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Abstract: In this paper we consider the linear quadratic differential game for descriptor systems that have index one. We derive both necessary and sufficient conditions for existence of an open-loop Nash equilibrium.

Keywords: linear quadratic differential games; open-loop information structure; descriptor systems.

Jel-codes: C61, C72, C73.

1 Introduction

Dynamic game theory brings together three features that are key to many situations in economy, ecology, and elsewhere: optimizing behavior, presence of multiple agents, and enduring consequences of decisions. For that reason this framework is often used to analyze various policy problems in these areas (see e.g. [2], [7] and [11]).

In applications one often encounters however systems described by a set of ordinary differential equations subject to some algebraic constraints. These systems are known as descriptor systems.

As far as the authors know, except for the work [12], a study of differential games for descriptor systems is lacking up to now.

In this paper we take a first step in trying to fill this gap. We consider the problem of two players who like to optimize their performance given by a usual quadratic cost function depending both on the state and control variables. The underlying system is described by a set of differential and algebraic equations.

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We assume that the information structure of the game is of the open-loop type. That is, both players only know the initial state and structure of the system, and the set of admissible control actions are functions of time.

Linear quadratic control problems play an important role in applications. Therefore the linear quadratic control problem for descriptor systems has been considered in the literature by various authors too. The theory on the autonomous linear quadratic control problem for descriptor systems is, e.g., well documented in [10]. Here one can find also many references to this literature. Like most approaches for solving the linear quadratic control problem for descriptor systems, in this paper we solve the corresponding game problem by first applying an appropriate transformation to the pencil $\lambda E - A$ (see (3)). Under some additional simplifying assumptions on the system it is possible then to solve the game, for both a finite and infinite planning horizon, using the theory for affine linear quadratic differential games as documented in [4] and [5].

The outline of the paper is as follows. The next section formalizes the problem statement and summarizes some basic properties about descriptor systems. In section three we present the main results for the finite planning horizon, whereas section four contains those about the infinite planning horizon. In section five we illustrate some of the theory by an example. Finally section six concludes.

2 Preliminaries

In this paper we assume that the dynamics of the game is described by

$$E\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t),\ x(0) = x_0, \tag{1}$$

where $E,\ A \in \mathbb{R}^{(n+r)\times(n+r)}$, $\text{rank}(E) = n$, $B_i \in \mathbb{R}^{(n+r)\times m_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the controls player i can use to manipulate the system. Each player i has a quadratic cost functional J_i given by:

$$\int_0^{t_f} \{x^T(t)\bar{Q}_ix(t) + u_i^T(t)\bar{R}_iu_i(t)\}dt + x^T(t_f)\bar{Q}_{it_f}x(t_f). \tag{2}$$

Here all matrices are constant in time, $\bar{Q}_i = \bar{Q}_i^T$, and $\bar{R}_i$ is positive definite (> 0).
The inclusion of player j's control efforts into player i's cost function is dropped because, due to the open-loop information structure, this term drops out in the analysis.
From, e.g., [1] we recall the following results for the differential algebraic equation

$$E\dot{x}(t) = Ax(t) + f(t),\ x(0) = x_0. \tag{DAE}$$

and the associated matrix pencil

$$\lambda E - A. \tag{3}$$

System (DAE) and (3) are said to be *regular* if the characteristic polynomial $\det(\lambda E - A)$ is not identically zero. If the pencil (3) is not regular, then the system (DAE) is under-determined in the sense that consistent initial conditions do not uniquely determine solutions (see [6]). If the pencil (3) is regular, then the roots of the characteristic polynomial are the finite eigenvalues of the pencil. If E is singular, the pencil is said to have infinite eigenvalues which may be identified as the zero eigenvalues of the inverse pencil $E - \lambda A$. From [6] we recall the so-called Weierstrass canonical form.

