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AN INEQUALITY FOR GENERALIZED HEXAGONS

ABSTRACT. We show that a generalized hexagon with $s + 1$ points on a line and $t + 1$ lines through a point satisfies $s = 1$ or $t \leq s^3$.

1. INTRODUCTION

A generalized n-gon is an incidence structure with points and lines, where every pair of elements (an element is a point or a line) is in an n-gon, but such that no m-gons for $m < n$ occur (see [4]–[7], [9]–[12]).

Throughout this paper we assume that every line is incident with $s + 1$ points and every point is incident with $t + 1$ lines, $s > 1$, $t > 1$. W. Feit and G. Higman [5] (see also [7]) proved that such generalized n-gons can exist only if $n \in \{2, 3, 4, 6, 8\}$. The case $n = 2$ is degenerate (every point is incident with every line). It is not difficult to verify that $n = 3$ corresponds to a projective plane of order s, thus $s = t$. D.G. Higman [7] proved that $t \leq s^2$ if $n = 4$ or $n = 8$. We shall prove (in two different ways) that $t \leq s^3$ for generalized hexagons. This bound can be attained, since J. Tits [12] showed the existence of generalized hexagons with $t = s^3$ for every prime power s.

2. PRELIMINARIES

Let H be a generalized hexagon. The distance between two points of H is the minimal number of lines it takes to go from one point to the other. It is well known (cf. [4], [9] or [10]), and straightforward to verify, that the distances between the points of H give rise to a 3-class association scheme, or equivalently a symmetric coherent configuration of rank 4 (cf. [7] or [8]). For $i = 0, 1, 2, 3$ let B_i denote the association matrix of distance i in this association scheme. Clearly $B_0 = I$ (identity matrix) and $\sum_{i=0}^{3} B_i = J$ (all-one matrix). It is known (cf. [1] or [3]) that the B_i's have a common basis of eigenvectors, and that each B_i has at most four distinct eigenvalues. Let $\lambda_0(i), \ldots, \lambda_3(i)$ denote the eigenvalues of B_i for $i = 0, 1, 2, 3$. These eigenvalues and their multiplicities are given in Table I; references for these values are [9], [10] or [11] (in [11] only the eigenvalues and multiplicities of $B_1 + (t + 1)I$ are given, however, all eigenvalues then follow immediately from $B_1^2 = s(t + 1)I + (s - 1)B_1 + B_2$ and $B_3 = J - I - B_1 - B_2$).

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TABLE I

Eigenvalue	$\lambda_0(i)$	$\lambda_1(i), \lambda_2(i)$	$\lambda_3(i)$
$i=0$	1	1	1
$i=1$	$s(t+1)$	$s-1\pm\sqrt{st}$	$-t-1$
$i=2$	$s^2t(t+1)$	$-s\pm(s-1)\sqrt{st}$	$t(t+1)$
$i=3$	t^2s^3	$\mp s\sqrt{st}$	$-t^2$
Multiplicity	1	$st(s+1)(t+1)\dfrac{1\pm\sqrt{st}+st}{2(s+t\pm\sqrt{st})}$	$s^3\dfrac{s^2t^2+st+1}{s^2+st+t^2}$

3. The first proof

This proof shows that the inequality $t \le s^3$ is a direct consequence of the Krein condition, see [7]–[10]. (Lemma 2.4 of [3] gives the same condition, but it is not in the explicit form quoted here.) Indeed, the Krein condition requires that

$$0 \le \sum_{i=0}^{3} \frac{\lambda_j(i)\lambda_k(i)\lambda_l(i)}{\lambda_0^2(i)}$$

for all $j, k, l \in \{1, 2, 3\}$. Take $j = k = 3$, and $l = 2$. Then we have

$$0 \le 1 + \frac{(-t-1)^2(s-1-\sqrt{st})}{s^2(t+1)^2} + \frac{t^2(t+1)^2(-s-(s-1)\sqrt{st})}{s^4t^2(t+1)^2} + \frac{t^4s\sqrt{st}}{t^4s^6}.$$

This yields

$$0 \le (s^2-1)(s+1)(s^2-\sqrt{st}).$$

We assumed $s > 1$, hence we have $s^2 \ge \sqrt{st}$. Thus $t \le s^3$.

4. The second proof

Define $B := B_2 - (s-1)B_1 + (s^2-s+1)I$. Then from Table I it follows that

$$\operatorname{rank}(B) = 1 + s^3(1+st+s^2t^2)/(s^2+st+t^2).$$

Let L be a line of H. Let S be the set of points at distance 1 from a point on L, but not on L. Clearly $|S| = (s+1)st$. Let B_1', B_2' and B' be the submatrices of B_1, B_2 and B, respectively, corresponding to S. We easily see that without loss of generality

$$I + B_1' = I_{st+t} \otimes J_s \text{ and } I + B_1' + B_2' = I_{s+1} \otimes J_{st},$$

where $\otimes$ denotes the Kronecker product and the indices of I and J indicate the size of these square matrices. Hence

$$B' = I_{s+1} \otimes J_{st} - s(I_{st+t} \otimes J_s) + s^2 I.$$

Because $I_{s+1} \otimes J_{st}$, $I_{st+t} \otimes J_s$ and $I_{st(s+1)}$ have a common basis of eigenvectors we can compute the eigenvalues of B' straightforwardly. They are s^2, 0 and st of multiplicity $(s^2 - 1)t$, $(s + 1)(t - 1)$ and $s + 1$, respectively. Thus

$$\text{rank}(B') = (s + 1)(ts - t + 1).$$

Using $\text{rank}(B') \leq \text{rank}(B)$ we get

$$(s + 1)(ts - t + 1)(s^2 + st + t^2) \leq s^2 + st + t^2 + s^3(1 + st + s^2t^2).$$

This yields

$$t^2(s^2 - 1)(t - s^3) \leq 0.$$

Thus $t \leq s^3$.

5. Final remarks

Because the dual of a generalized hexagon is again a generalized hexagon it also follows that $s \leq t^3$.

In [7] D.G. Higman studies coherent configurations on the flags of generalized n-gons. It is remarkable that, using the Krein condition for these configurations, he finds the desired inequalities for generalized quadrangles and octagons, but not for generalized hexagons.

In case of a 2-class association scheme it is known what the consequences are of equality in the Krein condition (cf. [2]). However, for schemes with more classes only a few results of that type are known (cf. [10]), but not enough to draw conclusion for generalized hexagons with $t = s^3$ via the first approach. But using the second approach one can obtain some additional regularity for generalized hexagons attaining our bound (see [6]).

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