho_2 = -a$ (by Lemma 2 and 3) and $(k_2 - \rho_2)(k_2 - \sigma_2) = (k_2 + \rho_2\sigma_2)v_2$ (because Γ_2 is strongly regular) we find after some computation:

$$\sigma_2 = \frac{a(ax + a - 2x)}{2ax - x - 1}.$$

Define

$$d = 4\sigma_2 - 2a + 3 = \frac{4a^2 + 2a - 3x - 3}{2ax - x - 1},$$

then $d \geq 0$, since $d \leq 1$ would imply $(2a - 1)(a + 1) \leq x(1 - a) < 0$, which contradicts $|a| \geq 2$. Moreover, using $2ax - x - 1 = k - 1 \geq a^2$, we have

$$d \leq 4 + \frac{2}{a} - \frac{3(x + 1)}{2ax - x - 1} < \begin{cases} 4 + \frac{2}{a} \leq 5 & \text{if } a \text{ and } x \text{ are positive} \\ 4 - \frac{3}{2a - 1} < 5 & \text{if } a \text{ and } x \text{ are negative.} \end{cases}$$

Since d is odd we can conclude that $d = 1$ or $d = 3$. If $d = 1$, then $x = 2a - 1$ and $v_1 = (2a - 1)(4a^2 - 3a + 1)/(3a - 1)$, which is no integer. Hence $d = 3$, so

$$x = \frac{2a+1}{3}, \quad \sigma_2 = \frac{a}{2}, \quad a \equiv 4 \bmod 6.$$

All other parameters now follow in a straightforward manner. □

If $a = (-2)^{n-1}$ we find the parameters of the example given in the introduction. No other examples are known. This leaves $a = 10$ (D is a 2-(533, 133, 33) design) as the smallest unsolved case. By Theorem 5, only for $a = -2$ is one of the graphs induced by the polarity complete or empty. For $a = -2$, however, the structure is unique. So $PG(2, 4)$ with its unitary polarity and the complement provide the only strongly regular polar decomposition for which one of the induced graphs is empty of complete. In this case, the polar structure is $AG(2, 3)$, the affine plane of order 3. If $|a| > 2$, it can be shown (by similar, but less complicated arguments as for Theorem 2.8 of Haemers and Higman (1989)) that the polar structure is a strongly regular design (see Higman (1988) for definition and notation) with parameters

$$(n_1, S_1, a_1, b_1, n_2, S_2, a_2, b_2) = (v_1, k\text{-}k_1, \lambda\text{-}2\text{-}\lambda_1, \lambda\text{-}\mu_1, v_2\ k\text{-}k_2, \lambda\text{-}\lambda_2, \lambda\text{-}\mu_2) \quad (3)$$

and that, conversely, a strongly regular design whose parameters satisfy (3) with v_1, k, k_1 etc. as given in Theorem 5, is the polar structure of some symmetric design with a strongly regular polar decomposition.

A symmetric design with a polarity can be seen as a strongly regular graph for which loops are admitted. More precisely, if we allow loops in graphs, a strongly regular graph is a simple strongly regular graph or a symmetric design with a polarity. In this more general setting, a strongly regular graph with strongly regular decomposition either belongs to the case without loops treated in Haemers and Higman (1989), or is a design with a strongly regular polar decomposition. This can be proved as follows. For a strongly regular decomposition of a symmetric design with a polarity it follows by eigenvalue arguments as in Lemma 3, that if Γ_1 is a symmetric design with a polarity, then so is Γ_2. This, however, is impossible, by Rahilly (1988). Thus both Γ_1 and Γ_2 must be strongly regular graphs without loops, so we have a strongly regular polar decomposition.

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