Theorem 2.1 *If (3) is regular, then there exist nonsingular matrices X and Y such that*

$$Y^TEX = \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} \text{ and } Y^TAX = \begin{bmatrix} J & 0 \\ 0 & I_r \end{bmatrix}, \tag{4}$$

where J is a matrix in Jordan form whose elements are the finite eigenvalues, $I_k \in \mathbb{R}^{k\times k}$ is the identity matrix and N is a nilpotent matrix also in Jordan form. J and N are unique up to permutation of Jordan blocks. □

If (3) is regular the solutions of (DAE) take the form

$$x(t) = X_1z_1(t) + X_2z_2(t)$$

where with $X = [X_1\ X_2]$, $Y = [Y_1^T\ Y_2^T]$, $X_1,\ Y_1^T \in \mathbb{R}^{(n+r)\times n}$, $X_2,\ Y_2^T \in \mathbb{R}^{(n+r)\times r}$ and

$$\begin{aligned} z_1(t) &= e^{Jt}z_1(0) + \int_0^t e^{J(t-s)}Y_1f(s)ds; \\ &\quad z_1(0) = [I_n\ 0]X^{-1}x_0 \\ z_2(t) &= -\sum_{i=0}^{k-1} N^iY_2\frac{d^i}{dt^i}f(t), \end{aligned}$$

under the consistency condition:

$$[0\ I_r]X^{-1}x_0 = -\sum_{i=0}^{k-1} N^iY_2\frac{d^i}{dt^i}f(0).$$

Here k is the degree of nilpotency of N. That is the integer k for which $N^k = 0$ and $N^{k-1} \neq 0$. The *index* of the pencil (3) and of the descriptor system (DAE) is the degree k of nilpotency of N. If E is nonsingular, we define the index to be zero.

From the above formulae it is obvious that the solution $x(t)$ will not contain derivatives of the function f if and only if $k \leq 1$. In that case the solution $x(t)$ is called *impulse free*. In general, the solution $x(t)$ involves derivatives of order $k-1$ of the forcing function f if (DAE) has index k.
Next, let $[V\ W]$ be an orthogonal matrix such that image V equals the image of E^T and image W equals the null space of E. Then $E = [E_1\ 0][V\ W]^T = E_1V^T$, where E_1 is full column rank. The next lemma characterizes pencils which have an index of at most one.

Lemma 2.2 *The following statements are equivalent:*

i) pencil (3) is regular and has at most index one.

ii) rank $\left(\begin{bmatrix} E \\ V^TA \end{bmatrix}\right) = n + r\ (= \text{rank}(E + VV^TA))$.

iii) rank $([E\ AW]) = n + r\ (= \text{rank}(E + AWW^T))$. □

Since we do not want to consider derivatives of the input function in this paper, we restrict the analysis to regular index one systems here. The above discussion motivates then the next assumptions.

Assumption 2.3 Throughout this paper the next assumptions are made w.r.t. system (1):

1. matrix E is singular;
2. $\det(\lambda E - A) \neq 0$;
3. $\text{rank}([E\ AW] = n + r$;
4. For every $x_0 \in \mathbb{R}^n$ there exist a $u_1(0)$ and $u_2(0)$ such that $G(Ax_0 + B_1u_1(0) + B_2u_2(0)) = 0$, or equivalently, image $GA \subset$ image $G[B_1\ B_2]$, where $G := [0\ I]Y^T$.

3 The finite planning horizon

In this section we consider the game (1,2) under the assumption that t_f is finite. Furthermore we assume that

$$X^T\bar{Q}_{it_f}X = \begin{bmatrix} Q_{it_f} & 0 \\ 0 & 0 \end{bmatrix},\ i = 1, 2, \text{where } Q_{it_f} \in \mathbb{R}^{n\times n}.$$

With

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := X^{-1}x,\ \text{with } x_1 \in \mathbb{R}^n \text{ and } x_2 \in \mathbb{R}^r \tag{5}$$

the game (1,2) has a set of open-loop Nash equilibrium actions $(u_1(.), u_2(.))$ if and only if $(u_1(.), u_2(.))$ are open-loop Nash equilibrium actions for the game

$$\begin{aligned} \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Y^TB_1u_1(t) + \\ +Y^TB_2u_2(t),\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = X^{-1}x_0, \end{aligned} \tag{6}$$

where player i has the quadratic cost functional J_i:

$$\begin{aligned} \int_0^{t_f} \{[x_1^T(t)\ x_2^T(t)]X^T\bar{Q}_iX\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u_i^T(t)\bar{R}_iu_i(t)\}dt + \\ +x_1^T(t_f)Q_{it_f}x_1(t_f). \end{aligned} \tag{7}$$

