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Abstract

In this paper big boss interval games are introduced and various characterizations are given. The structure of the core of a big boss interval game is explicitly described and plays an important role relative to interval-type bi-monotonic allocation schemes for such games. Specifically, each element of the interval core of a big boss interval game is extendable to a bi-monotonic allocation scheme. Furthermore, on the class of big boss interval games two interval solution concepts of value type are defined which can be seen as extensions of the compromise value and the AL-value to the interval setting. It turns out that these interval solutions coincide and generate a bi-monotonic allocation scheme for each big boss interval game.

JEL Classification: C71

Keywords: cooperative games, interval data, big boss games, bi-monotonic allocation schemes, the AL-value, the compromise value

1 Introduction

Big boss interval games are particular cooperative interval games inspired by the model of classical big boss games. A cooperative interval game is

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a pair consisting of a set of players and a worth function which assigns to each coalition a closed interval in $\mathbb{R}$ such that the empty coalition receives $[0, 0]$. Cooperative interval games are a useful tool for modeling real-life situations where payoffs for people or businesses are uncertain and decisions regarding their cooperation have to be made based on all possible realizations which belong to intervals whose lower and upper bounds are known with certainty. An interesting motivating example for the model of cooperative interval games game can be found in Bauso and Timmer (2006): a joint replenishment situation where each retailer faces a demand bounded by a minimum and a maximum value. The model of cooperative interval games is firstly introduced in Branzei, Dimitrov and Tijs (2003) to model bankruptcy situations under interval uncertainty of claims. Bauso and Timmer (2006) design robust dynamic rules for cooperative games incorporating interval uncertainty. Alparslan Gök, Miquel and Tijs (2008) consider cooperative interval games and look at the corresponding classical cooperative games, which are selections of the interval game. Based on classical solutions on the selections such as the core and the Shapley value they define then solutions for interval cooperative games. In Alparslan Gök, Branzei and Tijs (2008a) another approach is taken, where solutions are described with the aid of tuples of intervals, the focus being on interval cores and stable sets. Other interval solution concepts like the Shapley value and the Weber set are introduced on a special class of cooperative interval games in Alparslan Gök, Branzei and Tijs (2008b). The reader is referred to Alparslan Gök, Branzei and Tijs (2008c) for a discussion of how interval solutions can be handled in practice.

The class of classical big boss games (Muto et al. (1988)) has received much attention in cooperative game theory and various situations were modeled using such games. We refer here to information market situations (Muto, Potters and Tijs (1989)), information collecting situations (Branzei, Tijs and Timmer (2001a, b), Tijs, Timmer and Branzei (2006)) and holding situations (Tijs, Meca and López (2005)). In case such situations are described in terms of interval data the corresponding cooperative games are under restricting conditions big boss interval games.

The paper is organized as follows. We recall in Section 2 basic notions and facts from the theory of traditional cooperative games and cooperative interval games. Big boss interval games are introduced and some characterizations as well as a description of the interval core inspired from the classical theory are given (Muto et al. (1988)) in Section 3. In Section 4 we study bi-monotonic allocation schemes (bi-mas) for big boss interval games and show that interval core elements are bi-mas extendable. Further, we introduce a compromise-like solution concept, the $\mathcal{T}$-value, and the interval counterpart of the AL-value (Tijs (2005)). It turns out that on the class of big boss interval games these interval solutions coincide and generate an interval-type bi-mas for each game in this class.

2 Preliminaries

We start this section with basic definitions and results from the classical cooperative game theory.
A cooperative game in coalitional form is an ordered pair $< N, v >$, where $N = \{1, 2, ..., n\}$ is the set of players, and $v : 2^N \to \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$. Often, we also refer to such a game as a TU (transferable utility) game. We denote by G^N the family of all classical cooperative games with player set N. For a game $v \in G^N$ and a coalition $T \in 2^N \setminus \{\emptyset\}$, the subgame with player set T is the game v_T defined by $v_T(S) = v(S)$ for all $S \in 2^T$. In the sequel we denote such subgames by $< T, v >$. A game $v \in G^N$ is additive if $v(S \cup T) = v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$ (Branzei, Dimitrov and Tijs (2005)).
A game $< N, v >$ is called a big boss game with n as a big boss (Muto et al. (1988), Tijs (1990)) if the following conditions are satisfied:

(i) $v \in G^N$ is monotonic, i.e. $v(S) \leq v(T)$ if for each $S, T \in 2^N$ with $S \subset T$;

(ii) $v(S) = 0$ if $n \notin S$;

(iii) $v(N) - v(S) \geq \sum_{i \in N \setminus S}(v(N) - v(N \setminus \{i\}))$ for all S, T with $n \in S \subset N$.

