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SOME REMARKS ON SUBDESIGNS OF SYMMETRIC DESIGNS

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Abstract: Let $D(v, k, \lambda)$ be a symmetric design containing a symmetric design $D_1(v_1, k_1, \lambda_1)$ $(k_1 < k)$ and let $x = v_1(k-k_1)/(v-v_1)$. We show that $k \geq (k_1 - x)^2 + \lambda$. If equality holds, D_1 is called a tight subdesign of D. In the special case, $\lambda_1 = \lambda$, the inequality reduces to that of R.C. Bose and S.S. Shrikhande and tight subdesigns then correspond to their notion of Baer subdesigns. The possibilities for (v, k, λ) designs having Baer subdesigns are investigated.

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1. Introduction

Let $D(v, k, \lambda)$ be a symmetric design containing a symmetric design $D_1(v_1, k_1, \lambda_1)$ $(k_1 < k)$. We call D_1 a subdesign of D. Let $x = v_1(k-k_1)/(v-v_1)$. We show that $k \geq (k_1 - x)^2 + \lambda$ (Theorem 1). If equality holds, D_1 is called a tight subdesign of D. In the special case $\lambda_1 = \lambda$, our inequality reduces to that of R.C. Bose and S.S. Shrikhande [3] and tight subdesigns then correspond to their notion of Baer subdesigns. We give examples of tight subdesigns. We divide the possibilities for (v, k, λ) designs, having Baer subdesigns into three cases (Theorem 2), and give examples for each case.

Our major tool is the following result of Haemers [9].

Result 1. Let A be a complex hermitian matrix of size n, which is partitioned into block matrices:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

such that A_{ii} is a square matrix of size m for all $1 \leq i \leq m$. Let B be the square matrix of size m, each element, b_{ij} of which equals the average row sum of the block A_{ij}. Then the eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ of A and the eigenvalues $\beta_1 \geq \cdots \geq \beta_m$ of B satisfy $\alpha_{n-m+i} \leq \beta_i \leq \alpha_i$ for all $1 \leq i \leq m$.

Moreover, if for some M, $1 \leq M \leq m$, $\beta_i = \alpha_i$ for all $1 \leq i \leq M$ and $\beta_i = \alpha_{n-m+i}$ for all $M < i \leq m$, then A_{ij} has constant row and column sum for all $1 \leq i, j \leq m$.

2. Main results

Theorem 1. *Let $D_1(v_1, k_1, \lambda_1)$ be a symmetric subdesign of a symmetric design $D(v, k, \lambda)$. Put $x = v_1(k - k_1)/(v - v_1)$. Then*

(i) $k \geq (k_1 - x)^2 + \lambda$.

(ii) *If $k = (k_1 - x)^2 + \lambda$, then the points [blocks] of D_1 and the blocks [points] not on D_1 form a possibly degenerate block design $D_2(v_1, x, \lambda - \lambda_1)$.*

Proof. (i) Let D_1 and D_2 be the incidence matrices of the designs in question. Write

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

then x equals the average row sum of D_3. Form

$$A = \begin{bmatrix} O & D \\ D' & O \end{bmatrix} = \begin{bmatrix} O & O & D_1 & D_2 \\ O & O & D_3 & D_4 \\ D_1' & D_3' & O & O \\ D_2' & D_4' & O & O \end{bmatrix},$$

where D' denotes the transpose of D. Next, we construct the matrix B consisting of the average row sums of A corresponding to the given blocking. Then,

$$B = \begin{bmatrix} O & O & k_1 & k - k_1 \\ O & O & x & k - x \\ k_1 & k - k_1 & O & O \\ x & k - x & O & O \end{bmatrix}.$$

The eigenvalues of

$$\begin{bmatrix} k_1 & k - k_1 \\ x & k - x \end{bmatrix}$$

are k and $k_1 - x$. Hence the eigenvalues of B are $\pm k$ and $\pm(k_1 - x)$. The eigenvalues of A are $\pm k$ and $\pm\sqrt{k - \lambda}$. Using the inequality of Result 1, then gives $\sqrt{k - \lambda} \geq (k_1 - x)$. This yields $k \geq (k_1 - x)^2 + \lambda$.

