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2-Designs Having an Intersection Number $k - n$

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In this paper we examine 2-designs having an intersection number $k - n$. This intersection number gives rise to an equivalence relation on the blocks of the design. Conditions on the sizes of these equivalence classes and some properties of any further intersection numbers are obtained. If such a design has at most three intersection numbers then it gives rise to a strongly regular graph. This leads to a result on the embedding of quasi-residual designs. As as example a quasi-residual 2-(56, 12, 3) design is constructed and embedded in a symmetric 2-(71, 15, 3) design.

INTRODUCTION

Many interesting types of 2-designs do have an intersection number $k - n$, such as symmetric and affine designs, and those which admit a strong tactical decomposition, see [1]. Majumdar [12] showed that this intersection number $k - n$ gives rise to an equivalence relation on the blocks. The aim of this paper is to study 2-designs with an intersection number $k - n$.

We shall see that for a given value of $k - n \notin \{0, \lambda\}$ there are only finitely many such designs. Bounds are obtained for the size of an (equivalence) class and for the remaining intersection numbers; also the case of tightness is treated.

In section 4 we prove that all classes have equal size m, say, if the design has just three intersection numbers. In some special cases the converse is true, such as when $\lambda v/b$ and $n/m + k - n$ are consecutive integers. Goethals and Seidel [8] proved that any 2-design with just 2 intersection numbers

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"carries" a strongly regular graph. A similar situation exists in the case when the design has three intersection numbers, provided one of them equals $k - n$.

For the case when $k - n = 0$, and the design is quasi-residual we give conditions under which the design is embeddable in a symmetric 2-design. As an example we construct a 2-(56, 12, 3) design with intersection numbers 3, 2 and 0 ($=k - n$), and with non-trivial class graph. This design turns out to be embeddable in a symmetric 2-(71, 15, 3) design.

1. Preliminaries

For general references about designs we refer to [5], [7] and [10]. Suppose $\mathscr{D}$ is a 2-(v, k, λ) design of order n with b blocks and r blocks through a point, so $bk = vr$, $\lambda(v - 1) = r(k - 1)$, $n = r - \lambda$, $b \geqslant v$. The *intersection numbers* of $\mathscr{D}$ are the possible values of $|x \cap y|$, where x and y are two distinct blocks of $\mathscr{D}$. We do not allow $\mathscr{D}$ to have repeated blocks, so k is never an intersection number. A 2-design with one intersection number is a symmetric design ($b = v$). A 2-design with two intersection numbers is called quasi-symmetric. The following property of the intersection numbers will be frequently used, see [13].

Result 1.1. Let y be a block of $\mathscr{D}$, let m_i be the number of other blocks in $\mathscr{D}$ intersecting y in i points, $0 \leqslant i < k$. Then

$$\sum_{i=0}^{k-1} m_i = b - 1, \qquad \sum_{i=0}^{k-1} i m_i = k(r - 1),$$

$$\sum_{i=0}^{k-1} i^2 m_i = k(r - 1) + k(k - 1)(\lambda - 1).$$

A *resolution* of $\mathscr{D}$ is a partition of its blocks into classes $B_1, \ldots, B_c$ such that any point occurs a constant number of times in each class. Note that the constant equals $k\,|B_i|/v$. A resolution is called *strong* if the number of points in the intersection of two distinct blocks depends only on whether these blocks are from different classes or from the same class. So if $\mathscr{D}$ is strongly resolvable, which means $\mathscr{D}$ admits a strong resolution, $\mathscr{D}$ is quasi-symmetric or symmetric. The following is proved in [11].

Result 1.2. (i) $\mathscr{D}$ is strongly resolvable if and only if $\mathscr{D}$ has a resolution with $b - v + 1$ classes.

(ii) If $\mathscr{D}$ is strongly resolvable then two distinct blocks of the same class have $k - n$ points in common.