From (6) it follows that

$$\begin{aligned} x_2(t) &= -[0\ I_r]Y^T(B_1u_1(t) + B_2u_2(t)) \\ &= -Y_2(B_1u_1(t) + B_2u_2(t)). \end{aligned} \tag{8}$$

Substitution of (8) into the cost functions (7) shows that $(u_1(.), u_2(.))$ are open-loop Nash equilibrium actions for the game (1,2) if and only if $(u_1(.), u_2(.))$ are open-loop Nash equilibrium actions for the game

$$
\begin{aligned}
\dot{x}_1(t) = Jx_1(t) + Y_1B_1u_1(t) + Y_1B_2u_2(t),\\
x_1(0) = [I_n\ 0]X^{-1}x_0,
\end{aligned} \tag{9}
$$

with cost functional J_i for player i given by

$$
\begin{aligned}
&\int_0^{t_f} \{z^T(t) \begin{bmatrix} I_n & 0 \\ 0 & -B_1^TY_2^T \\ 0 & -B_2^TY_2^T \end{bmatrix} X^T\bar{Q}_iX \begin{bmatrix} I_n & 0 & 0 \\ 0 & -Y_2B_1 & -Y_2B_2 \end{bmatrix} \\
&\qquad\qquad z(t) + u_i^T\bar{R}_iu_i(t)\}dt + x_1^T(t_f)Q_{it_f}x_1(t_f) \\
&=: \int_0^{t_f} \{z^T(t)M_iz(t)\}dt + x_1^T(t_f)Q_{it_f}x_1(t_f),
\end{aligned} \tag{10}
$$

where $z^T = [x_1^T(t)\ u_1^T(t)\ u_2^T(t)]$ and

$$
M_i = \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix}. \tag{11}
$$

For a spelling of the matrices defined in (11) see Appendix A. In Appendix B we introduced some additional notation that will be used throughout this paper. Using this notation we obtain the next result. An outline of the proof is given in Appendix C.

Theorem 3.1 *Assume that the two Riccati differential equations*

$$
\begin{aligned}
\dot{K}_i(t) = -J^TK_i(t) - K_i(t)J + (K_i(t)Y_1B_i + V_i)R_{ii}^{-1}\\
(B_i^TY_1^TK_i(t) + V_i^T) - Q_i;\ K_i(T) = Q_{it_f},
\end{aligned} \tag{12}
$$

have a symmetric solution $K_i(.)$ on $[0, t_f],\ i = 1, 2$.
Then the linear quadratic differential game (1,2) has an open-loop Nash equilibrium for every initial state if and only if matrix

$$
H(t_f) = [I\ 0\ 0]e^{-Mt_f} \begin{bmatrix} I \\ Q_{1t_f} \\ Q_{2t_f} \end{bmatrix}
$$

is invertible.
Moreover, if for every x_0 there exists an open-loop Nash equilibrium then the solution is unique. The unique equilibrium actions as well as the associated state trajectory can be calculated from the linear two-point boundary value problem

$$
\dot{y}(t) = My(t),\ \text{with}\ Py(0) + Qy(t_f) = [x_1^T(0)\ 0\ 0]^T. \tag{13}
$$

Here

$$
P = \begin{bmatrix} I_n\ 0\ 0 \\ 0\ 0\ 0 \\ 0\ 0\ 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 & 0 \\ -Q_{1t_f} & I_n & 0 \\ -Q_{2t_f} & 0 & I_n \end{bmatrix}.
$$