(ii) From $DD' = (k - \lambda)I + \lambda J$ and $D_1 D_1' = (k_1 - \lambda_1)I + \lambda_1 J$, it follows that

$D_2D_2^t = ((k-k_1)-(\lambda-\lambda_1))I + (\lambda-\lambda_1)J$. On the other hand if $k=(k_1-x)^2+\lambda$, Result 1 gives that D_2 has constant column sum. This proves (ii).

Remark (i) In proving (i) of Theorem 1, we did not use that D_1 represents a block design, but only that k_1 is the average row sum of D_1.

(ii) It is easily seen that if D is non-trivial then $k-k_1 \neq \lambda-\lambda_1$. But then $D_2D_2^t$ is a non-singular matrix, thus $v_1 \leq v - v_1 \leq \frac{1}{2}v$.

Corollary ([3] or [10]). *Let $D_1(v_1, k_1, \lambda)$ be a subdesign of $D(v, k, \lambda)$. Then $k \geq (k_1-1)^2+\lambda$.*

This follows immediately upon noting that in this case $x \leq 1$.

Definition $D_1(v_1, k_1, \lambda_1)$ is a *tight* subdesign of $D(v, k, \lambda)$ if $k=(k_1-x)^2+\lambda$.

Example 1. Let D be the design formed by the points and hyperplanes of PG(n, q), $2 \leq m \leq n-1$. Let X and Y be m and $n-m-1$ dimensional subspaces of PG(n, q) respectively, which do not have a point in common. The points of X and the hyperplanes containing Y form a subdesign D_1 of D. This subdesign is not tight.

Example 2. Let H_1 be a regular Hadamard matrix of size $m \geq 4$. This means that H_1 is a Hadamard matrix of size m and that in addition, $HJ = \rho J$, where ρ is a constant and J is the all one matrix. Using $H_1H_1^t = mI$ and $m \equiv 0 \pmod 4$, it follows that $m = \rho^2 = 4n^2$ for some positive integer n. Then H_1 is equivalent to a symmetric design $D_1(4n^2, n(2n-1), n(n-1))$. Put

$$H = \begin{bmatrix} H_1 & -H_1 & -H_1 & -H_1 \\ -H_1 & H_1 & -H_1 & -H_1 \\ -H_1 & -H_1 & H_1 & -H_1 \\ -H_1 & -H_1 & -H_1 & H_1 \end{bmatrix}.$$

Then H is a regular Hadamard matrix of size $16n^2$ and is equivalent to a symmetric design $D(16n^2, 2n(4n+1), 2n(2n+1))$. It is easily checked that D_1 is a tight subdesign of D. For examples of regular Hadamard matrices cf. [8].

Remarks. Let $D_1(v_1, k_1, \lambda_1)$ be a tight subdesign of $D(v, k, \lambda)$. Then

(i) $k-\lambda$ is a square.

(ii) The complement of D_1 is a tight subdesign of the complement of D.

3. Baer subdesigns

If $D_1(v_1, k_1, \lambda)$ is a tight subdesign of $D(v, k, \lambda)$ then D_1 is called a Baer subdesign of D([3]). For $\lambda = 1$, Baer subdesigns are just Baer subplanes of projective planes. In this case many things have been investigated [6]. Using $\bar{D}$ to denote the complement of D, we have:

Example 3. Let $D_1(v_1, k_1, 1)$ be a Baer subplane of $D(v, k, 1)$. Then $\bar{D}_1(v_1, v_1 - k_1, v_1 - 2k_1 + 1)$ is a tight subdesign of $\bar{D}(v, v-k, v-2k+1)$.

Theorem 2. *Let $D_1(v_1, k_1, \lambda)$ be a Baer subdesign of $D(v, k, \lambda)$. Then one of the following holds:*

(a) $v = \lambda(\lambda^2 - 2\lambda + 2)$, *$D$ has parameters $(\lambda(\lambda^2 - 2\lambda + 2), \lambda^2 - \lambda + 1, \lambda)$ and D_1 is the trivial design $(\lambda, \lambda, \lambda)$.*

(b) $v = \lambda^2(\lambda + 2)$, *$D$ has parameters $(\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda)$ and D_1 is the trivial design $(\lambda + 2, \lambda + 1, \lambda)$.*

(c) $v > \lambda^2(\lambda + 2)$.