Now let $\mathscr{D}$ be a symmetric 2-(v, k, λ) design (i.e., $b = v = 1 + k(k - 1)/\lambda$) with $2k < v$. Let x be a block of $\mathscr{D}$ and let $\mathscr{D}_x$ be the incidence structure

which is obtained from $\mathscr{D}$ by deleting the points of x from the points of $\mathscr{D}$ and the blocks of $\mathscr{D}$ (disregarding the empty block). Then $\mathscr{D}_x$ is a 2-$(v-k, k-\lambda, \lambda)$ design of order $k-\lambda$, having $v-1$ blocks and k blocks through a point. We call $\mathscr{D}_x$ the *residual* design of $\mathscr{D}$ with respect to x. A design having the parameters of $\mathscr{D}_x$ is called a *quasi-residual* design. It is easily seen that a 2-(v, k, λ) design of order n is quasi-residual if and only if $k = n$.

We shall also need some definitions and results on strongly regular graphs for which we refer to [5], [8] or [15]. Throughout this paper I, J and $\mathbf{j}$ will denote respectively, the identity matrix, the all-one matrix, and the all-one vector of appropriate size.

Result 1.3. Given a graph Γ with adjacency matrix A. Γ is regular if and only if $\mathbf{j}$ is an eigenvector of A; the corresponding eigenvalue is the valency. Γ is strongly regular if and only if Γ is regular and $A \mid \mathbf{j}^{\perp}$ has just two eigenvalues.

The eigenvalues of a graph are the eigenvalues of its adjacency matrix.

Result 1.4. Let Γ be a strongly regular graph on p vertices with eigenvalues θ_1, θ_2 and the valency θ_0. The complement of Γ has eigenvalues $-1-\theta_1$, $-1-\theta_2$ and $p-1-\theta_0$ and the same multiplicities.

The disjoint union of d complete graphs on p vertices is a strongly regular graph, which we shall denote by $\Gamma(d, p)$. The complement of $\Gamma(d, p)$ is the complete d-partite graph on dp vertices.

Result 1.5. A strongly regular graph with eigenvalues θ_1, θ_2 $(\theta_1 > \theta_2)$ and the valency θ_0, is a $\Gamma(d, p)$ if and only if $\theta_0 = \theta_1$.

2. The Intersection Number $k-n$

We quote the following result due to Majumdar [12].

Result 2.1. Given any two blocks x and y of a 2-design, the number of points in their intersection is at least $k-n$. Two distinct blocks, x and y intersect in exactly $k-n$ points if and only if $|x \cap z| = |y \cap z|$ for all blocks z different from x and y.

Hence if a 2-design, $\mathscr{D}$, has $k-n$ as an intersection number, and x and y are two blocks of $\mathscr{D}$, we may define $x \sim y$ if and only if $x = y$ or $|x \cap y| = k-n$. Then $\sim$ is an equivalence relation on the blocks of $\mathscr{D}$. The partition of the blocks into equivalence classes will be called the *maximal decomposition* of $\mathscr{D}$. A refinement of the maximal decomposition is called a *decomposition* of the design. A decomposition of a 2-design has the property that the number of points in the intersection of two blocks depends only on their decomposition classes. A decomposition is said to be *regular* if each class

contains the same number of blocks. We shall show in subsequent sections that if a 2-design has just two or three intersection numbers then the maximal decomposition is regular.

Before taking a closer look at decompositions we first examine what type of designs do admit an intersection number $k - n$.

THEOREM 2.2. *If a 2-design admits an intersection number $k - n$ then*

(i) $k - n \geqslant 0$

(ii) $v - k - n \geqslant 0$

(iii) $v - 1 \geqslant b - r$

(iv) $v - 1 \geqslant r$

(v) $k > \frac{1}{2}r$.

Proof. (i) is trivial. (ii): Take two blocks having $k - n$ points in common, then the number of points in neither of these blocks equals $v - k - n$. (iii): Multiply (i) by $(v - 1)/k$. (iv): Multiply (ii) by $(v - 1)/(v - k)$. (v): Add (iii) and (iv). ∎

From $b \geqslant v$ it follows $k - n \leqslant \lambda$ hence $0 \leqslant k - n \leqslant \lambda$. If $k - n = \lambda$ then $b = v$ and the 2-design is symmetric. We know that in a symmetric 2-design any two blocks have λ points in common. If $k - n = 0$ then the 2-design is quasi-residual; not all quasi-residual designs have an intersection number $k - n = 0$, but for instance all affine planes do.

Since there are infinitely many affine planes and symmetric 2-designs, there exist infinitely many 2-designs with an intersection number $k - n$ whenever $k - n = 0$ or $k - n = \lambda$. However, following a suggestion by J. H. van Lint, we can prove that for other values of $k - n$ this is not true.