Note that big boss games form a cone in G^N. Further, a game $< N, v >$ is a total big boss game with big boss n if and only if $< T, v >$ is a big boss game for each $T \in 2^N$ with $n \in T$.

Definition 2.1. *Let $v \in G^N$ and $n \in N$. Then, this game is a total big boss game with n as a big boss if the following conditions are satisfied:*

(i) *$v \in G^N$ is monotonic, i.e. $v(S) \leq v(T)$ if for each $S, T \in 2^N$ with $S \subset T$;*

(ii) $v(S) = 0$ *if* $n \notin S$;

(iii) $v(T) - v(S) \geq \sum_{i \in T \setminus S}(v(T) - v(T \setminus \{i\}))$ *for all* S, T *with* $n \in S \subset T$.

In this paper we only consider total big boss games and call them shortly big boss games. Now, we give some useful definitions for solutions from the classical cooperative game theory.
The imputation set of a game $< N, v >$ is defined by

$$I(N,v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for each } i \in N\}.$$

The core (Gillies (1959)) is a central solution concept on G^N. The core $C(N,v)$ of $v \in G^N$ is defined by

$$C(N,v) = \left\{ x \in I(N,v) | \sum_{i \in S} x_i \geq v(S) \text{ for each } S \in 2^N \right\}.$$

A game whose core is nonempty is called here a balanced game. We recall that the core $C(N,v)$ of a traditional big boss game is always nonempty and equal to $\{x \in I(N,v) | 0 \leq x_i \leq M_i(N,v) \ \text{ for each } i \in N \setminus \{n\}\}$, where for each $i \in N$, $M_i(N,v) = v(N) - v(N \setminus \{i\})$.
A game $v \in G^N$ is called quasi-balanced if $m(N,v) \leq M(N,v)$ and $\sum_{i=1}^{n} m_i(N,v) \leq v(N) \leq \sum_{i=1}^{n} M_i(N,v)$, where for each $i \in N$,

$$m_i(N,v) = \max \{R(S,i) | i \in S, S \subset N\}$$

with

$$R(S,i) = v(S) - \sum_{j \in S \setminus \{i\}} M_j(N,v).$$

The τ-value or compromise value (Tijs (1981)) is defined on the class of quasi-balanced games. Specifically, for each quasi-balanced game $< N, v >$ its τ-value, $\tau(N,v)$, is a feasible compromise between the upper vector $M(N,v) = (M_i(N,v))_{i \in N}$ and the lower vector $m(N,v) = (m_i(N,v))_{i \in N}$ of a game satisfying $\sum_{i \in N} \tau_i(N,v) = v(N)$.
For a big boss game with n as a big boss the τ-value of v is given by

$$\tau(N,v) = (\frac{1}{2}M_1(N,v), \frac{1}{2}M_2(N,v), \ldots, v(N) - \sum_{i \in N \setminus \{n\}} \frac{1}{2}M_i(N,v)).$$

Now, let $\sigma = (\sigma(1), \sigma(2) \ldots, \sigma(k), \sigma(k+1), \ldots, \sigma(n))$ be an ordering of the players in $N = \{1, 2, \ldots, n\}$. The lexicographic maximum of the core $C(N, v)$ of a balanced game $< N, v >$ with respect to σ is denoted by $L^\sigma(N, v)$. Then, the average lexicographic value $AL(N, v)$ (Tijs (2005)) of $v \in G^N$ is the average of all lexicographically maximal vectors of the core of the game, i.e. $AL(N, v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L^\sigma(N, v)$. For a big boss game $L^\sigma(N, v)$ is equal to

$$L^\sigma_{\sigma(i)}(N, v) = \begin{cases} M_{\sigma(i)}(N, v), & i < k \\ 0, & i > k \\ v(N) - \sum_{i=1}^{k-1} M(N, v), & i = k. \end{cases}$$

It is known that the AL-value coincides with the τ-value on the class of (total) big boss games.

In the sequel we recall basic interval calculus and useful notions and results regarding cooperative interval games (Alparslan Gök, Branzei and Tijs (2008a,b)).

Let $I, J \in I(\mathbb{R})$ with $I = [\underline{I}, \overline{I}]$, $J = [\underline{J}, \overline{J}]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then,

(i) $I + J = [\underline{I}, \overline{I}] + [\underline{J}, \overline{J}] = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$;

(ii) $\alpha I = \alpha [\underline{I}, \overline{I}] = [\alpha \underline{I}, \alpha \overline{I}]$.

