Tilburg University

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Haemers, W. H.

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AN UPPER BOUND FOR THE SHANNON CAPACITY OF A GRAPH

W. HAEMERS

ABSTRACT

In [2] it is shown that the Shannon capacity of the Schläfli-graph is strictly less than Lovász's bound. To this purpose a different upper bound for the Shannon capacity of a graph has been deduced. In this note we shall investigate this upper bound for its own merits. As a result the Shannon capacity of some graphs will be determined.

INTRODUCTION

The purpose of this note is to investigate the upper bound for the Shannon capacity as given in [2]. We were not able to fit things into a nice theory, like Lovász did for his bound – see [4]. This paper just gives some examples illustrating how the bound can be used. We shall quote results about association schemes, Lovász's bound, etc. without giving many details. Often the necessary details can be found in [7], which appears in these proceedings.

Let G be a graph, with vertex set $\{1, \dots, \nu\}$. We say that a $\nu \times \nu$-matrix B *fits* G if $(B)_{ii} \neq 0$, and $(B)_{ij} = 0$ if i and j are adjacent, for all $i, j \in \{1, \dots, \nu\}$. Let us write $B^{\otimes k}$ for the Kronecker product of k copies of B, and G^k for the normal product of k copies of G. So, if B fits G, then $B^{\otimes k}$ fits G^k. Let, as usual, $\alpha(G)$ denote the size of the largest coclique (independent set) of G. Now $\Theta(G)$, the *Shannon capacity* of G, is defined by

$$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)}.$$

This number has an interpretation in coding theory – see [8]. The following theorem is proved in [2].

Theorem. *If a matrix B (over any field) fits a graph G, then* $\Theta(G) \leqslant \operatorname{rank}(B)$.

Proof. Since $B^{\otimes k}$ fits G^k, $B^{\otimes k}$ has a diagonal matrix, of size $\alpha(G^k)$, with non-zero diagonal entries, as a submatrix. Hence $\operatorname{rank}(B^{\otimes k}) \geqslant \alpha(G^k)$. On the other hand it is well-known (see, e.g., [5] p. 28), that $\operatorname{rank}(B^{\otimes k}) = (\operatorname{rank}(B))^k$. Thus we have

$$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)} \leqslant \sup_k \sqrt[k]{\operatorname{rank}(B^{\otimes k})} = \operatorname{rank}(B). \blacksquare$$

For a graph G we introduce the following number

$$R(G) := \min\{\operatorname{rank}(B) \mid \text{the matrix } B \text{ fits } G\}.$$

The theorem gives us $\Theta(G) \leqslant R(G)$. Write $\bar{G}$ for the complementary graph of G, and let $\gamma(G)$ denote the colouring number of G; then

(1) $\qquad R(G) \leqslant \gamma(\bar{G})$.

Proof. Let $\bar{G}$ be coloured with $\gamma(\bar{G})$ colours, and define the matrix B by

$$(B)_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are in the same colour class,} \\ 0, & \text{otherwise.} \end{cases}$$

Then B fits G, and $\operatorname{rank}(B) = \gamma(\bar{G})$. Thus our theorem gives the desired result. $\blacksquare$

This determines $R(G)$ for all graphs G with $\alpha(G) = \gamma(\bar{G})$. However, inequality (1) does not lead to new results concerning the Shannon capacity since Shannon [8] already proved that $\Theta(G) \leqslant \gamma(\bar{G})$ for all graphs G. In order to compare our upper bound $R(G)$ with Lovász's bound $\vartheta(G)$ we shall need the following results, which are proved in [4].

$$(2) \qquad \vartheta(G) \cdot \vartheta(\bar{G}) \geqslant \nu.$$

If G is regular of degree k then

$$(3) \qquad \vartheta(G) \leqslant \frac{-\nu\lambda_\nu}{k-\lambda_\nu},$$

where λ_ν denotes the smallest eigenvalue of the adjacency matrix of G.

APPLICATIONS

Let A be the adjacency matrix of G. Let λ be a non-zero eigenvalue of A, with multiplicity μ. Then $A - \lambda I$ fits G and has rank $\nu - \mu$ (over the field of real numbers). Hence

$$(4) \qquad R(G) \leqslant \nu - \mu.$$

Result (4) may be useful if one of the eigenvalues has a large multiplicity. This happens if G is a strongly regular graph (see [3]), that is, if A satisfies

$$(5) \qquad (A - rI)(A - sI) = (k + rs)J,$$

for certain numbers k, r and s $(k \geqslant r \geqslant s)$. Here I denotes the identity matrix, and J denotes the all-one matrix. By looking at the diagonal of both sides of (5) we see that G is regular of degree k. If G is non-trivial (i.e., if G and $\bar{G}$ is not a disjoint union of complete graphs), (5) implies that r and s are non-zero eigenvalues of G. Let f and g be their multiplicities. Then rank $(J) = 1$ implies $f + g = \nu - 1$. Hence by (4)

$$(6) \qquad R(G) \leqslant \min\{1 + f, 1 + g\}.$$

The eigenvalues of $\bar{G}$ are $\nu - k - 1$, $-s - 1$ and $-r - 1$; hence by (3) we have

$$\vartheta(G) \leqslant \frac{-\nu s}{k-s} \quad \text{and} \quad \vartheta(\bar{G}) \leqslant \frac{\nu(r+1)}{\nu - k + r}.$$

Hence

$$\vartheta(G) \cdot \vartheta(\bar{G}) \leqslant \frac{-s\nu^2(r+1)}{(k-s)(\nu-k+r)}.$$