Denoting $[y_0^T(t),\ y_1^T(t),\ y_2^T(t)]^T := y(t)$, *with* $y_i \in \mathbb{R}^n, i = 0,1,2,$ *the state and equilibrium actions are*

$$x^*(t) = X \begin{bmatrix} x_1^*(t) \\ x_2^*(t) \end{bmatrix} \text{ where } x_1^*(t) = y_0(t),$$
$$x_2^*(t) = Y_2[B_1\ B_2]G^{-1}\left(Zy_0(t) + \tilde{B}^T \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}\right)$$
$$\text{and } \begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1}\left(Zy_0(t) + \tilde{B}^T \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}\right), t > 0,$$

respectively. □

Similar as in [4, Theorem 7.2 and Proposition 7.5] one can relate the existence of open-loop Nash equilibria for this game also to the existence of solutions of a set of coupled Riccati equations. Following the lines of the proofs provided in [4] we obtain the next analogues.

Theorem 3.2
A. *Assume that*

i. The set of (coupled) Riccati differential equations

$$\dot{P}(t) = -\tilde{A}_2^T P(t) - P(t)\tilde{A} + P(t)BG^{-1}\tilde{B}^T P(t) - \tilde{Q};$$
$$P^T(t_f) = [Q_{1t_f}^T,\ Q_{2t_f}^T]$$

has a solution P *on* $[0, t_f]$, *and*

ii. The two Riccati differential equations (12) have a symmetric solution $K_i(.)$ *on* $[0, t_f]$.

Then the differential game (1,2) has a unique open-loop Nash equilibrium for every initial state. Moreover, the equilibrium actions are

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1}(Z + \tilde{B}^T P(t))\tilde{\Phi}(t,0)[I\ 0]X^{-1}x_0, \tag{14}$$

where $\tilde{\Phi}(t,0)$ *is the solution of the transition equation*

$$\dot{\tilde{\Phi}}(t,0) = (A - BG^{-1}(Z + \tilde{B}^T P(t)))\tilde{\Phi}(t,0);\ \tilde{\Phi}(0,0) = I.$$

The corresponding state trajectory is given by

$$x^*(t) = X \begin{bmatrix} x_1^*(t) \\ x_2^*(t) \end{bmatrix} \text{ where } x_1^*(t) = \tilde{\Phi}(t,0)[I\ 0]X^{-1}x_0,$$
$$x_2^*(t) = Y_2[B_1\ B_2]G^{-1}(Z + \tilde{B}^T P(t))x_1^*(t).$$

B. *For all* $t_f \in [0, t_1)$ *there exists for all* x_0 *a unique open-loop Nash equilibrium for the game (1,2) if and only if the above Riccati differential equations i. and ii. have an appropriate solution for all* $t_f \in [0, t_1)$.
In case the game has a unique equilibrium the actions are given by (14). □

4 The infinite planning horizon

In this section we assume that the cost functional player $i = 1, 2$, likes to minimize is:

$$\lim_{t_f \to \infty} J_i(x_0, u_1, u_2, t_f), \tag{15}$$

where

$$J_i(x_0, u_1, u_2, t_f) = \int_0^{t_f} \{x^T(t)\bar{Q}_i x(t) + u_i^T(t)\bar{R}_i u_i(t)\} dt,$$

subject to (1).
We assume that the matrix pairs (A, B_i), $i = 1, 2$, are finite dynamics stabilizable. That is, if $\sigma(H)$ denotes the spectrum of matrix H; $\mathbb{C}^- = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$; $\mathbb{C}_0^+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$, then $\mathrm{rank}([\lambda E - A, B_i]) = n + r, \forall \lambda \in \mathbb{C}_0^+$. It can be easily shown that this assumption is equivalent with the assumption that the matrix pairs $(J, Y_1 B_i)$, $i = 1, 2$, are stabilizable in (9). So, in principle, each player is capable to stabilize the system (1) on his own.

We assume that the players choose control functions belonging to the set $\mathcal{U}_s(x_0)$ of square integrable functions yielding a stable closed-loop system (see also e.g. [13]):

$$\Big\{u \in L_2(0, \infty) \mid \lim_{t_f \to \infty} J_i(t_f, x_0, u) \in \mathbb{R} \cup \{-\infty, \infty\}, \\ \lim_{t \to \infty} x(x_0, u, t) = 0\Big\}.$$

Here $x(x_0, u, t)$ is the solution of (1)[1]. Notice that the assumption that the players use simultaneously stabilizing controls introduces the cooperative meta-objective of both players to stabilize the system (see e.g. [4] for a discussion). For simplicity of notation we will omit from now on the dependency of $\mathcal{U}_s$ on x_0.