By (i) and (ii) we see that $I(\mathbb{R})^N$ has a cone structure.

In this paper we also need a partial substraction operator. We define $I - J$, only if $|I| \geq |J|$, by $I - J = [\underline{I}, \overline{I}] - [\underline{J}, \overline{J}] = [\underline{I} - \underline{J}, \overline{I} - \overline{J}]$. Note that $\underline{I} - \underline{J} \leq \overline{I} - \overline{J}$. We recall that I is weakly better than J, which we denote by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. We also use the reverse notation $J \preccurlyeq I$, if and only if $\underline{J} \leq \underline{I}$ and $\overline{J} \leq \overline{I}$.

We recall that a *cooperative interval game* in coalitional form (Alparslan Gök, Miquel and Tijs (2008)) is an ordered pair $< N, w >$ where $N = \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$, where $I(\mathbb{R})$ is the set of all closed intervals in $\mathbb{R}$. For each $S \in 2^N$, the worth set (or worth interval) $w(S)$ of the coalition S in the interval game $< N, w >$ is of the form $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the lower bound and $\overline{w}(S)$ is the upper bound of $w(S)$. We denote by IG^N the family of all interval games with player set N. Note that if all the worth intervals are degenerate intervals, i.e. $\underline{w}(S) = \overline{w}(S)$ for each $S \in 2^N$, then the interval game $< N, w >$ corresponds in a natural way to the classical cooperative

game $< N, v >$ where $v(S) = \underline{w}(S)$.
Some classical TU-games associated with an interval game $w \in IG^N$ will play a key role, namely the *border* games $< N, \underline{w} >$, $< N, \overline{w} >$ and the *length* game $< N, |w| >$, where $|w|(S) = \overline{w}(S) - \underline{w}(S)$ for each $S \in 2^N$. Note that $\overline{w} = \underline{w} + |w|$.
For $w_1, w_2 \in IG^N$ we say that $w_1 \preccurlyeq w_2$ if $w_1(S) \preccurlyeq w_2(S)$, for each $S \in 2^N$.
For $w_1, w_2 \in IG^N$ and $\lambda \in \mathbb{R}_+$ we define $< N, w_1 + w_2 >$ and $< N, \lambda w >$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\lambda w)(S) = \lambda \cdot w(S)$ for each $S \in 2^N$. So, we conclude that IG^N endowed with $\preccurlyeq$ is a partially ordered set and has a cone structure with respect to addition and multiplication with non-negative scalars described above. For $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $< N, w_1 - w_2 >$ is defined by $(w_1 - w_2)(S) = w_1(S) - w_2(S)$.
Now, we recall the definitions of the interval imputation set and the interval core of an interval game.
The interval imputation set $\mathcal{I}(N, w)$ of the interval-valued game w, is defined by

$$\mathcal{I}(N, w) = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i \in N} I_i = w(N), w(i) \preccurlyeq I_i, \text{ for all } i \in N \right\}.$$

The interval core $\mathcal{C}(N, w)$ of the interval game w is defined by

$$\mathcal{C}(N, w) = \left\{ (I_1, \ldots, I_n) \in \mathcal{I}(N, w) | \sum_{i \in S} I_i \succcurlyeq w(S), \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

Here, $\sum_{i \in N} I_i = w(N)$ is the efficiency condition and $\sum_{i \in S} I_i \succcurlyeq w(S)$, $S \in 2^N \setminus \{\emptyset\}$, are the stability conditions of $(I_1, \ldots, I_n)$.
We recall that for a game $w \in IG^N$ and a coalition $S \in 2^N \setminus \{\emptyset\}$, the *subgame* with player set T is the game w_T defined by $w_T(S) = w(S)$ for all $S \in 2^T$. So, w_T is the restriction of w to the set 2^T. Next, we refer to such subgames by $< T, w >$.
A game $w \in IG^N$ is called $\mathcal{I}$-balanced if and only if the interval core $\mathcal{C}(N, w)$ is nonempty (Theorem 3.1 in Alparslan Gök, Branzei and Tijs (2008a)) and a game $w \in IG^N$ whose all subgames are $\mathcal{I}$-balanced is called a *totally $\mathcal{I}$-balanced* interval game.
We call a game $< N, w >$ *size monotonic* if $< N, |w| >$ is monotonic, i.e. $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SMIG^N$ the class of size monotonic interval games with player set N.

