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Published in:
Annals of Discrete Mathematics

Publication date:
1982

Annals of Discrete Mathematics 15 (1982) 205–212

A PARTIAL GEOMETRY pg(9, 8, 4)

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Dedicated to N.S. Mendelsohn on the occasion of his 65th birthday

We describe a construction of a partial geometry pg(9, 8, 4) based on binary words of length 9 and PG(1, 8).

1. Introduction

A (finite) *partial geometry* $S = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is an incidence structure with a symmetric incidence relation satisfying the following axioms:

(i) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(ii) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line (here the second assertion is implied by (i)).

(iii) If x is a point and L a line, such that $x \not\mathrm{I}\, L$, then there are exactly α ($\alpha \geq 1$) points $x_1, x_2, \ldots, x_\alpha$ and α lines $L_1, L_2, \ldots, L_\alpha$ such that $x \mathrm{~I~} L_i \mathrm{~I~} x_i \mathrm{~I~} L$ ($i = 1, 2, \ldots, \alpha$).

Although the numbers s, t and α are called the parameters of the partial geometry, we denote such a partial geometry by pg$(s+1, t+1, \alpha)$ because the numbers in brackets correspond to the objects which are counted by (i), (ii) and (iii).

We call the points x and y *collinear* if there is a line incident with x and y. We denote this by $x \sim y$.

If $S = \mathrm{pg}(s+1, t+1, \alpha)$ then the corresponding graph $\Gamma(S)$ is defined as follows. A vertex of $\Gamma(S)$ is a point of S and $\{x, y\}$ is an edge iff $x \sim y$. It is easy to check that $\Gamma(S)$ is a strongly regular graph with parameters

$$n = (s+1)(st+\alpha)/\alpha\,, \qquad k = s(t+1)\,, \qquad \lambda = (s-1) + t(\alpha-1)\,,$$
$$\mu = (t+1)\alpha$$

and that the eigenvalues of the adjacency matrix of $\Gamma(S)$ are $s(t+1)$, $s-\alpha$, $-t-1$ of multiplicity 1, $f := st(s+1)(t+1)/\alpha(s+t+1-\alpha)$ and $n-f-1$ respectively (cf. [1, 7]).

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A partial geometry is called a *proper* partial geometry if $1 < \alpha < \min\{s, t\}$. Until a few months ago the only known infinite classes of proper partial geometries were those constructed using arcs in projective planes of order 2^h. All these geometries have odd α. The only other known proper partial geometry was pg(6, 6, 2) constructed by Van Lint and Schrijver (cf. [6]) using codes.

Subsequently Cohen constructed a pg(9, 8, 4) using group theory (cf. [2]). In this note we describe a much simpler representation of such a geometry. These two constructions inspired De Clerck, Dye and Thas (cf. [3]) to construct a new infinite class of partial geometries $\mathrm{pg}(2^{2n-1}+1, 2^{2n-1}, 2^{2n-2})$ which is more or less naturally embedded in projective geometries (e.g. pg(9, 8, 4) is associated with a quadric in PG(7, 2)). The idea of this construction was generalized by Thas, producing a method for constructing $\mathrm{pg}(3^{2n-1}+1, 3^{2n-1}, 2.3^{2n-2})$ which works for at least two (and probably for all) values of n (cf. [8]). At that point (early 1980) the geometry pg(6, 6, 2) was once again the only 'sporadic' proper partial geometry. Quite recently (June 1980) Haemers constructed a second one, namely pg(5, 18, 2) (cf. [5]).

2. Some lemmas

Partial geometries were introduced by Bose [1] to study large cliques in strongly regular graphs. We first give a bound on the size of a (co-)clique which is well known. For a proof using matrix techniques we refer to [4]. We give a proof by a double counting method (which we have seen referred to as Mendelsohn's method).

Lemma 1. *Let G be a strongly regular graph with parameters n, k, λ, μ and let σ be the negative eigenvalue of the adjacency matrix of G, i.e. the negative root of the equation $x^2+(\mu-\lambda)x+\mu-k=0$. Let C be a coclique of size c. Then,*

$$c \leq -n\sigma/(k-\sigma)$$

and equality holds iff each vertex outside C is adjacent to $kc/(n-c)$ vertices of C..