In the rest of the paper the algebraic Riccati equations

$$J^T K_i + K_i J - (K_i Y_1 B_i + V_i) R_{ii}^{-1} (B_i^T Y_i^T K_i + V_i^T) \\ + Q_i = 0, \ i = 1, 2, \tag{16}$$

and the set of (coupled) algebraic Riccati equations

$$0 = \widetilde{A}_2^T P + P\widetilde{A} - PBG^{-1}\widetilde{B}^T P + \widetilde{Q} \tag{17}$$

or, equivalently,

$$0 = A_2^T P + PJ - (PB + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}) G^{-1} (\widetilde{B}^T P + Z) + Q$$

play a crucial role.

Definition 4.1 A solution $P^T =: (P_1^T, P_2^T)$, with $P_i \in \mathbb{R}^n$, of the set of algebraic Riccati equations (17) is called

[1] $\lim_{t_f \to \infty} J_i(t_f, x_0, u) = -\infty(\infty)$ if $\forall r \in \mathbb{R}, \exists T_f \in \mathbb{R}$ such that $t_f \geq T_f$ implies $J_i(t_f, x_0, u) \leq r (\geq r)$.

a. *stabilizing*, if $\sigma(\tilde{A} - BG^{-1}\tilde{B}^T P) \subset \mathbb{C}^-$;

b. *left-right stabilizing*[2](LRS) if

i. it is a stabilizing solution, and

ii. $\sigma(-\tilde{A}_2^T + PBG^{-1}\tilde{B}^T) \subset \mathbb{C}_0^+$; □

The next relationship between certain invariant subspaces of matrix M and solutions of the Riccati equation (17) is well-known (see e.g. [4, Chapter 7.3]). This property can also be used to calculate the (left-right) stabilizing solutions of (17).

Lemma 4.2 *Let $C \subset \mathbb{R}^{3n}$ be an n-dimensional invariant subspace of M, and let $C_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, 2$, be three real matrices such that*

$$C = \mathrm{Im}\left[C_0^T,\ C_1^T,\ C_2^T\right]^T.$$

If C_0 is invertible, then $P_i := C_i C_0^{-1}$, $i = 1, 2$, solves (17) and $\sigma(A - BG^{-1}(Z + \tilde{B}^T P)) = \sigma(M|_C)$. Furthermore, (P_1, P_2) is independent of the specific choice of basis of C. □

The next lemma summarizes the relationship between the LRS solution of (17) and the stable graph subspace of matrix M. A proof of it can be found in [5] and [8].

Lemma 4.3

1. *The set of algebraic Riccati equations (17) has a LRS solution (P_1, P_2) if and only if matrix M has an n-dimensional stable graph subspace and M has $2n$ eigenvalues (counting algebraic multiplicities) in $\mathbb{C}_0^+$.*

2. *If the set of algebraic Riccati equations (17) has a LRS solution, then it is unique.* □

Following [5] the following theorem can be proved.

Theorem 4.4 *If the differential game (3,1) has an open-loop Nash equilibrium for every initial state, then*

1. *M has at least n stable eigenvalues (counted with algebraic multiplicities). More in particular, there exists a p-dimensional stable M-invariant subspace S, with $p \geq n$, such that*

$$\mathrm{Im}\begin{bmatrix} I \\ \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} \subset S,$$

for some $\tilde{V}_i \in \mathbb{R}^{n\times n}$.

[2] In [4] such a solution is called strongly stabilizing.