3 Big boss interval games

Let $w \in IG^N$ and let $< N, |w| >$ be the corresponding length game. Then, we call a game $< N, w >$ a *big boss interval game* if its border game $< N, \underline{w} >$ and the game $< N, |w| >$ are classical (total) big boss games. We denote by $BBIG^N$ the set of all big boss interval games with player set N (without loss of generality we denote the big boss by n). Note that $BBIG^N$ is a subcone of IG^N.
The interval game in the next example is not a big boss interval game since the related length game is not a big boss game.

EXAMPLE 3.1. *Let $< N, w >$ be a three-person interval game with $w(1) = w(2) = w(3) = w(1,2) = [0,0], w(2,3) = [5,6], w(1,3) = [6,6]$ and $w(N) = [9,11]$. Here, $< N, \underline{w} >$ is a big boss game, but the length game $< N, |w| >$ is not because it does not satisfy the condition* (iii) *in Definition 2.1 (take $S = \{1\}$).*

In the following proposition and theorem characterizations for big boss interval games are given.

PROPOSITION 3.1. *Let $w \in IG^N$ and its related games $|w|, \underline{w}, \overline{w} \in G^N$. Then, $w \in BBIG^N$ if and only if its length game $< N, |w| >$ and its border games $< N, \underline{w} >$, $< N, \overline{w} >$ are big boss games.*

Proof. The proof is straightforward. Note that $\overline{w} = \underline{w} + |w|$ is a big boss game because classical big boss games form a cone. □

THEOREM 3.1. *Let $w \in SMIG^N$. Then, the following two conditions are equivalent:*

(i) $w \in BBIG^N$;

(ii) $< N, w >$ *satisfies:*

 (a) *Big boss property:*
 $w(S) = [0,0]$ *for each* $S \in 2^N$ *with* $n \notin S$;

 (b) *Monotonicity property:*
 $w(S) \preccurlyeq w(T)$ *for each* $S, T \in 2^N$ *with* $n \in S \subset T$;

 (c) *Union property:*

$$w(T) - w(S) \succcurlyeq \sum_{i \in T \setminus S} (w(T) - w(T \setminus \{i\})) \text{ for all } S, T \text{ with } n \in S \subset T.$$

Proof. By Proposition 3.1, $w \in BBIG^N$ if and only if $< N, \underline{w} >$, $< N, \overline{w} - \underline{w} >$ and $< N, \overline{w} >$ are classical big boss games. Now, using Definition 2.1, $w \in BBIG^N$ if and only if $< N, w >$ satisfies (a), (b) and (c). $\square$

Now, we define for each $w \in SMIG^N$ and each $i \in N$, the *marginal contribution* of i in the game w by $M_i(N, w) = w(N) - w(N \setminus \{i\})$.
Further, we give a concavity property for big boss interval games with n as a big boss.

(d) n-concavity property:

$$w(S \cup \{i\}) - w(S) \succcurlyeq w(T \cup \{i\}) - w(T),$$

for all $S, T \in 2^N$ with $n \in S \subset T \subset N \setminus \{i\}$.

The following theorem which is inspired by (Branzei, Tijs and Timmer (2001b)) shows that (c) and (d) are equivalent if (a), (b) hold.

THEOREM 3.2. *Let $w \in IG^N$ and let* (a), (b) *hold. Then,* (c) *implies* (d) *and conversely.*

Proof. (i) Suppose that (d) holds. Let S, T be such that $n \in S \subset T$. Suppose $T \setminus S = \{i_1, \dots, i_h\}$. Then,

$$\begin{aligned} w(T) - w(S) &= w(S \cup \{i_1\}) - w(S) \\ &+ \sum_{r=2}^{h} (w(S \cup \{i_1, \dots, i_r\}) - w(S \cup \{i_1, \dots, i_{r-1}\})) \\ &= \sum_{r=1}^{h} M_{i_r}(S \cup \{i_1, \dots, i_r\}, w) \\ &\succcurlyeq \sum_{r=1}^{h} M_{i_r}(T, w) = \sum_{i \in T \setminus S} M_i(T, w), \end{aligned}$$

where "the inequality" follows from (d). So, (d) implies (c).

