A system of probabilistic inequalities

Ten Raa, M.H.

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PROBLEMS AND SOLUTIONS

EDITED BY MURRAY S. KLAMKIN

COLLABORATING EDITORS: HENRY E. FETTIS, CECIL C. ROUSSEAU, YUDELL L. LUKE, OTTO G. RUEHR

All problems and solutions should be sent, typewritten in duplicate, to Murray S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1. An asterisk placed beside a problem number indicates that the problem was submitted without solution. Proposers and solvers whose solutions are published will receive 10 *reprints of the corresponding problem section. Other solvers will receive just one reprint provided a self-addressed stamped (U.S.A. or Canada) envelope is enclosed. Proposers and solvers desiring acknowledgment of their contributions should include a self-addressed stamped postcard (no stamp necessary outside the U.S.A. and Canada). Solutions should be received by February* 15, 1984.

PROBLEMS

Two Legendre Polynomial Identities

Problem 83-16, *by* A. D. RAWLINS (Brunel University, Middlesex, UK).

Show that

$$(1)\qquad P_n\{x(x^2-\alpha^2)^{-1/2}\} = \frac{(-1)^n}{n!}(x^2-\alpha^2)^{(n+1)/2}\frac{d^n}{dx^n}(x^2-\alpha^2)^{-1/2}, \qquad n \geqq 0,$$

$$(2)\qquad P_n\left\{\frac{1+Bx}{B+x}\right\} = B^{-1}(n!)^{-2}\left\{\frac{-(B+x)}{2}\right\}^{n+1} \cdot \prod_{p=1}^{n}\left\{-\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]-p(p-1)\right\}\left\{\frac{x-B}{x+B}\right\}, \qquad n \geqq 1.$$

A System of Probabilistic Inequalities

Problem 83-17*, *by* T. TEN RAA (Erasmus University, Rotterdam, The Netherlands).

Consider n locations, $n \geqq 2$:

$$\underset{1}{\bigcirc}\ \underset{2}{\bigcirc} \cdots \underset{i}{\bigcirc} \cdots \underset{j}{\bigcirc} \cdots \underset{n}{\bigcirc}.$$

Initially, there is *one* particle at each location. Then there are two consecutive transitions, governed by real nonnegative matrices (p_{ij}) and (x_{ij}), respectively. Thus p_{ij} is the probability that the particle which is originally at location i will move to j in the first round. Similarly, x_{ij} is the probability that a particle which after the first round is at location i will move to j in the second round.

It is assumed that the system is *shaky*. More precisely, all particles move with certainty in both rounds. Furthermore, in the second transition any particle has a positive probability to reach any other location. Formally,

$$(1)\qquad p_{ii}=0, \qquad p_{ij}>0 \quad \text{for some } j \neq i, \quad i=1,\cdots,n,$$

$$(2)\qquad x_{ii}=0, \qquad x_{ij}>0 \quad \text{for all } j \neq i, \qquad i=1,\cdots,n.$$

(It is not necessary to assume that transition probabilities add up to unity.)

Consider the particle which originates from any location i. The probability that it ends up at location j is $\sum_{k=1}^{n} p_{ik} x_{kj}$. Now *suppose* that the system is so shaky that our particle is at least as likely to end up in any other location j as to return to its origin i:

$$\sum_{k=1}^{n} p_{ik} x_{kj} \geqq \sum_{k=1}^{n} p_{ik} x_{ki} \quad \text{for all } j \neq 1, \quad i = 1, \cdots, n. \tag{3}$$

It is *conjectured* that this is impossible: the real nonnegative system (1), (2) and (3) is *inconsistent*. In other words, some particle must be more likely to return than to end up at some other location.

This problem arose in an attempt to extend a generalization of Kakutani's fixed point theorem to certain unbounded regions.

An Infinite System of Differential Equations

Problem 83-18*, *by* T. D. ROGERS (University of Alberta).

M. von Smoluchowski [1] obtained the differential system

$$\frac{dN_k}{dt} = \sum_{i+j=k} a_{ij} N_i N_j - N_k \sum_{j=1}^{\infty} a_{kj} N_j$$

in the analysis of the coagulation of colloidal particles as a consequence of Brownian motion. The physically implausible assumption that a_{ij} = constant enables one to derive a solution in closed form. Can anything be said about the form of the solutions in the more general case? Even the cases $a_{ij} = a_i$ or $a_{ij} = a_j$ would be of interest. This equation has had recent application as a model for cell aggregation kinetics, and related numerical work has been carried out by T. D. Rogers and J. R. Sampson [2], [3].

REFERENCES

[1] S. CHANDRASEKHAR, *Stochastic problems in physics and astronomy*, in Papers on Noise and Stochastic Processes, Nelson Wax, ed., Dover, New York, 1954, pp. 1–89.

[2] T. D. ROGERS, *Local models of cell aggregation kinetics*, Bull. Math. Biol., 39 (1977), pp. 23–42.

[3] T. D. ROGERS AND J. R. SAMPSON, *A random walk model of cellular kinetics*, Int. J. Biomed. Comp., 1, 8 (1977), pp. 45–60.

Properties of an Operator

Problem 83-19*, *by* P. SCHWEITZER (University of Rochester).

Determine whether or not the operator $T: E^{NR+} \rightarrow E^{NR+}$,

$$(Tx)_{ir} \equiv \frac{a_{ir} + \sum_{j=1}^{N} M(r)_{ij} x_{jr}}{\max\left[1, \sum_{k=1}^{R} b_{ik}\left[a_{ik} + \sum_{l=1}^{N} M(k)_{il} x_{lk}\right]\right]}, \qquad x_{ir} \geqq 0,\ 1 \leqq r \leqq R, \quad 1 \leqq i \leqq N$$

(where $a_{ir} \geqq 0$, $b_{ir} > 0$, $M(r)_{ij} \geqq 0$, $\operatorname{Spr} M(r) < 1$, $\forall\ r, i, j$) (Spr = Spectral radius) is a contraction operator in some norm, or a n-step contraction operator for some n. (Empirically, T has a unique fixed point x^* and $T^m x \rightarrow x^*$ geometrically for any x.)

Remark. for the special case $R = 1$, T *is* a contraction operator. T reduces to (suppressing the index r or k)

$$(Tx)_i = \min\left[\frac{1}{b_i}, a_i + \sum_{j=1}^{N} M_{ij} x_j\right], \qquad 1 \leqq i \leqq N$$