Proof. Let D be a non-trivial design having a Baer subdesign D_1. Since $\lambda = \lambda_1$, we have $x \leq 1$. Since D_1 is tight, x must be an integer by Theorem 1 (ii). Hence,

$$x = v_1(k - k_1)/(v - v_1) = 1.$$

This gives

(1) $v = v_1(k - k_1 + 1)$, and

(2) $k = (k_1 - 1)^2 + \lambda$.

If D_1 is trivial, then $v_1 = k_1 = \lambda_1$ or $v_1 = k_1 + 1 = \lambda_1 + 2$. Using (1) and (2) we see that these two trivial cases lead to (a) and (b), respectively. If $v_1 > k_1 + 1$, then (1) and (2) gives

$$v > (k_1 + 1)((k_1 - 1)(k_1 - 2) + \lambda).$$

Using $k_1 > \lambda + 1$ we obtain $v > \lambda^2(\lambda + 2)$.

We now give examples to show that in each of the above cases, there exist symmetric designs with Baer subdesigns.

Example 4. A symmetric design $D(\lambda(\lambda^2 - 2\lambda + 2), \lambda^2 - \lambda + 1, \lambda)$ has the parameters of the symmetric design on the points and planes of PG$(3, \lambda - 1)$ which exist for all prime powers $\lambda - 1$. Moreover the points on a given line and all planes containing it form a Baer subdesign $D_1(\lambda, \lambda, \lambda)$.

Example 5. From Ahrens and Szekeres [1], the existence of symmetric designs D with parameters $(\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda)$ is known for all prime powers λ. From their construction it can be easily seen that D has a Baer subdesign $D_1(\lambda + 2, \lambda + 1, \lambda)$, corresponding to the $\lambda + 2$ points of a line in the corresponding geometry. Before giving an example to show the existence of a design satisfying (c) of

Theorem 2, we make some observations:

If we consider designs (v, k, λ) with $v > \lambda^2(\lambda+2)$, then according to [5], p. 105 the only known examples are projective planes of prime power order and biplanes ($=$symmetric designs with $\lambda = 2$) on 37, 56 and 79 points; as far as we know meanwhile one other example is found, a (71, 15, 3) design, see [2].

Note that if $D_1(v_1, k_1, \lambda)$ is a Baer subdesign of $D(v, k, \lambda)$, then v cannot be prime. Thus if we are to find a Baer subdesign $D_1(v_1, k_1, \lambda)$ of $D(v, k, \lambda)$ which is not a Baer subplane, it is easily seen from above that $D(56, 11, 2)$ is the only possible candidate available to us. Any Baer subdesign of D has parameters (7, 4, 2), those of the complement of the Fano plane (7, 3, 1). The next example shows that there is a (56, 11, 2) design with a Baer subdesign.

Example 6. We follow [7] Denniston who gives constructions of (56, 11, 2) designs some of which are based on Cameron's description [4] of biplanes. Namely, one block b^* is fixed and all the other blocks are in 1–1 correspondence with the unordered pairs of points of b^*. Each point not on b^* is represented by a disjoint union of polygons on the points of b^*. The block represented by $\{p, q\}$ is incident with p and q and with the points represented by graphs in which p and q are joined. Let us represent the points of b^* by $0, \ldots, 10$ then according to [7] in at least two of the constructed biplanes (the "nice" one due to Gewirtz, Hall, Lane and Wales, and another design due to Assmus and others), there exist three points off b^* whose polygons are:

$$\begin{array}{lll} (9\ 8\ 10\ 9) & (0\ 2\ 4\ 6\ 0) & (1\ 3\ 5\ 7\ 1) \\ (0\ 4\ 10\ 0) & (9\ 2\ 8\ 6\ 9) & (1\ 3\ 7\ 5\ 1) \\ (2\ 6\ 10\ 2) & (9\ 4\ 8\ 0\ 9) & (1\ 7\ 3\ 5\ 1). \end{array}$$

It is easily seen that these 3 points together with the points 1, 3, 5 and 7 from b^* form a (7, 4, 2) design which is a Baer subdesign of the (56, 11, 2) design.

Using the above example and considering the complementary designs, we have:

Example 7. There exists a $D(56, 45, 36)$ which has the Fano plane ($=(7, 3, 1)$ design) as a tight subdesign.

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