(ii) Suppose that (c) holds. Then,

$$w(U \cup \{j\}) - w(U \setminus \{i\}) \succcurlyeq M_j(U \cup \{j\}, w) + M_i(U \cup \{j\}, w). \quad (1)$$

But,

$$w(U \cup \{j\}) - w(U \setminus \{i\}) = w(U \cup \{j\}) - w(U) + w(U) - w(U \setminus \{i\})$$

$$= M_j(U \cup \{j\}, w) + M_i(U, w). \tag{2}$$

From (1) and (2) we obtain

$$M_i(U, w) \succcurlyeq M_i(U \cup \{j\}, w) \tag{3}$$

for all $U \subset N$ and $i, j \in N \setminus \{n\}$ such that $\{i, n\} \subset U \subset N \setminus \{j\}$. Now, take $S, T \subset N$ with $\{i, n\} \subset S \subset T$ and suppose that $T \setminus S = \{i_1, \ldots, i_h\}$. If we apply (3) h times then we have $M_i(S, w) \succcurlyeq M_i(S \cup \{i_1\}, w) \succcurlyeq M_i(S \cup \{i_1, i_2\}, w) \succcurlyeq \ldots \succcurlyeq M_i(T, w)$. So, (c) implies (d). □

We define the set $\mathcal{K}(T, w)$ for each subgame $< T, w >$ of $< N, w >$ by

$$\mathcal{K}(T, w) = \{(I_1, \ldots, I_n) \in \mathcal{I}(T, w) \mid [0, 0] \preccurlyeq I_i \preccurlyeq M_i(T, w) \text{ for each } i \in T \setminus \{n\}\}.$$

The next proposition gives a characterization of the interval core of a big boss interval game by using marginal contributions of the players.

PROPOSITION 3.2. *Let $w \in BBIG^N$. Then,*

$$\mathcal{C}(T, w) = \mathcal{K}(T, w). \tag{4}$$

Proof. It is sufficient to show $\mathcal{C}(T, w) = \mathcal{K}(T, w)$ for $T = N$.

(i) Suppose that $I = (I_1, \ldots, I_n) \in C(N, w)$.
Then, $w(N) = \sum_{i \in N} I_i$ and $\sum_{j \in N \setminus \{i\}} I_j \succcurlyeq w(N \setminus \{i\})$, for all $i \in N \setminus \{n\}$. Further,

$$I_i = \sum_{j \in N} I_j - \sum_{j \in N \setminus \{i\}} I_j = w(N) - \sum_{j \in N \setminus \{i\}} I_j \preccurlyeq w(N) - w(N \setminus \{i\}) = M_i(N, w),$$

where the second equality follows from efficiency and "the inequality" follows from stability. Clearly, $I_i \succcurlyeq [0, 0] = w(i)$ for $i \in N \setminus \{n\}$. So, $I \in \mathcal{K}(N, w)$. Therefore $\mathcal{C}(N, w) \subset \mathcal{K}(N, w)$ holds.

(ii) Suppose that $I = (I_1, \ldots, I_n) \in \mathcal{K}(N, w)$. Then, for a coalition S which does not contain n, one finds that $\sum_{i \in S} I_i \succcurlyeq [0,0] = w(S)$. To prove that $\sum_{i \in S} I_i \succcurlyeq w(S)$ for S such that $n \in S$ we first show that $w(N) - w(S) \succcurlyeq \sum_{i \in N \setminus S} M_i(N, w)$. Let $N \setminus S = \{i_1, \ldots, i_k\}$. Then, in a similar way as in the proof of Theorem 3.2 (i) with N in the role of T we have

$$\begin{aligned}
w(N) - w(S) &= w(S \cup \{i_1\}) - w(S) \\
&+ \sum_{s=2}^{k} (w(S \cup \{i_1, \ldots, i_s\}) - w(S \cup \{i_1, \ldots, i_{s-1}\})) \\
&= \sum_{s=1}^{k} M_{i_s}(S \cup \{i_1, \ldots, i_s\}, w) \\
&\succcurlyeq \sum_{s=1}^{k} M_{i_s}(N, w) = \sum_{i \in N \setminus S} M_i(N, w),
\end{aligned}$$

where "the inequality" follows from the n-concavity property. Then, using the definition of $\mathcal{K}(N, w)$ we have

$$w(S) \preccurlyeq w(N) - \sum_{i \in N \setminus S} M_i(N, w) \preccurlyeq w(N) - \sum_{i \in N \setminus S} I_i = \sum_{i \in S} I_i.$$

So, $I \in \mathcal{C}(N, w)$. Therefore $\mathcal{K}(N, w) \subset \mathcal{C}(N, w)$ holds.