Proof. Let x_i be the number of vertices outside C which are adjacent to i vertices of C. Then we have

$$\sum x_i = n-c, \quad \sum ix_i = kc, \quad \sum \binom{i}{2}x_i = \mu\binom{c}{2}.$$

Hence,

$$\sum \left(i - \frac{kc}{n-c}\right)^2 x_i = \mu c(c-1) + kc - \frac{k^2c^2}{n-c}$$

is nonnegative. This implies

$$\mu(c-1) + k - k^2c/(n-c) \geqslant 0 \, .$$

Write $x := -kc/(n-c)$. By use of $k(k-\lambda-1) = \mu(n-k-1)$ we find

$$x^2 + (\mu - \lambda)x + \mu - k \leqslant 0 \, ,$$

hence $\sigma \leqslant x$ and both results follow. □

A strongly regular graph G with parameters n, k, λ, μ is called *pseudo-geometric* if there are integers s, t, α such that a partial geometry $S = \mathrm{pg}(s+1, t+1, \alpha)$ would have a $\Gamma(S)$ with the parameters of G.

Lemma 2. *If G is a pseudo-geometric strongly regular graph corresponding to the parameters s, t, α, then for a clique L in G we have*

$$|L| \leq s+1 \, ,$$

and if equality holds, then every vertex not in L is adjacent to α vertices of L.

Proof. This follows from Lemma 1 applied to the complement of G. □

The following lemma is known, but not well enough !

Lemma 3. *Let G be a pseudo-geometric strongly regular graph corresponding to the parameters s, t, α. Let $\mathscr{L}$ be a collection of cliques of G, each of size $s+1$, such that*

(i) $|\mathscr{L}| = (t+1)(st+\alpha)/\alpha$,

(ii) *each edge of G is in at least one element of $\mathscr{L}$.*

Then the incidence structure with the vertices of G as points, the cliques of $\mathscr{L}$ as lines and inclusion as incidence is a $\mathrm{pg}(s+1, t+1, \alpha)$.

Proof. Since $\binom{s+1}{2}|\mathscr{L}|$ equals the number of edges of G, each edge is in exactly one element of $\mathscr{L}$. Since G is pseudo-geometric this implies that axioms (i) and (ii) of a partial geometry are satisfied. The third axiom follows from Lemma 2. □

In our construction of a partial geometry pg(9, 8, 4) a function φ defined on the 4-subsets of points of PG(1, 8) plays a central role.

Definition. Let PG(1, 8) be described in the usual way as $\mathbb{F}_8 \cup \{\infty\}$. Define

$$\varphi(a, b, c, d) := \left(\frac{abc + abd + acd + bcd}{a + b + c + d}\right)^{1/2}.$$

First observe that φ depends only on the 4-tuple $\{a, b, c, d\}$ and not on the order of the elements. Furthermore

$$\begin{aligned}
&\varphi(1 + a, 1 + b, 1 + c, 1 + d) = 1 + \varphi(a, b, c, d),\\
&\varphi(a^{-1}, b^{-1}, c^{-1}, d^{-1}) = (\varphi(a, b, c, d))^{-1},\\
&\varphi(\alpha a, \alpha b, \alpha c, \alpha d) = \alpha\varphi(a, b, c, d) \quad \text{if } \alpha \neq 0.
\end{aligned}$$

Since PSL(2, 8) is generated by the transformations $x \to x + 1$, $x \to 1/x$, $x \to \alpha x$ we see that φ and PSL(2, 8) commute in the above sense.

Let V be the set of nine points of PG(1, 8). We can consider φ as a function on $\binom{V}{4}$, i.e. the set of 4-subsets of V.

Lemma 4. *For every $x \in V$, the points of $V\backslash\{x\}$ and the blocks $\{X: \varphi(X) = x\}$ form the* 3-(8, 4, 1) *design of points and planes of* AG(3, 2).

Proof. W.l.o.g. we can take $x = \infty$. The definition of φ implies that the sets $X = \{a, b, c, d\}$ with $\varphi(X) = \infty$ satisfy $a + b + c + d = 0$, so they are the planes of AG(3, 2) in its representation as $\mathbb{F}_8$. □

Corollary 5. *φ has the following properties*:

(i) $\varphi(X) \notin X$,

(ii) $\varphi(X) = \varphi(V\backslash(X \cup \{\varphi(X)\}))$,

(iii) $\varphi(X) = \varphi(Y) \Rightarrow |X \cap Y| \in \{0, 2, 4\}$.