THEOREM 2.3. *For a given value of $k - n \notin \{0, \lambda\}$ there exist only finitely many 2-designs admitting the intersection number $k - n$.*

Proof. Take $0 < k - n < \lambda$. From $bk = vr$ and $\lambda(v - 1) = r(k - 1)$ it follows $b = r(rk - n)/k\lambda$, hence $k \mid rn$, thus $k \mid (r - k)(k - n)$. So we have

$$k \leqslant (r - k)(k - n) = (\lambda - (k - n))(k - n) \leqslant \tfrac{1}{4}\lambda^2.$$

So given λ we have finitely many possibilities for k, hence for n, and hence for all parameters of the 2-design. This implies that for a given λ the number of such 2-designs is finite. Now put $\alpha := (r - k)(k - n)/k$. With 2.2.v we have $r - k < k$ hence $0 < \alpha < k - n$. Now using $\lambda(v - 1) = r(k - 1)$ we have

$$\begin{aligned} v - 1 = {} & ((\lambda - k + n)(k - n) - \alpha) \\ & \times (\alpha\lambda - \alpha(k - n) + (\lambda - k + n)(k - n))/\alpha^2\lambda. \end{aligned}$$

Thus

$$\lambda \mid ((k-n)^2+\alpha)(k-n)(k-n+\alpha).$$

Now with $0 < \alpha < k-n$ we have $\lambda < 2(k-n)^3(k-n+1)$. So for a given value of $k-n$, only finitely many values of λ apply. With the former result it follows that then the number of such 2-designs is finite. ∎

From the proof of 2.3 we have $0 < \alpha < k-n$. This implies that $k-n=1$ cannot occur. Also we have seen

COROLLARY 2.4. *For a given value of λ there exist only finitely many 2-designs with an intersection number $k-n \notin \{0, \lambda\}$.*

We also point out that the total number of 2-designs admitting an intersection number $k-n$, satisfying $0 < k-n < \lambda$, is not finite. Indeed the complements of affine planes do the job.

Remark. One could ask whether a t-design with $t > 2$ can have an intersection number $k-n$. The answer is affirmative if and only if it is a Hadamard 3-design. This follows easily from 2.2 and some properties of Hadamard 3-designs, cf. [5].

Designs admitting a nontrivial strong tactical decomposition, see [1], [2], [3] and [14] are examples of 2-designs with an intersection number $k-n$. This includes all strongly resolvable designs. In the next section we will obtain conditions for a 2-design, having an intersection number $k-n$, to be strongly resolvable.

3. INEQUALITIES

THEOREM 3.1. *If $B_1, \ldots, B_c$ are the classes of a decomposition of a 2-design $\mathscr{D}$ then*

$$|B_j| \leqslant \frac{b}{b-v+1}$$

with equality for all j if and only if $B_1, \ldots, B_c$ are the classes of a resolution of $\mathscr{D}$.

Proof. Let A_j be the matrix of size $v \times |B_j|$ giving the incidence between the points of $\mathscr{D}$ and the blocks of B_j. Then

$$A_j^T A_j = (k-n)J + nI, \qquad A_j^T J = kJ.$$

Hence

$$\begin{aligned} 0 &\leqslant \mathbf{j}^T \left(A_j - \frac{k}{v} J\right)^T \left(A_j - \frac{k}{v} J\right) \mathbf{j} \\ &= \mathbf{j}^T (nI + (k-n-k^2/v)\, J)\, \mathbf{j} \\ &= |B_j|\, (n + |B_j|\, (k-n-k^2/v)). \end{aligned}$$

With $bk = vr$ and $\lambda(v - 1) = r(k - 1)$ this implies

$$n - n \mid B_j \mid (b - v + 1)/b \geqslant 0,$$

which yields the desired inequality.

Clearly equality holds if and only if

$$\left(A_j - \frac{k}{v} J\right) \mathbf{j} = 0,$$

which is equivalent to $A_j\mathbf{j} = k \mid B_j \mid /v\mathbf{j}$. This means that every point of $\mathscr{D}$ is in $k \mid B_j \mid /v$ blocks of B_j. Hence we have equality for all j if and only if $B_1, \ldots, B_c$ are the classes of a resolution of $\mathscr{D}$. ∎

COROLLARY 3.2. *If a decomposition of a 2-design is regular then*

$$v + c \geqslant b + 1$$

with equality if and only if the decomposition is a resolution and hence a strong resolution.