□

Next, we define for a (big boss) subgame $< T, w >$ (with n as a big boss) of $w \in BBIG^N$ two particular elements of its interval core, which we call the big boss interval point and the union interval point. These points will play an important role regarding the characterization of big boss games using the description of the interval core. The *big boss interval point* $\mathcal{B}(T, w)$ is defined by

$$\mathcal{B}_j(T, w) = \begin{cases} [0,0], & j \in T \setminus \{n\} \\ w(T), & j = n, \end{cases}$$

and the *union interval point* $\mathcal{U}(T, w)$ is defined by

$$\mathcal{U}_j(T, w) = \begin{cases} M_j(T, w), & j \in T \setminus \{n\} \\ w(T) - \sum_{i \in T \setminus n} M_i(T, w), & j = n. \end{cases}$$

THEOREM 3.3. *Let $w \in IG^N$ be such that property* (a) *in Theorem 3.1 holds. Then, $w \in BBIG^N$ if and only if for each $T \subset N$ with $n \in T$ the big boss interval point $\mathcal{B}(T,w)$ and the union interval point $\mathcal{U}(T,w)$ belong to the interval core of $< T, w >$.*

Proof. If $w \in BBIG^N$ then by Proposition 3.2 it is clear that $\mathcal{B}(T,w)$ and $\mathcal{U}(T,w) \in \mathcal{C}(T,w)$ for each $T \subset N$ with $n \in T$.
Conversely, assume that for each $T \subset N$ with $n \in T$ the points $\mathcal{B}(T,w)$ and $\mathcal{U}(T,w)$ belong to the interval core. Since by hypothesis $< N, w >$ satisfies (a), we only need to show that (b) and (c) hold.
First, take $n \in T$. Since $\mathcal{B}(T,w) \in \mathcal{C}(T,w)$, we have

$$w(S) \preccurlyeq \sum_{i\in S} \mathcal{B}_i(T,w) = \mathcal{B}_n(T,w) + \sum_{i\in S\setminus n} \mathcal{B}_i(T,w) = w(T) + [0,0] = w(T).$$

So, (b) is satisfied.
Second, take S such that $n \in S \subset T$. Since $\mathcal{U}(T,w) \in \mathcal{C}(T,w)$ we have,

$$w(S) \preccurlyeq \sum_{i\in S} \mathcal{U}_i(T,w) = \mathcal{U}_n(T,w) + \sum_{i\in S\setminus n} \mathcal{U}_i(T,w) =$$
$$(w(T) - \sum_{i\in T\setminus n} M_i(T,w)) + \sum_{i\in S\setminus n} M_i(T,w) = w(T) - \sum_{i\in T\setminus S} M_i(T,w).$$

So, (c) is satisfied. □

From the above theorem we learn that big boss interval games are totally $\mathcal{I}$-balanced games. Note that $\mathcal{B} : BBIG^N \to I(\mathbb{R})^N$ and $\mathcal{U} : BBIG^N \to I(\mathbb{R})^N$ are additive maps.
Next we give an example with economic flavour leading to a big boss interval game.

EXAMPLE 3.2. *Let us consider a production economy with one landlord and many peasants. Let $N = \{1, 2, \ldots, n\}$ be the player set, where 1 is the landlord who can not produce anything alone, and $2, 3, \ldots, n$ are landless peasants.*
Let $f : [0,n] \twoheadrightarrow I(\mathbb{R})$ be the production function with interval data, where $f(s)$ is the interval reward $[f_1(s), f_2(s)]$ if s peasants are hired by the landlord, where $f(0) = [0,0]$, f_1 and $f_2 - f_1$ are concave with $f_2 - f_1 \geq 0$. The

situation can be modeled by the big boss interval game $< N, w >$, where $N = \{1, 2, \dots, n\}$ and the characteristic function is given by

$$w(S) = \begin{cases} [0,0], & 1 \notin S \\ f(|S|-1), & 1 \in S. \end{cases}$$

4 Bi-monotonic allocation schemes

In this section we introduce bi-monotonic allocation schemes (bi-mas) for big boss interval games. We denote by P_n the set $\{S \subset N | n \in S\}$ of all coalitions containing the big boss.

