Structure and uniqueness of the (81, 20, 1, 6) strongly regular graph
Brouwer, A.E.; Haemers, W.H.

Published in:
Discrete Mathematics

Publication date:
1992

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright, please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Structure and uniqueness of the (81, 20, 1, 6) strongly regular graph

A.E. Brouwer
Faculteit Wiskunde en Informatica, Technische Universiteit Eindhoven, Postbus 513, 5600 MB Eindhoven, Netherlands

W.H. Haemers
Faculteit der Econometry, Universiteit Tilburg, Postbus 90153, 5000 LE Tilburg, Netherlands

Received 12 November 1991
Revised 4 March 1992

Abstract

We prove that there is a unique graph (on 81 vertices) with spectrum $20^12^{60}(-7)^{20}$. We give several descriptions of this graph, and study its structure.

Let $\Gamma = (X, E)$ be a strongly regular graph with parameters $(v, k, \lambda, \mu) = (81, 20, 1, 6)$. Then $\Gamma$ (that is, its $0-1$ adjacency matrix $A$) has spectrum $20^12^{60}(-7)^{20}$, where the exponents denote multiplicities. We will show that up to isomorphism there is a unique such graph $\Gamma$. More generally we give a short proof for the fact (due to Ivanov and Shpectorov [9]) that a strongly regular graph with parameters $(v, k, \lambda, \mu) = (q^4, (q^2 + 1)(q - 1), q - 2, q(q - 1))$ that is the collinearity graph of a partial quadrangle (that is, in which all maximal cliques have size $q$) is the second subconstituent of the collinearity graph of a generalized quadrangle $GQ(q, q^2)$. In the special case $q = 3$ this will imply our previous claim, since $\lambda = 1$ implies that all maximal cliques have size 3, and it is known (see Cameron et al. [5]) that there is a unique generalized quadrangle $GQ(3, 9)$ (and this generalized quadrangle has an automorphism group transitive on the points). The proof will use spectral techniques very much like those found in...
Haemers [7] and Brouwer and Haemers [3]. For completeness let us explicitly formulate the tools we use.

**Tool 1.** Let $A$ and $B$ be real symmetric matrices of orders $n$ and $m$ (where $m \leq n$) and with eigenvalues $\theta_1 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \cdots \geq \eta_m$, respectively. We say that the eigenvalues of $B$ interlace those of $A$ when $\theta_j \geq \eta_j \geq \theta_{n-m+j}$ for all $j$ ($1 \leq j \leq m$). We say that the interlacing is tight when for some integer $l$ we have $\eta_j = \theta_j$ for $1 \leq j \leq l$ and $\eta_j = \theta_{n-m+j}$ for $l+1 \leq j \leq m$. If $B$ is a principal submatrix of $A$ then the eigenvalues of $B$ interlace those of $A$. Another case of interlacing is the following result: Given a symmetric partition of the rows and columns of a symmetric matrix $A$, let $B$ be the matrix with as entries the average row sums of the parts of $A$. Then the eigenvalues of $B$ interlace those of $A$, and when the interlacing is tight, the parts of $A$ have constant row sums.

**Tool 2.** Given a symmetric partition of a symmetric matrix $A$ with two eigenvalues into four submatrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

the eigenvalues of $A_{22}$ can be computed from those of $A_{11}$. If $A$ has eigenvalues $\alpha$ and $\beta$ (where $\alpha > \beta$) with multiplicities $f$ and $n-f$, respectively, and $A_{11}$ (of order $m$) has eigenvalues $\theta_1 \geq \cdots \geq \theta_m$, then $A_{22}$ (of order $n-m$) has eigenvalues $\eta_1 \geq \cdots \geq \eta_{n-m}$, where

$$\eta_i = \begin{cases} \alpha & \text{if } 1 \leq i \leq f - m, \\ \beta & \text{if } f + 1 \leq i \leq n - m, \\ \alpha + \beta - \theta_{f-i+1} & \text{otherwise}. \end{cases}$$

For undefined concepts and notation, see Brouwer et al. [2]. For surveys on strongly regular graphs, see Hubaut [8] and Brouwer and van Lint [4].