Proof. Since the decomposition is regular we have $\mid B_j \mid = b/c$ for all j. Hence 3.1 gives the result. ∎

From now on m will denote the number of blocks in a class of a regular decomposition. The number of classes is always denoted by c.

THEOREM 3.3. *If $\mathscr{D}$ is a nonsymmetric 2-design with a regular decomposition then*

$$k - n < \lambda v/b < n/m + k - n.$$

Proof. With $bk = vr$ we have $\lambda v/b - (k - n) = (r - \lambda)(b - v)/b$. Hence by $b > v$ we have $k - n < \lambda v/b$.

Theorem 3.1 implies $m \leqslant b/(b - v + 1)$, hence

$$\begin{aligned} n/m + k - n &\geqslant n(b - v + 1)/b + k - n \\ &= (-nv + n + bk)/b = (-nv + n + vr)/b \\ &= \lambda v/b + n/b > \lambda v/b. \quad ∎ \end{aligned}$$

It will return out that the numbers $\lambda v/b$ and $n/m + k - n$ play a special role with respect to regular decompositions. The next theorem shows that $n/m + k - n$ is an upper bound for the intersection numbers.

THEOREM 3.4. *Let $\mathscr{D}$ be a 2-design with a regular decomposition and let ρ_{ij} be the number of points in the intersection of two blocks from different classes B_i and B_j. Then*

$$\text{(i)} \quad \rho_{ij} \geqslant 2k^2/v - n/m - k + n$$

with equality if and only if all points of $\mathscr{D}$ are in the same number of blocks from $B_i \cup B_j$, and

$$\text{(ii)} \quad \rho_{ij} \leqslant n/m + k - n$$

with equality if and only if each point occurs the same number of times in blocks from B_i as in blocks from B_j.

Proof. As in the proof of Theorem 3.1 let A_i denote the matrix describing the incidence between the points of $\mathscr{D}$ and the blocks of B_i. Put

$$N = [A_i - k/v\, J \mid A_j - k/v\, J]$$

then

$$N^T N = \left[\begin{array}{c|c} nI + (k - n - k^2/v)\, J & (\rho_{ij} - k^2/v)\, J \\ \hline (\rho_{ij} - k^2/v)\, J & nI + (k - n - k^2/v)\, J \end{array}\right]$$

So we have

$$0 \leqslant \mathbf{j}^T N^T N \mathbf{j} = 2m^2(k - n - 2k^2/v + \rho_{ij}) + 2mn,$$

hence

$$\rho_{ij} \geqslant 2k^2/v - k + n - n/m.$$

Clearly equality holds if and only if $N\mathbf{j} = 0$, which is equivalent to $[A_i\, A_j]\mathbf{j} = 2mk/v\, \mathbf{j}$. This proves (i). Now define

$$\tilde{\mathbf{j}} := (\mathbf{j}^T \ -\mathbf{j}^T)^T, \qquad \text{with } \mathbf{j} \text{ and } -\mathbf{j} \text{ both of length } m.$$

Then $0 \leqslant \tilde{\mathbf{j}}^T N^T N \tilde{\mathbf{j}} = 2m^2(k - n - \rho_{ij}) + 2mn$, hence $\rho_{ij} \leqslant n/m + k - n$. Equality holds if and only if $N\tilde{\mathbf{j}} = 0$, which means $A_i\mathbf{j} = A_j\mathbf{j}$. This proves (ii). ∎

Majumdar [12] proved that any intersection number ρ of a 2-design satisfies

$$k - n \leqslant \rho \leqslant 2\lambda v/b - k + n.$$

We have already mentioned the equivalence relation on the blocks having $k - n$ points in common. In case of a regular decomposition we can do a

similar thing. Define two classes B_i and B_j to be related if and only if $i = j$ or $\rho_{ij} = n/m + k - n$, then from 3.4.ii it follows immediately that we have an equivalence relation.

In the case when $\mathscr{D}$ is strongly resolvable it can be seen that both bounds of Theorem 3.4 are tight.