Lemma 6. *If $A \in \binom{V}{3}$ and $\varphi_A : V\backslash A \to V\backslash A$ is defined by $\varphi_A(x) := \varphi(A \cup \{x\})$, then $\varphi_A^3 = I$.*

Proof. Since PSL(2, 8) is 3-transitive on V we may take $A = \{0, 1, \infty\}$. Then $\varphi_A(x) = x^4$, so $\varphi_A^3(x) = x$. □

Corollary 7. *If $A \in \binom{V}{3}$, then A and the two orbits of φ_A are a partition of V into three* 3-*subsets.*

Lemma 8. *If $\{A, B, C\}$ is the partition determined by φ_A, then φ_B determines the same partition.*

Proof. Let $x \in C$, i.e., $C = \{x, \varphi_A(x), \varphi_A^2(x)\}$. By (ii) of Corollary 5 and by Lemma 6 we have $\varphi_B(x) = \varphi_A^2(x) \in C$, i.e., C is an orbit of φ_B. $\square$

3. Construction of pg(9, 8, 4)

We first construct a strongly regular graph G with parameters (135, 64, 28, 32). As vertices of G we consider all binary words of length 9 with weight 4 or 8. There are $\binom{9}{8} + \binom{9}{4} = 135$ such words. Two words are joined by an edge iff their distance is 2 or 6. It is elementary to check that this yields the required graph. In fact this description of the graph differs only slightly from the usual description (using the quadratic form $\Sigma_{i<j} x_i x_j$).

The difficulty in finding the corresponding partial geometry lies in the fact that G has far too many cliques of size 9. These are of four types:

Type 1. The rows of $(J-I)_9$, i.e., the nine words of weight 8.

Type 2. The rows of a matrix which has the form

$$\left(\begin{array}{c|cccc|cccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & & & & & & & & \\
1 & & (J-I)_4 & & & & 0 & & \\
1 & & & & & & & & \\
1 & & & & & & & & \\ \hline
1 & & & & & & & & \\
1 & & 0 & & & & (J-I)_4 & & \\
1 & & & & & & & & \\
1 & & & & & & & &
\end{array}\right).$$

There are $9 \cdot \binom{8}{4}/2 = 315$ of these.

Type 3. The rows of a matrix which has the form

$$\begin{pmatrix} J & I & 0 \\ 0 & J & I \\ I & 0 & J \end{pmatrix}$$

where all submatrices are 3 by 3. There are $\binom{8}{2} \cdot \binom{6}{3} = 560$ of these.

Type 4. The rows of a matrix of the form

$$\left(\begin{array}{c|c} (J-I)_3 & J \\ \hline J & I_6 \end{array}\right).$$

There are $\binom{9}{3} = 84$ of these.

The partial geometry which we wish to construct has 120 lines, so we have to pick 120 out of 960 maximal cliques. We shall take the clique of Type 1, 63 cliques of Type 2, and 56 cliques of Type 3. It may be interesting for the reader to know that the numbers of types of cliques led us to this choice and subsequently the problem of finding a 'natural' way to pick the required subsets led us to the lemmas of Section 2.

We identify the nine positions with the set V. There are 9 choices for the first row of a matrix of type 2, depending on the position of the 0. If 0 is in position x we shall take only those partitionings of the remainder into two 4-subsets which correspond to the 3-design of Lemma 4. This yields 63 cliques of Type 2. A clique of Type 3 is chosen only if the three 3-subsets of V (used in forming the matrix) are as in Corollary 7. This yields $2\binom{9}{3}/3 = 56$ such cliques. The cliques chosen above are defined to be the lines.

We now claim that we have constructed pg(9, 8, 4). By Lemma 3 it suffices to show that every edge of G is in at least one of the lines. There are four cases to distinguish.

Case 1. Two words of weight 8 are in the line of Type 1.

Case 2. A word of weight 8 and a word of weight 4 with distance 6 have three 1's in common. By Lemma 4 these three positions uniquely determine a line of Type 2 containing this edge of G.