Take a game $w \in BBIG^N$ with n as a big boss. We call a scheme $B = (B_{iS})_{i \in S, S \in P_n}$ an (interval) allocation scheme for w if $(B_{iS})_{i \in S}$ is an interval core element of the subgame $< S, w >$ for each coalition $S \in P_n$. Such an allocation scheme $B = (B_{iS})_{i \in S, S \in P_n}$ is called a *bi-monotonic (interval) allocation scheme (bi-mas)* for w if for all $S, T \in P_n$ with $S \subset T$ we have $B_{iS} \succcurlyeq B_{iT}$ for all $i \in S \setminus \{n\}$, and $B_{nS} \preccurlyeq B_{nT}$. In a bi-mas the big boss is weakly better off in larger coalitions, while the other players are weakly worse off.

We say that for a game $w \in BBIG^N$ with n as a big boss an imputation $I = (I_1, \dots, I_n) \in \mathcal{I}(w)$ is *bi-mas extendable* if there exists a bi-mas $B = (B_{iS})_{i \in S, S \in P_n}$ such that $B_{iN} = I_i$ for each $i \in N$. The next theorem is inspired by Voorneveld, Tijs and Grahn (2003).

THEOREM 4.1. *Let $w \in BBIG^N$ with n as a big boss and let $I \in \mathcal{C}(N, w)$. Then, I is bi-mas extendable.*

Proof. Since $I \in \mathcal{C}(N, w)$, by (4), we can find for each $i \in N \setminus \{n\}$ an $\alpha_i \in [0, 1]$, such that $I_i = \alpha_i M_i(N, w)$, and then $I_n = w(N) - \sum_{i \in N \setminus \{n\}} \alpha_i M_i(N, w)$. We will show that $(B_{iS})_{i \in S, S \in P_n}$, defined by $B_{iS} = \alpha_i M_i(S, w)$ for each S and i such that $i \in S \setminus \{n\}$, and $B_{nS} = w(S) - \sum_{i \in S \setminus \{n\}} \alpha_i M_i(S, w)$ is a bi-mas. Take $S, T \in P_n$ with $S \subset T$ and $i \in S \setminus \{n\}$. We have to prove that $B_{iS} \succcurlyeq B_{iT}$ and $B_{nS} \preccurlyeq B_{nT}$. First, $B_{iS} = \alpha_i M_i(S, w) \succcurlyeq \alpha_i M_i(T, w) = B_{iT}$, where "the inequality" follows from (d). Second,

$$\begin{aligned}
B_{nT} &= w(T) - \sum_{i \in T \setminus \{n\}} \alpha_i M_i(T, w) \\
&\succcurlyeq (w(S) + \sum_{i \in T \setminus S} M_i(T, w)) - \sum_{i \in T \setminus \{n\}} \alpha_i M_i(T, w) \\
&= (w(S) - \sum_{i \in S \setminus \{n\}} \alpha_i M_i(T, w)) + \sum_{i \in T \setminus S} (1 - \alpha_i) M_i(T, w) \\
&\succcurlyeq (w(S) - \sum_{i \in S \setminus \{n\}} \alpha_i M_i(S, w)) + \sum_{i \in T \setminus S} (1 - \alpha_i) M_i(T, w) \\
&= B_{nS} + \sum_{i \in T \setminus S} (1 - \alpha_i) M_i(T, w) \succcurlyeq B_{nS}
\end{aligned}$$

where the first follows from (c), the second follows from (d), and the third follows from $\alpha_i \leq 1$ and the nonincreasing of the interval marginal contribution vectors. So, $B_{nT} \succcurlyeq B_{nS}$. □

Now, we introduce on the class of big boss interval games an interval compromise-like solution concept, called the $\mathcal{T}$-value, and the interval AL-value inspired by Tijs (2005), and show that the $\mathcal{T}$-value equals the interval AL-value.

Let $w \in BBIG^N$. The $\mathcal{T}$-value of w is defined by

$$\mathcal{T}(N, w) = \frac{1}{2}(\mathcal{U}(N, w) + \mathcal{B}(N, w)).$$

Note that $\mathcal{T} : BBIG^N \to I(\mathbb{R})^N$ has some trade-off flavour because $\mathcal{T}(N, w)$ is the average of the union point $\mathcal{U}(N, w)$ and the big boss interval point $\mathcal{B}(N, w)$ for each $w \in BBIG^N$.