4. Regular Decompositions

We now show that every 2-design with exactly three intersection numbers has a regular maximal decomposition.

Lemma 4.1. *Let $\mathscr{D}$ be a 2-design with three intersection numbers ρ_1, ρ_2 and ρ_3. Then the number of blocks that intersect a given block in ρ_i points ($i = 1, 2, 3$) is a constant independent of the block chosen.*

Proof. Let y be any block of $\mathscr{D}$ and let y intersect x_1 blocks in ρ_1 points and x_2 blocks in ρ_2 points. Then by result 1.1 we have

$$k + x_1\rho_1 + x_2\rho_2 + (b - 1 - x_1 - x_2)\,\rho_3 = rk$$
$$k^2 + x_1\rho_1^2 + x_2\rho_2^2 + (b - 1 - x_1 - x_2)\,\rho_3^2 = \lambda k^2 + nk.$$

Eliminating one of the x_i ($i = 1, 2$) from the two equations above we find

$$x_i = \frac{\lambda k^2 + nk - k^2 + (b - 1)\,\rho_j\rho_k - k(r - 1)(\rho_j + \rho_k)}{(\rho_i - \rho_j)(\rho_i - \rho_k)} \qquad (*)$$

where i, j, k are all distinct values 1, 2 or 3. Hence x_i is independent of the chosen block. ∎

Corollary 4.2. *Let $\mathscr{D}$ be a 2-design with exactly three intersection numbers $k - n$, ρ_1 and ρ_2. The maximal decomposition is regular, and the size m of the classes equals*

$$\frac{\lambda k^2 - kn + n^2 + b\rho_1\rho_2 - \lambda v(\rho_1 + \rho_2)}{(k - n - \rho_1)(k - n - \rho_2)}$$

Proof. Take $\rho_i = k - n$ in (*). The size of the class containing a given block y equals $x_i + 1$. ∎

It is clear that this method of proof cannot be used to prove regularity of the maximal decomposition of a 2-design with four or more intersection numbers one of which is $k - n$. In fact, the design of Bhattacharya provides a counterexample; cf. [12].

COROLLARY 4.3. *Let $\mathscr{D}$ be a 2-design with exactly three intersection numbers: $k-n$, ρ_1 and ρ_2. Then $\rho_1 = n/m + k - n$ if and only if $\rho_2 = \lambda v/b$.*

Proof. This follows from Corollary 4.2 by straightforward verification. ∎

Although we have no counterexample it is not likely that a 2-design with a regular maximal decomposition needs to have three or less intersection numbers. However in some special cases this is true. One of these is a result of Singhi and Shrikhande [16].

Result 4.4. If a quasi-residual 2-design has a regular decomposition with $\lambda - 1$ blocks in each class, then the only possible other intersection numbers are λ and $\lambda - 1$.

Another case is the corollary of the next theorem.

THEOREM 4.5. *Let $\mathscr{D}$ be a design with a block x, which intersects $m_x - 1$ blocks in $k-n$ points. Suppose $\lambda v/b$ and $n/m_x + k - n$ are consecutive integers. Then for any other block y of $\mathscr{D}$ we have $|x \cap y| = \lambda v/b$ or $|x \cap y| = n/m_x + k - n$.*

Proof. Let x_i be the number of blocks that intersect x in i points. Then $x_{k-n} = m_x - 1$ and $x_i = 0$ if $0 \leqslant i < k - n$.

Using result 1.1 we have

$$\begin{aligned}
&\sum_{i=k-n+1}^{k-1} (i - \lambda v/b)(i - (n/m_x + k - n))\, x_i \\
&\quad = \lambda k^2 + nk - k^2 - (m_x - 1)(k-n)^2 - (v/b + n/m_x + k - n) \\
&\qquad \times (rk - m_x k + m_x n - n) + \lambda v/b(n/m_x + k - n)(b - m_x) \qquad (*) \\
&\quad = nrk - n^2 - \lambda vn - rk^2 + nk + \lambda vk - (nrk - n^2 - \lambda vn)/m_x \\
&\quad = 0 \quad \text{on applying } \lambda(v-1) = r(k-1).
\end{aligned}$$

Thus if $\lambda v/b$ and $n/m_x + k - n$ are consecutive integers all terms of (*) are nonnegative and so we must have $x_i = 0$ for all $i \neq k - n$, $\lambda v/b$ or $n/m_x + k - n$. ∎

In the case when all m_x are equal Theorem 4.5 implies:

COROLLARY 4.6. *If $\mathscr{D}$ is a design with a regular maximal decomposition and $\lambda v/b$ and $n/m + k - n$ are consecutive integers, then $\mathscr{D}$ has at most three intersection numbers.*

The family of 2-designs admitting a strong tactical decomposition which was constructed in [3] yields examples of both 4.4 and 4.6.