Let us first give a few descriptions of our graph on 81 vertices.

(A) Let $X$ be the point set of AG(4, 3), the 4-dimensional affine space over $\mathbb{F}_3$, and join two points when the line connecting them hits the hyperplane at infinity (a PG(3, 3)) in a fixed elliptic quadric $Q$. This description shows immediately that $v = 81$ and $k = 20$ (since $|Q| = 10$). Also $\lambda = 1$ since no line meets $Q$ in more than two points, so that the affine lines are the only triangles. Finally $\mu = 6$, since a point outside $Q$ in PG(3, 3) lies on 4 tangents, 3 secants and 6 exterior lines with respect to $Q$, and each secant contributes 2 to $\mu$. We find that the group of automorphisms contains $G = 3^4 \cdot \text{PGO}_7(3) \cdot 2$, where the last factor 2 accounts for the linear transformations that do not preserve the quadratic form $Q$, but multiply it by a constant. In fact this is the full group, as will be clear from the uniqueness proof.
(B) A more symmetric form of this construction is found by starting with $X = \mathbb{Z}^2$ provided with the standard bilinear form. The corresponding quadratic form ($Q(x) = \text{wt}(x)$, the number of nonzero coordinates of $x$) is elliptic, and if we join two vertices $x + (1), y + (1)$ of $X$ when $Q(x - y) = 0$, i.e., when their difference has weight 3, we find the same graph as under A. This construction shows that the automorphism group contains $G = 3^4 \cdot (2 \times \text{Sym}(6)) \cdot 2$, and again this is the full group.

(C) There is a unique strongly regular graph $\Sigma$ with parameters $(v, k, \lambda, \mu) = (112, 30, 2, 10)$, the collinearity graph of the unique generalized quadrangle with parameters $GQ(3, 9)$. Its second subconstituent is strongly regular (since $\Sigma$ is a Smith graph), and hence is isomorphic to our graph $\Gamma$. (See Cameron et al. [5].) We find that $\text{Aut} \Gamma$ contains (and in fact it equals) the point stabilizer in $U_4(3) : D_5$ acting on $GQ(3, 9)$.

(D) In the McLaughlin graph $\Lambda$ (the unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (275, 112, 30, 56)$) let $x, y$ be two adjacent vertices. The subgraph of $\Lambda$ induced by the neighbours of $y$ is isomorphic to $\Sigma$; the subgraph $T$ induced by the nonneighbours of $y$ is the unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (162, 56, 10, 24)$. (Again, see Cameron et al [5].) Thus, by (C) above, we may identify $\Gamma$ with the subgraph of $\Lambda$ induced by the vertices adjacent to $y$ but not to $x$. Let $\Gamma'$ be the subgraph induced by the vertices nonadjacent to both $x$ and $y$, so that $T$ is partitioned by the vertex sets of $\Gamma$ and $\Gamma'$. Then also $\Gamma'$ is a strongly regular graph with parameters $(v, k, \lambda, \mu) = (81, 20, 1, 6)$ (its spectrum can be computed from that of $T$ and that of $\Gamma$). We find that $\text{Aut} \Gamma'$ contains the edge stabilizer in $\text{Aut} \Lambda = \text{McL}.2$—in fact as an index 2 subgroup.

(E) The graph $\Gamma$ is the coset graph of the truncated ternary Golay code $C$: take the $3^3$ cosets of $C$ and join two cosets when they contain vectors differing in only one place.

(F) The graph $\Gamma$ is the Hermitean forms graph on $F_9^2$; more generally, take the $q^4$ matrices $M$ over $F_q^2$ satisfying $M^T = M$, where $^\circ$ denotes the field automorphism $x \mapsto x^q$ (applied entrywise), and join two matrices when their difference has rank 1. This will give us a strongly regular graph with parameters $(v, k, \lambda, \mu) = (q^4, (q^2 + 1)(q - 1), q - 2, q(q - 1))$.

