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A NON-EXISTENCE RESULT FOR QUASI-SYMMETRIC DESIGNS

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SUMMARY. We prove that for a quasi-symmetric 2-$(\frac{(m-1)(m-2)}{2}, \frac{a(m-2)}{2}, \frac{a(am-2a-2)}{2(m-3)})$ design with block intersection sizes $\frac{a(a-1)}{2}$ and $\frac{a(am-4a+2)}{2(m-3)}$, the number $(m-2)^{m(m-1)/2} \times (2a(m-3)(m-a-1))^{m(m+1)/2}$ must be the square of an integer.

1. Results

A 2–(v, k, λ)-design (with b blocks and r blocks through a given point) is said to be quasi-symmetric if any two distinct blocks meet in x or y points ($x > y$). The block graph of a quasi-symmetric design is the graph defined on the blocks, where two blocks are adjacent whenever they meet in x points. Such a block graph is known to be strongly regular having eigenvalues $(rk+y-k-by)/(x-y)$, $(r-\lambda+y-k)/(x-y)$ and $(y-k)/(x-y)$ with multiplicities 1, $v-1$ and $b-v$ respectively, see Goethals and Seidel [4]. So, in the terminology of Bose [1], a quasi-symmetric design is a block design for which the dual is a partially balanced design.

Seidel [6] determined all strongly regular graphs for which the (0,1)–adjacency matrix has an eigenvalue 1 or (complementary) -2. The question which of these graphs are block graphs of quasi-symmetric designs is partially answered in [4] and partly straightforward. Only for $\overline{T}(m)$, the complement of the triangular graph $T(m)$ (i.e. the line graph of the complete graph on m vertices), the answer is difficult. Such designs have parameters as given in the next theorem.

Theorem: *A quasi-symmetric* 2–$(\frac{(m-1)(m-2)}{2}, \frac{a(m-2)}{2}, \frac{a(am-2a-2)}{2(m-3)})$ *design* D *with block intersection sizes* $\frac{a(a-1)}{2}$ *and* $\frac{a(am-4a+2)}{2(m-3)}$, *can exist only if*

$$(m-2)^{m(m-1)/2}(2a(m-3)(m-a-1))^{m(m+1)/2}$$

is the square of an integer.

Proof: The block graph Γ of D has eigenvalues $(m-2)(m-3)/2$, $3-m$ and 1 with multiplicities 1, $m-1$ and $m(m-3)/2$, respectively. For $m = 8$ the result is obvious, so assume $m \neq 8$. Then, by Chang [3] and Hoffman [5], Γ can only be $\overline{T}(m)$. Thus Γ has $\overline{T}(m-1)$ as an induced subgraph. Let $M = [M_1 \;\; M_2]$ denote the incidence matrix of D, such that M_1 corresponds to $\overline{T}(m-1)$. Then M_1 is a square matrix of size v and we have

$$M_1^T M_1 = kI + xA_1 + y(J - A_1 - I),$$

where A_1 is the adjacency matrix of $\overline{T}(m-1)$. The eigenvalues of A_1 are $(m-3)(m-4)/2$, $4-m$ and 1 with multiplicities 1, $m-2$ and $(m-1)\times(m-4)/2$, respectively. This implies that $M_1^T M_1$ has eigenvalues

$$\frac{a^2(m-2)^2}{4}, \quad \frac{a(m-a-1)}{2(m-3)}, \quad \text{and} \quad \frac{a(m-a-1)(m-2)}{2(m-3)},$$

with the same multiplicities as above. The product of these eigenvalues is the determinant of $M_1^T M_1$ which is the square of an integer. Hence

$$(2a(m-a-1)(m-3))^{m-2}(2a(m-a-1)(m-2)(m-3))^{(m-1)(m-4)/2}$$

is a square. This completes the proof.

Remarks. (1) Replacing a by $m-a-1$ leads to the complementary parameter set. So for the investigation of these designs we can restrict ourselves to $a \leq (m-1)/2$.

(2) For $a = 2$ we have the residual designs of the biplanes. They exist for $m = 4,5,6,9,11$ and 13. These designs and their complements are the only known examples of designs that satisfy the parameter conditions above. For $a = 2$ the theorem requires that $m-2$ is a square if $v+m$ is even. This also follows from the Bruck-Ryser-Chowla theorem applied to the corresponding biplane.

(3) The smallest biplane for which existence has not been settled has $m = 16$. For $a > 2$ $(a \leq (m-1)/2)$ the four smallest feasible cases are:

$$(m,a) = (15,6), \ (18,5), \ (23,10), \ (24,9).$$

The first three are ruled out by the above theorem (non-existence of the first one also follows from a result of Calderbank [2]). The fourth one is the smallest unsettled case for $2 < a < m-3$.

(4) Although the above result excludes a very large part of the feasible parameters with $2 < a < m-3$, still infinitely many possibilities survive. It is doubtful that any such design exists for $a \neq 2$ and $a \neq m-3$.

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