5. Strongly Regular Graphs

Let $\mathscr{D}$ be any quasi-symmetric 2-design with intersection numbers ρ_1, and ρ_2 (take $\rho_1 > \rho_2$). Define the *block graph* of $\mathscr{D}$ to be the graph whose vertices are the blocks of $\mathscr{D}$, two vertices being adjacent if and only if the corresponding blocks have ρ_1 points in common. Goethals and Seidel proved the following result (cf. [5] or [8]).

Result 5.1. The block graph of a quasi-symmetric 2-design is strongly regular.

We shall show later on that a similar method may be used to obtain a strongly regular graph from a design with three intersection numbers, provided one of them is $k - n$. First we consider the case of two intersection numbers one of which is $k - n$.

Lemma 5.2. *Let $\mathscr{D}$ be a quasi-symmetric 2-design. Then*

(i) *the number of blocks that intersect a given block in ρ_1 points equals*

$$\frac{k(r-1) - \rho_2(b-1)}{\rho_1 - \rho_2},$$

(ii) $\rho_1 = k^2/v$ *if and only if* $\rho_2 = k - n$.

Proof. Let y be any block of $\mathscr{D}$ and let x_1 denote the number of blocks that intersect y in ρ_1 points (we know this number is a constant by result 5.1). By result 1.1

$$x_1\rho_1 + (b - x_1 - 1)\,\rho_2 = k(r-1),$$

i.e.,

$$x_1 = \frac{k(r-1) - \rho_2(b-1)}{\rho_1 - \rho_2}. \tag{1}$$

Also by result 1.1 we have

$$k^2 + x_1\rho_1^2 + (b - x_1 - 1)\,\rho_2^2 = \lambda k^2 + nk. \tag{2}$$

Hence eliminating x_1 in (1) and (2) gives

$$\rho_1(rk - k - b\rho_2 + \rho_2) = \lambda k^2 + nk + k\rho_2 - rk\rho_2 - k^2,$$

from which (ii) follows by straightforward verification. ∎

Theorem 5.3. *For a 2-design $\mathscr{D}$ the following are equivalent*

(i) *$\mathscr{D}$ is quasi-symmetric with an intersection number $k - n$,*

(ii) *$\mathscr{D}$ is strongly resolvable,*

(iii) *$\mathscr{D}$ is quasi-symmetric and its block graph is a complete c-partite graph.*

Proof. (i) → (ii): According to Lemma 5.2, $\mathscr{D}$ has a maximal decomposition in which every class has size

$$\frac{k(r-1)-(b-1)\,k^2/v}{k-n-k^2/v}+1=b/(b-v+1)$$

on applying $bk = vr$. Thus by Theorem 3.1 the maximal decomposition is a resolution which must be strong by Corollary 3.2.

(ii) → (iii): By definition $\mathscr{D}$ is quasi-symmetric and two vertices of its block graph are adjacent if and only if the corresponding blocks belong to different classes, so it is a complete c-partite graph.

(iii) → (i): Take two nonadjacent vertices of the complete c-partite graph. We know that every other vertex is either adjacent or nonadjacent to both. This means that for the corresponding two blocks of $\mathscr{D}$ we have the property that every further block intersects both blocks in the same number of points. With result 2.1 it follows that the two blocks intersect in $k-n$ points. ∎

From now on $\mathscr{D}$ is a 2-design with possibly an intersection number $k-n$ and just two further intersection numbers ρ_1 and ρ_2 ($\rho_1 > \rho_2$). Let the *class graph* of $\mathscr{D}$ be the graph whose vertices are the classes of the maximal decomposition, where two vertices are adjacent if and only if two blocks, one from each of the corresponding classes, meet in ρ_1 points. So the number of vertices of the class graph equals the number c of classes.