(G) The graph $\Gamma$ is the graph with vertex set $F_q^3$, where two vertices are joined when their difference is a fourth power. (This construction was given by Van Lint and Schrijver [10].)

Now let us embark upon the uniqueness proof. Let $\Gamma = (X, E)$ be a strongly regular graph with parameters $(v, k, \lambda, \mu) = (q^4, (q^2 + 1)(q - 1), q - 2, q(q - 1))$ and assume that all maximal cliques (we shall just call them lines) of $\Gamma$ have size $q$.

Let $\Gamma$ have adjacency matrix $A$. Using the spectrum of $A$—it is $k^1(q - 1)^2(q - 1 - q^2)^3$, where $f = q(q - 1)(q^2 + 1)$ and $g = (q - 1)(q^2 + 1)$—we can obtain some structure information. Let $T$ be the collection of subsets of $X$ of cardinality $q^3$ inducing a subgraph that is regular of degree $q - 1$. 
Step 1. If $T \in \mathcal{T}$, then each point of $X \setminus T$ is adjacent to $q^2$ points of $T$.

Look at the matrix $B$ of average row sums of $A$, with sets of rows and columns partitioned according to $(T, X \setminus T)$. We have

$$B = \begin{pmatrix} q^2(q - 1) & q^2 \\ q^2 & q^2(q - 1) \end{pmatrix}$$

with eigenvalues $k$, $q - 1 - q^2$, so interlacing is tight, and by Tool 1 it follows that the row sums are constant in each block of $A$.

Step 2. Given a line $L$, there is a unique $T_L \in T$ containing $L$.

Let $Z$ be the set of vertices in $X \setminus L$ without a neighbour in $L$. Then $|Z| = q^4 - q - q(k - q + 1) = q^3 - q$. Let $T = L \cup Z$. Each vertex of $Z$ is adjacent to $q^4 = q^2(q - 1)$ vertices with a neighbour in $L$, so $T$ induces a subgraph that is regular of degree $q - 1$.

Step 3. If $T \in \mathcal{T}$ and $x \in X \setminus T$, then $x$ is on at least one line $L$ disjoint from $T$, and $T_L$ is disjoint from $T$ for any such line $L$.

The point $x$ is on $q^2 + 1$ lines, but has only $q^2$ neighbours in $T$. Each point of $L$ has $q^2$ neighbours in $T$, so each point of $T$ has a neighbour on $L$ and hence is not in $T_L$.

Step 4. Any $T \in \mathcal{T}$ induces a subgraph $\Delta$ isomorphic to $q^2 K_q$.

It suffices to show that the multiplicity $m$ of the eigenvalue $q - 1$ of $\Delta$ is (at least) $q^2$ (it cannot be more). By interlacing we find $m \geq q^2 - q$, so we need some additional work. Let $M := A - (q - 1/q^2)I$. Then $M$ has spectrum $(q - 1)^{q^2}(q - 1 - q^2)^q$, and we want that $M_T$, the submatrix of $M$ with rows and columns indexed by $T$, has eigenvalue $q - 1$ with multiplicity (at least) $q^2 - 1$, or, equivalently (by Tool 2), that $M_{X \setminus T}$ has eigenvalue $q - 1 - q^2$ with multiplicity (at least) $q - 2$. But for each $U \in \mathcal{T}$ with $U \cap T = \emptyset$ we find an eigenvector $x_U := (2 - q)x_U + x_{X \setminus (T \cup U)}$ of $M_{X \setminus T}$ with eigenvalue $q - 1 - q^2$. A collection $\{x_U \mid U \in \mathcal{U}\}$ of such eigenvectors cannot be linearly dependent when $U = \{U_1, U_2, \ldots\}$ can be ordered such that $U_i \not\subseteq \bigcup_{j<i} U_j$ and $\bigcup U \neq X \setminus T$, so we can find (using Step 3) at least $q - 2$ linearly independent such eigenvectors, and we are done.

Step 5. Any $T \in \mathcal{T}$ determines a unique partition of $X$ into members of $T$.

Indeed, we saw this in the proof of the previous step.