Given a game $w \in BBIG^N$, and an ordering

$$\sigma = (\sigma(1), \sigma(2) \ldots, \sigma(k), \sigma(k+1), \ldots, \sigma(n))$$

of the players in $N = \{1, 2, \ldots, n\}$, the lexicographic maximum of the interval core $\mathcal{C}(N, w)$ of $< N, w >$ with respect to σ, which we denote by $L^\sigma(N, w)$, is defined as follows:

$$L^\sigma_{\sigma(i)}(N, w) = \begin{cases} M_{\sigma(i)}(N, w), & i < k \\ 0, & i > k \\ w(N) - \sum_{j=1}^{k-1} M_j(N, w), & i = k. \end{cases} \tag{5}$$

We notice that L^σ is additive on $BBIG^N$.
The *interval average lexicographic value* $AL(N,w)$ of $w \in BBIG^N$ is defined by

$$AL(N,w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L^\sigma(N,w),$$

where $\Pi(N)$ is the set of permutations $\sigma : N \to N$.
Applying (5) we obtain

$$AL(N,w) = (\frac{1}{2}M_1(N,w), \ldots, \frac{1}{2}M_{n-1}(N,w), w(N) - \frac{1}{2}\sum_{i=1}^{n-1} M_i(N,w)).$$

So, we have $AL(N,w) = \mathcal{T}(N,w)$.

Remark 4.1. *Let $w \in BBIG^N$ with n as a big boss. Then,*

$$\mathcal{T}(N,w) = AL(N,w) \in \mathcal{C}(N,w)$$

and the (total) AL-value generates a bi-mas for $w \in BBIG^N$.

References

[1] Alparslan Gök S.Z., Miquel S. and Tijs S., "Cooperation under interval uncertainty", *preprint no.* ***73****, Institute of Applied Mathematics, METU* (2007) and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 09* (2008) (to appear in Mathematical Methods of Operations Research).

[2] Alparslan Gök S.Z., Branzei R. and Tijs S., "Cores and stable sets for interval-valued games", *preprint no.* ***90****, Institute of Applied Mathematics, METU* and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 17* (2008a).

[3] Alparslan Gök S.Z., Branzei R. and Tijs S., "Convex interval games", *preprint no.* ***100****, Institute of Applied Mathematics, METU* and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 37* (2008b).

[4] Alparslan Gök S.Z., Branzei R. and Tijs S., "How to handle interval solutions for cooperative interval games", Working paper (2008c).

[5] Bauso D. and Timmer J.B., "Robust Dynamic Cooperative Games", *Memorandum 1813 Department of Applied Mathematics, University of Twente, Enschede*, ISSN 0169-2690 (2006)

[6] Branzei R., Dimitrov D. and Tijs S., Shapley-like values for interval bankruptcy games, Economics Bulletin 3 (2003) 1-8.

[7] Branzei R., Dimitrov D. and Tijs S., "Models in Cooperative Game Theory: Crisp, Fuzzy and Multi-Choice Games", *Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin* Vol. **556** (2005).

[8] Branzei R., Tijs S. and Timmer J.,"Collecting information to improve decision-making", *International Game Theory Review*, Vol.**3**, No.1 (2001a) 1-12.

[9] Branzei R., Tijs S. and Timmer J., "Information collecting situations and bi-monotonic allocation schemes", *Mathematical Methods of Operations Research* **54** (2001b) 303-313.

[10] Gillies D. B., "Solutions to general non-zero-sum games." In: Tucker, A.W. and Luce, R.D. (Eds.), *Contributions to theory of games IV, Annals of Mathematical Studies **40**. Princeton University Press, Princeton* (1959) pp. 47-85.

[11] Muto S., Nakayama M., Potters J. and Tijs S., "On big boss games", *The Economic Studies Quarterly* Vol.**39**, No. 4 (1988) 303-321.

[12] Muto S., Potters J. and Tijs S., "Information market games", *International Journal of Game Theory* **18** (1989) 209-226.

[13] Tijs S., "Bounds for the core and the τ-value." In: Moeschlin O., Pallaschke D. (eds.), *Game Theory and Mathematical Economics*, North Holland, Amsterdam (1981) pp. 123-132.

[14] Tijs S., "Big boss games, clan games and information market games." In: Ichiishi T., Neyman A., Tauman Y. (eds.), *Game Theory and Applications.* Academic Press, San Diego (1990) pp.410-412.

[15] Tijs S., "The first steps with Alexia, the average lexicographic value", *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 123* (2005).

[16] Tijs S., Meca A. and López M.A., "Benefit sharing in holding situations", *European Journal of Operational Research* **162(1)** (2005) 251-269.

[17] Tijs S., Timmer J. and Branzei R., "Compensations in information collecting situations", *Journal of Public Economic Theory*, **8** (2006) 181-191.

[18] Voorneveld M., Tijs S. and Grahn M., "Monotonic allocation schemes in clan games ", *Mathematical Methods of Operations Research* **56** (2003) 439-449.