THEOREM 5.4. *The class graph of $\mathscr{D}$ is strongly regular with eigenvalues*

$$\theta_0=\frac{\lambda vc-b(k-n-\rho_2+\rho_2 c)}{b(\rho_1-\rho_2)}$$

$$\theta_1=\frac{\rho_2-k+n}{\rho_1-\rho_2}$$

$$\theta_2=\frac{b(\rho_2-k+n)-cn}{b(\rho_1-\rho_2)}$$

and multiplicities 1, $c-b+v-1$ *and* $b-v$, *respectively.*

Proof. Let N be the incidence matrix of $\mathscr{D}$. Then N satisfies $NN^T = \lambda J + nI$. This implies that NN^T has eigenvalues n with multiplicity $v-1$ and rk with multiplicity one. Hence $N^TN - nI$ has eigenvalues $-n$, 0 and λv where λv has multiplicity one. By Result 2.1 and Corollary 4.2 we can write

$$N^TN - nI = D^TBD$$

where B is a symmetric $c \times c$ matrix with $k - n$ on the diagonal and ρ_1 and ρ_2 elsewhere, and D is the $c \times b$ matrix:

$$D = \begin{array}{c} \begin{array}{ccc} \overbrace{\quad\quad}^{m} & \overbrace{\quad\quad}^{m} & \overbrace{\quad\quad}^{m} \end{array} \\ \begin{bmatrix} 1 \cdots 1 & & & \\ & 1 \cdots 1 & & \\ & & \ddots & \\ & & & 1 \cdots 1 \end{bmatrix} \end{array}.$$

Suppose α is an eigenvalue of B with eigenvector $\mathbf{a}$. Then $\alpha\mathbf{a} = B\mathbf{a} = (1/m)\, BDD^T\mathbf{a}$, since $DD^T = mI$. Thus $D^TBDD^T\mathbf{a} = \alpha m D^T\mathbf{a}$. So any eigenvalue of B multiplied by m is an eigenvalue of $N^TN - nI$.

Hence the only possible eigenvalues of B are $(\lambda - r)/m$, 0 and $\lambda v/m$, where $\lambda v/m$ is a simple eigenvalue belonging to the all-one eigenvector $\mathbf{j}$. Define

$$A = \frac{1}{\rho_1 - \rho_2}(B - \rho_2 J - (k - n - \rho_2)\, I).$$

Then A is the adjacency matrix of the class graph of $\mathscr{D}$. Clearly

$$\theta_0 = \frac{\lambda vc - b(k - n - \rho_2 + \rho_2 c)}{b(\rho_1 - \rho_2)}$$

is an eigenvalue of A belonging to the eigenvector $\mathbf{j}$. The only two possible eigenvalues of $A \mid \mathbf{j}^\perp$ are

$$\theta_1 = \frac{\rho_2 - k + n}{\rho_1 - \rho_2} \quad \text{and} \quad \theta_2 = \frac{b(\rho_2 - k + n) - cn}{b(\rho_1 - \rho_2)}.$$

Now let f_1 and f_2 be the multiplicities of θ_1 and θ_2 respectively. Then we have $f_1 + f_2 + 1 = c$ and $\theta_1 f_1 + \theta_2 f_2 + \theta_0 = \text{trace}(A) = 0$, hence $f_1 = (\theta_2 - c\theta_2 - \theta_0)/(\theta_1 - \theta_2) = c - b - 1 + (bk - \lambda v)/n = v + c - b - 1$ on applying $bk = vr$. Thus $f_2 = c - 1 - (v + c - b - 1) = b - v$. By Theorem 5.3 $\mathscr{D}$ is not strongly resolvable and hence Corollary 3.2 gives $v + c - b - 1 > 0$. $\mathscr{D}$ is clearly not symmetric, so $b - v > 0$, and both eigenvalues θ_1 and θ_2 do occur. Now result 1.3 implies that our graph is strongly regular. ∎

COROLLARY 5.5. *If two blocks x and y of $\mathscr{D}$ intersect in ρ_i points then the number of blocks intersecting x in ρ_j points and y in ρ_k points is a constant dependent on i, j and k but independent of x and y, for all $i, j, k \in \{1, 2\}$.*

Proof. This follows immediately from Theorem 5.4 and the definition of strongly regular graphs. The values of these constants can be readily calculated from the eigenvalues of the graph. ∎

If all classes of the maximal decomposition have size one, so $c = b$, then $\mathscr{D}$ is quasi-symmetric and Theorem 5.4 reduces to Result 5.1.