Case 3. Two words of weight 4 with distance 2 look like

$$\underbrace{\begin{matrix}1&1&1\\1&1&1\end{matrix}}_{D}\quad \begin{matrix}1&0\\0&1\\ a&b\end{matrix}\quad \underbrace{\begin{matrix}0&0&0&0\\0&0&0&0\end{matrix}}_{X}.$$

If $\varphi(X) \in D$ the edge is in a line of Type 2 with the 0 of row 1 in position $\varphi(X)$ and X as one of the 4-tuples. If $\varphi(X) = a$, then $\varphi_D(b) = a$, i.e., a and b are in the same class of the partition determined by D (Corollary 7). Hence the edge in question is in a line of Type 3.

Case 4. Two words of weight 4 with distance 6 look like

$$\begin{matrix}0&0&1\\0&0&1\\ a&b&c\end{matrix}\quad \underbrace{\begin{matrix}1&1&1\\0&0&0\end{matrix}}_{D}\quad \underbrace{\begin{matrix}0&0&0\\1&1&1\end{matrix}}_{E}.$$

(i) Clearly there is a Type 2 line containing this edge if $\varphi_D^{-1}(c) \in \{a, b\}$;

(ii) Next, suppose that $\varphi_D^{-1}(c) = g$ for some $g \in E$. Then (iii) of Corollary 5 implies that $\varphi_E^{-1}(c) \in D$. Now if $\varphi_D^2(g) = \varphi_D(c)$ is in E we are finished because then we have a Type 3 line containing the edge in question. The same is true if

$\varphi_E(c) \in D$. So it remains to check whether it is possible that $\varphi_D(c) \in \{a, b\}$ and $\varphi_E(c) \in \{a, b\}$. Assume that $\varphi_D(c) = a$. Then by (iii) of Corollary 5 we have $\varphi_E(c) = b$. Let $E = \{g, h, i\}$. The partition determined by φ_D is $\{D, \{g, c, a\}, \{b, h, i\}\}$. By Lemma 8 it follows that $\varphi_{\{b,h,i\}}(g) \in \{a, c\}$. On the other hand $\varphi_{\{b,h,i\}}(g) = \varphi_E(b) = \varphi_E^2(c) = \varphi_E^{-1}(c) \in D$, a contradiction. So the remaining possibility does not occur and the proof is complete.

4. Remarks

The authors of [3] conjecture, and we agree, that the pg(9, 8, 4) constructed by Cohen, the one constructed in this paper and the one associated with the hyperbolic quadric in PG(7, 2) are isomorphic. In fact they consider it likely that pg(9, 8, 4) is unique. These questions are still open.

A nice feature of our description is that one immediately sees that PSL(2, 8) is an automorphism group of the geometry which stabilizes a line. If the conjecture above is true the full group should be A_9.

Let us consider the *dual* of pg(9, 8, 4) (i.e., we interchange the roles of points and lines). This is a pg(8, 9, 4) which has the nice property that it admits a parallelism, that is the lines can be partitioned into parallel classes such that Euclid's axiom holds. Indeed, for each $x \in \mathrm{PG}(1, 8) = V$ define

$$C_x := \{X \in \tbinom{V}{4} \colon \varphi(X) = x\} \cup \{V \backslash \{x\}\}\,.$$

Then it is straightforward to verify that the sets C_x produce the required partitioning of the points of pg(9, 8, 4), i.e., the lines of pg(8, 9, 4), into parallel classes.

A rather small candidate for a new partial geometry is pg(5, 8, 2) with 75 points and 120 lines. Since the geometry has 8 lines per point and the same number of lines as pg(9, 8, 4) one might hope that such a pg(5, 8, 2) is a partial subgeometry of the larger one which means that it is obtainable by deleting a suitably chosen set of 60 points of pg(9, 8, 4) (four on each line). The points corresponding to four parallel classes in the dual geometry actually have the property that there are four on each line. This construction does not work ! In fact the whole idea has no chance because of the following theorem.

Theorem 9. *Let S be a* pg(9, 8, 4) *and let S' be a* pg(5, 8, 2). *Then S' cannot be a partial subgeometry of S.*

Proof. Let A and A' be the adjacency matrices of the graphs $\Gamma(S)$ and $\Gamma(S')$. Then A has an eigenvalue 4 of multiplicity 84 and hence $\mathrm{rank}(A - 4I) = 51$. If S'

is a subgeometry of S, then A' is a principal submatrix of A and this implies that $\mathrm{rank}(A'-4I) \leq 51$.

From this it follows that A' has an eigenvalue 4 of multiplicity at least 24 but A' does not have an eigenvalue 4 at all ! Therefore S' is not a subgeometry of S. $\square$

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