Let $\Pi$ be the collection of partitions of $X$ into members of $T$. We have $|T| = q(q^2 + 1)$ and $|\Pi| = q^2 + 1$. Construct a generalized quadrangle $\mathrm{GQ}(q, q^2)$ with point set $\{\infty\} \cup T \cup X$ as follows: The $q^2 + 1$ lines on $\infty$ are $\{\infty\} \cup \pi$ for $\pi \in \Pi$. The $q^2$ remaining lines on each $T \in \mathcal{T}$ are $\{T\} \cup L$ for $L \subseteq T$. It is completely straightforward to check that we really have a generalized quadrangle $\mathrm{GQ}(q, q^2)$.

Other graphs. Some of our arguments can be generalized a little. Given a strongly regular graph $\Gamma = (X, E)$ with parameters $(v, k, \lambda, \mu)$ and spectrum $k^{1-r}s^r$, suppose that there is a subset $L$ of $X$ inducing a strongly regular subgraph
of \( \Gamma \) with parameters \((u, r, \lambda, \mu)\). Then \( k = r + u\mu \) and \( v = u(k - \lambda) \). Each point outside \( L \) has at most one neighbour in \( L \). Let \( Z \) be the set of points in \( X \setminus L \) without neighbour in \( L \). Each point of \( Z \) has \( u\mu \) neighbours outside \( Z \), and hence \( Z \), and also \( T := L \cup Z \), is regular of valency \( r \). In a few cases one can show using multiplicity arguments that \( T \) must consist of a number of copies of \( L \). For example:

(a) Starting with a single point in a complete multipartite graph \( K_{m \times n} \) (with spectrum \((m - 1)n^0 m^{(m-1)}(-n)^{m-1}\)) we find a coclique of size \( n \).

(b) Starting with an edge in the Petersen graph, we find a subgraph \( 3K_2 \). (Similarly, an edge in the complement of the Clebsch graph is contained in a unique \( 4K_2 \), but this is the special case \( q = 2 \) of our result above.)

(c) Starting with a pentagon in the Hoffman–Singleton graph, we find a subgraph \( 5C_5 \).

(d) Starting with a quadrangle in the Gewirtz graph, we find a subgraph \( 6C_4 \). (This was the starting point of Brouwer and Haemers [3]; also the uniqueness of the \((162, 56, 10, 24)\) strongly regular graph (Cameron et al. [5]) relies on this fact.)

(e) Starting with a grid \( 3 \times 3 \) in the Berlekamp–van Lint–Seidel graph (Berlekamp et al. [1]), we find a subgraph \( 9 \ (3 \times 3) \). Maybe one could prove uniqueness (for strongly regular graphs with parameters \((v, k, \lambda, \mu) = (243, 22, 1, 2)\)) using this?

(f) Starting with a triangle in a \((57, 14, 1, 4)\) graph \( \Gamma \), we find (under the assumption that \( \Gamma \) does not contain a 15-coclique) a subgraph \( 7K_3 \). This implies that \( \Gamma \) is embeddable in a non-existing \( GQ(3, 6) \) (see Dixmier and Zara [6], or Payne and Thas [11]). Thus, the non-existence proof for \( \Gamma \) in Wilbrink and Brouwer [12] can be shortened considerably.

Our strongly regular graph on 81 vertices might have distance regular antipodal 2-, 3- and 6-covers of diameter 4. Maybe one can prove non-existence for the 2- and 6-covers and uniqueness for the 3-cover (e.g., by proving that a grid \( 3 \times 3 \) must lift to a grid again)?

**p-Rank and Smith normal form.** Writing \( S(M) \) for the Smith normal form of a matrix \( M \), we find for the adjacency matrix \( A \) of our 81-point graph: \( S(A) = \text{diag}(1^{20}, 2^{41}, 14^{19}, 140^1) \) and \( S(A - 2I) = \text{diag}(1^{19}, 3^1, 6^1, 0^{40}) \) and \( S(A + 7I) = \text{diag}(1^{19}, 3^2, 9^{39}, 27^1, 0^{20}) \). In particular, \( A + I \) has 3-rank 19.

**References**