Remark. It does not often happen that a 2-design with more than 2 intersection numbers has the property that the block incidence structure gives an association scheme (or equivalently the dual of the design is a *PBIBD*). However from Corollary 5.5 it follows that $\mathscr{D}$ has this property.

The class graph of a 2-design constructed in [1], [3] or [14] is always a $\Gamma(d, p)$. This is easily checked with the help of the following lemma.

LEMMA 5.6. *The class graph of $\mathscr{D}$ is a $\Gamma(d, p)$ if and only if $\rho_2 = \lambda v/b$.*

Proof. By Result 1.5 the graph is a $\Gamma(d, p)$ if and only if $\theta_1 = \theta_0$, i.e., if and only if $\rho_2 = \lambda v/b$. ∎

We remark that the "if" part of Lemma 5.6 is also a consequence of 3.4.ii and 4.3. Lemma 5.6 implies that the class graph of a 2-design satisfying Corollary 4.6 is a $\Gamma(d, p)$. In Section 7 we construct a design $\mathscr{D}$ whose class graph is not a $\Gamma(d, p)$.

6. QUASI-RESIDUAL DESIGNS

Throughout this section let $\mathscr{D}$ be a quasi-residual $2 - ((k-1)(k-\lambda)/\lambda, k-\lambda, \lambda)$ design with intersection numbers ρ_1, ρ_2 ($\rho_1 > \rho_2$) and possibly $k - n = 0$. Then $\mathscr{D}$ has $k(k-1)/\lambda$ blocks and k blocks through a point, and the symmetric design into which it possibly embeds is a 2-(v, k, λ) design where $\lambda(v-1) = k(k-1)$.

LEMMA 6.1. *Let $\bar{\mathscr{D}}$ be a quasi-symmetric 2-$(k, \lambda, (\lambda-1)/m)$ design (m is the size of the classes of $\mathscr{D}$). Then the block graph of $\bar{\mathscr{D}}$ has the same eigenvalues as the complement of the class graph of $\mathscr{D}$ if and only if the intersection numbers of $\bar{\mathscr{D}}$ are $\lambda - \rho_1$ and $\lambda - \rho_2$.*

Proof. With Theorem 5.4 and Result 1.4 it follows that the eigenvalues of the complement of the class graph of $\mathscr{D}$ are $(\rho_1(c-1) - (k-1)(k-\lambda)/m)/(\rho_1 - \rho_2)$, $((k-\lambda)/m - \rho_1)/(\rho_1 - \rho_2)$ and $-\rho_1/(\rho_1 - \rho_2)$ with multiplicities 1, $k-1$ and $c-k$ respectively. Let $\bar{\rho}_1$ and $\bar{\rho}_2$ be the intersection numbers of $\bar{\mathscr{D}}$. Again using Theorem 5.4 we find that the block graph of $\bar{\mathscr{D}}$ has eigenvalues $((\lambda - \bar{\rho}_2)(c-1) - (k-1)(k-\lambda)/m)/(\bar{\rho}_1 - \bar{\rho}_2)$, $((k-\lambda)/m + \bar{\rho}_2 - \lambda)/(\bar{\rho}_1 - \bar{\rho}_2)$ and $(\bar{\rho}_2 - \lambda)/(\bar{\rho}_1 - \bar{\rho}_2)$ with multiplicities 1, $k-1$ and $c-k$ respectively. We observe that the eigenvalues of both graphs are the same if and only if $\bar{\rho}_1 = \lambda - \rho_2$ and $\bar{\rho}_2 = \lambda - \rho_1$. ∎

THEOREM 6.2. *$\mathscr{D}$ is embeddable in a symmetric 2-design $\mathscr{S}$ if and only if there exists a quasi-symmetric 2-$(k, \lambda, (\lambda - 1)/m)$ design $\bar{\mathscr{D}}$, whose block graph is isomorphic to the complement of the class graph of $\mathscr{D}$.*