A system of probabilistic inequalities

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PROBLEMS AND SOLUTIONS

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All problems and solutions should be sent, typewritten in duplicate, to Murray S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1. An asterisk placed beside a problem number indicates that the problem was submitted without solution. Proposers and solvers whose solutions are published will receive 10 *reprints of the corresponding problem section. Other solvers will receive just one reprint provided a self-addressed stamped (U.S.A. or Canada) envelope is enclosed. Proposers and solvers desiring acknowledgment of their contributions should include a self-addressed stamped postcard (no stamp necessary outside the U.S.A. and Canada). Solutions should be received by February* 15, 1984.

PROBLEMS

Two Legendre Polynomial Identities

Problem 83-16, *by* A. D. RAWLINS (Brunel University, Middlesex, UK).

Show that

$$(1)\qquad P_n\{x(x^2-\alpha^2)^{-1/2}\}=\frac{(-1)^n}{n!}(x^2-\alpha^2)^{(n+1)/2}\frac{d^n}{dx^n}(x^2-\alpha^2)^{-1/2},\qquad n\geqq 0,$$

$$(2)\qquad P_n\left\{\frac{1+Bx}{B+x}\right\}=B^{-1}(n!)^{-2}\left\{\frac{-(B+x)}{2}\right\}^{n+1}\cdot\prod_{p=1}^{n}\left\{-\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]-p(p-1)\right\}\left\{\frac{x-B}{x+B}\right\},\qquad n\geqq 1.$$

A System of Probabilistic Inequalities

Problem 83-17*, *by* T. TEN RAA (Erasmus University, Rotterdam, The Netherlands).

Consider n locations, $n\geqq 2$:

$$\underset{1}{\bigcirc}\ \underset{2}{\bigcirc}\cdots\underset{i}{\bigcirc}\cdots\underset{j}{\bigcirc}\cdots\underset{n}{\bigcirc}.$$

Initially, there is *one* particle at each location. Then there are two consecutive transitions, governed by real nonnegative matrices (p_{ij}) and (x_{ij}), respectively. Thus p_{ij} is the probability that the particle which is originally at location i will move to j in the first round. Similarly, x_{ij} is the probability that a particle which after the first round is at location i will move to j in the second round.

It is assumed that the system is *shaky*. More precisely, all particles move with certainty in both rounds. Furthermore, in the second transition any particle has a positive probability to reach any other location. Formally,

$$(1)\qquad p_{ii}=0,\qquad p_{ij}>0\quad\text{for some } j\neq i,\quad i=1,\cdots,n,$$

$$(2)\qquad x_{ii}=0,\qquad x_{ij}>0\quad\text{for all } j\neq i,\quad i=1,\cdots,n.$$

(It is not necessary to assume that transition probabilities add up to unity.)

Consider the particle which originates from any location i. The probability that it ends up at location j is $\sum_{k=1}^{n} p_{ik}x_{kj}$. Now *suppose* that the system is so shaky that our particle is at least as likely to end up in any other location j as to return to its origin i:

$$(3) \qquad \sum_{k=1}^{n} p_{ik}x_{kj} \geqq \sum_{k=1}^{n} p_{ik}x_{ki} \quad \text{for all } j \neq 1, \quad i = 1, \cdots, n.$$

It is *conjectured* that this is impossible: the real nonnegative system (1), (2) and (3) is *inconsistent*. In other words, some particle must be more likely to return than to end up at some other location.

This problem arose in an attempt to extend a generalization of Kakutani's fixed point theorem to certain unbounded regions.

An Infinite System of Differential Equations

Problem 83-18*, *by* T. D. ROGERS (University of Alberta).

M. von Smoluchowski [1] obtained the differential system

$$\frac{dN_k}{dt} = \sum_{i+j=k} a_{ij}N_iN_j - N_k \sum_{j=1}^{\infty} a_{kj}N_j$$

in the analysis of the coagulation of colloidal particles as a consequence of Brownian motion. The physically implausible assumption that a_{ij} = constant enables one to derive a solution in closed form. Can anything be said about the form of the solutions in the more general case? Even the cases $a_{ij} = a_i$ or $a_{ij} = a_j$ would be of interest. This equation has had recent application as a model for cell aggregation kinetics, and related numerical work has been carried out by T. D. Rogers and J. R. Sampson [2], [3].

REFERENCES

[1] S. CHANDRASEKHAR, *Stochastic problems in physics and astronomy*, in Papers on Noise and Stochastic Processes, Nelson Wax, ed., Dover, New York, 1954, pp. 1–89.

[2] T. D. ROGERS, *Local models of cell aggregation kinetics*, Bull. Math. Biol., 39 (1977), pp. 23–42.

[3] T. D. ROGERS AND J. R. SAMPSON, *A random walk model of cellular kinetics*, Int. J. Biomed. Comp., 1, 8 (1977), pp. 45–60.

Properties of an Operator

Problem 83-19*, *by* P. SCHWEITZER (University of Rochester).

Determine whether or not the operator $T: E^{NR+} \longrightarrow E^{NR+}$,

$$(Tx)_{ir} \equiv \frac{a_{ir} + \sum_{j=1}^{N} M(r)_{ij}x_{jr}}{\max\left[1, \sum_{k=1}^{R} b_{ik}\left[a_{ik} + \sum_{l=1}^{N} M(k)_{il}x_{lk}\right]\right]}, \qquad x_{ir} \geqq 0, 1 \leqq r \leqq R, \quad 1 \leqq i \leqq N$$

(where $a_{ir} \geqq 0$, $b_{ir} > 0$, $M(r)_{ij} \geqq 0$, $\operatorname{Spr} M(r) < 1$, $\forall\, r, i, j$) (Spr = Spectral radius) is a contraction operator in some norm, or a n-step contraction operator for some n. (Empirically, T has a unique fixed point x^* and $T^m x \longrightarrow x^*$ geometrically for any x.)

Remark. for the special case $R = 1$, T *is* a contraction operator. T reduces to (suppressing the index r or k)

$$(Tx)_i = \min\left[\frac{1}{b_i}, a_i + \sum_{j=1}^{N} M_{ij}x_j\right], \qquad 1 \leqq i \leqq N$$