If we multiply both sides of (5) by the all-one vector j we obtain $(k-r)(k-s) = (k+rs)\nu$, which yields $-\nu s(r+1) = (k-s)(\nu-k+r)$. Thus we have $\vartheta(G) \cdot \vartheta(\bar{G}) \leqslant \nu$. Now with (2) we have

$$\vartheta(G) \cdot \vartheta(\bar{G}) = \nu \quad \text{and} \quad \vartheta(G) = \frac{-\nu s}{k-s}.$$

These equalities also follow directly from results of Schrijver [6]. In [3] a lot of strongly regular graphs are described. Using this it is not difficult to find strongly regular graphs with $1 + g < \frac{-\nu s}{k-s}$, so $R(G) < \vartheta(G)$. Let us give one example. Let G be the graph whose vertices are the points of an elliptic quadric in PG$(5, q)$, two vertices being adjacent iff the line through the two points is on the quadric. Then $\nu = (q^3+1)(q+1)$, $k = q(q^2+1)$, $r = q-1$, $s = -q^2-1$, $g = q(q^2-q+1)$. Thus $R(G) \leqslant 1 + q(q^2-q+1) < q^3 + 1 = \vartheta(G)$.

The generalization of (6) to association schemes (see [1], [6] or [7]) is as follows. Let G be a graph corresponding to one class in a primitive association scheme. Let μ be the multiplicity of any eigenvalue of the scheme, not corresponding to the vector j. Then

(7) $\qquad R(\bar{G}) \leqslant 1 + \mu.$

Proof. Let $D_0, \ldots, D_n$ be the adjacency matrices of the association scheme. Let E be the idempotent corresponding to μ. Then rank $(E) = \mu$ and $E = \sum_{i=0}^{n} p_i D_i$, for certain numbers p_i. If D_j is the adjacency matrix of G then $E - p_j J$ fits $\bar{G}$, and has rank at most $\mu + 1$. Thus our theorem gives the result. ∎

Now we shall give an application of our theorem in case the matrix B is taken over a finite field. We define a graph G as follows. The vertices are the m-subsets of a fixed n-set. Let p be a prime not dividing m.

Let two vertices x and y be adjacent iff $|x \cap y| \not\equiv 0 \pmod{p}$. Then

(8) $\qquad R(G) \leqslant n.$

Proof. Let M be the $n \times \binom{n}{m}$ incidence matrix of the m-subsets of the n-set. Then, in the field GF (p), the matrix $B := M^T M$ fits G, and rank $(B) \leqslant n$.∎

For several values of m, n and p it is possible to find a coclique of size n in G. For example, take $p = 2$, $m = 3$, $n \equiv 0 \pmod 4$; partition the underlying n-set into classes of size 4: then all 3-subsets which are subsets of one of these classes form a coclique in G of size n. Hence by (8) $\alpha(G) = \Theta(G) = R(G) = n$. The graph G in this example is a graph corresponding to one class in the Johnson scheme $J(n, m) = J(n, 3)$ (see [1], [6] or [7]). One of the multiplicities of these association schemes is equal to $n - 1$. So (7) gives $R(\bar{G}) \leqslant n$. Thus $R(G) \cdot R(\bar{G}) \leqslant n^2$. From (2) we have $\vartheta(G) \cdot \vartheta(\bar{G}) \geqslant \binom{n}{3}$. Thus if $n > 8$ at least one of $R(G)$ and $R(\bar{G})$ must be smaller than Lovász's bound. In fact, using the techniques of [6] and [7] it follows that $\vartheta(G) = \dfrac{n(n-2)(2n-1)}{3(3n-14)} > n \geqslant R(G)$ if $n > 8$.

We hope that this note illustrates that our bound can be a useful tool in determining or estimating the Shannon capacity of a graph. Sometimes our bound is easier to compute than Lovász's bound. A disadvantage is that $R(G)$ always is an integer. However, we might use

$$\Theta(G) \leqslant \sqrt[k]{R(G^k)}$$

to obtain non-integer bounds, since it may very well be the case that $R(G^k) < (R(G))^k$.

For most graphs Lovász's bound seems to be smaller than ours, but there are many cases in which the two bounds coincide. For instance for the Kneser-graphs (n, m) Roos (private communication) proved that $R(G) = \binom{n-1}{m-1}$ if n is large enough with respect to m.

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REFERENCES

[1] Ph. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Repts. suppl.,* 10 (1973).

[2] W. Haemers, On the problems of Lovász concerning the Shannon capacity of a graph, *IEEE Trans. Inform. Theory,* 25 (1979), 231-232.

[3] X.L. Hubaut, Strongly regular graphs, *Discrete Math.,* 13 (1975), 357-381.

[4] L. Lovász, On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory,* 25 (1979), 1-7.

[5] M. Marcus – H. Minc, *A survey of matrix theory and matrix inequalities,* Alyn and Bacon, Boston 1964.

[6] A. Schrijver, A comparison of the Delsarte and Lovász bounds, *IEEE Trans. Inform. Theory,* 25 (1979), 425-429.

[7] A. Schrijver, Association schemes and the Shannon capacity: Eberlein polynomials and the Erdős – Ko – Rado theorem, *these proceedings.*

[8] C.E. Shannon, The zero-error capacity of a noisy channel, *IRE Trans. Inform. Theory,* 3 (1956), 3-15.

W. Haemers

University of Technology, Eindhoven, The Netherlands.