2. *the two algebraic Riccati equations (16) have a stabilizing solution.*

Conversely, if the two algebraic Riccati equations (16) have a stabilizing solution and $v^T(t) =: [x^T(t), \psi_1^T(t), \psi_2^T(t)]$ *is an asymptotically stable solution of*

$$\dot{v}(t) = Mv(t), \;\; x(0) = x_0,$$

then, with $\psi^T(t) := [\psi_1^T(t), \psi_2^T(t)]$,

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1}\left[\tilde{B}^T\psi(t) + Zx(t)\right], \tag{18}$$

provides an open-loop Nash equilibrium for the linear quadratic differential game (3,1). □

Remark 4.5 Similar conclusions as in [5] can be drawn now. A general conclusion is that the number of equilibria depends critically on the eigenstructure of matrix M. With s denoting the number (counting algebraic multiplicities) of stable eigenvalues of M we have.
1. If $s < n$, still for some initial state there may exist an open-loop Nash equilibrium.
2. In case $s \geq 2$, the situation might arise that for some initial states there exists an infinite number of equilibria.
3. If M has a stable graph subspace, S, of dimension $s > n$, for every initial state x_0 there exists, generically, an infinite number of open-loop Nash equilibria. □

The next theorem shows that in case the set of coupled algebraic Riccati equations (17) have a stabilizing solution, the game always has at least one equilibrium.

Theorem 4.6 *Assume that*

1. *the set of coupled algebraic Riccati equations (17) has a set of stabilizing solutions* $P_i,\; i = 1, 2$; *and*

2. *the two algebraic Riccati equations (16) have a stabilizing solution* $K_i(.)$, $i = 1, 2$.

Then the linear quadratic differential game (3,1) has an open-loop Nash equilibrium for every initial state.
Moreover, one set of equilibrium actions is (for $t > 0$*) given by:*

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1}(Z + \tilde{B}^T P)\tilde{\Phi}(t, 0)[I \;\; 0]X^{-1}x_0, \tag{19}$$

where $\tilde{\Phi}(t, 0)$ *is the solution of the transition equation*

$$\dot{\tilde{\Phi}}(t, 0) = (J - BG^{-1}(Z + \tilde{B}^T P))\tilde{\Phi}(t, 0); \;\; \tilde{\Phi}(0, 0) = I.$$

The corresponding state trajectory is given by

$$x^*(t) = X \begin{bmatrix} x_1^*(t) \\ x_2^*(t) \end{bmatrix} \text{ where } x_1^*(t) = \tilde{\Phi}(t, 0)[I \;\; 0]X^{-1}x_0,$$
$$x_2^*(t) = Y_2[B_1 \;\; B_2]G^{-1}(Z + \tilde{B}^T P)x_1^*(t).$$

Furthermore, the costs by using the actions (19) for the players are $([I\ 0]X^{-1}x_0)^T \bar{M}_i [I\ 0]X^{-1}x_0,\ i = 1,2$, *where, with* $A_{cl} := J - BG^{-1}(Z + \tilde{B}^T P)$, $\bar{M}_i$ *is the unique solution of the Lyapunov equation*

$$[I,\ (-G^{-1}(Z+\tilde{B}^TP))^T]M_i[I,\ (-G^{-1}(Z+\tilde{B}^TP))^T]^T + A_{cl}^T\bar{M}_i + \bar{M}_iA_{cl} = 0. \tag{20}$$

□

Corollary 4.7 *An immediate consequence of Lemma 4.2 and Theorem 4.6 is that if* M *has a stable invariant graph subspace and the two algebraic Riccati equations (16) have a stabilizing solution, the game will have at least one open-loop Nash equilibrium.* □

The equilibrium actions (19) can be implemented also as a state feedback by considering the system:

$$\dot{x}_1(t) = (J - BG^{-1}(Z + \tilde{B}^TP))x_1(t),\ x_1(0) = [I\ 0]X^{-1}x_0.$$

Then, $\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1}(Z + \tilde{B}^TP)x_1(t)$.

Notice that in case the set of algebraic Riccati equations (17) has more than one set of stabilizing solutions, there exists more than one open-loop Nash equilibrium. Matrix M has then a stable subspace of dimension larger than n. Consequently (see Remark 4.5, item 3) for every initial state there will exist, generically, an infinite number of open-loop Nash equilibria. This point was first noted by Kremer in [8] in case matrix A is stable.

The above reflections raise the question whether it is possible to find conditions under which the game has a unique equilibrium for every initial state. The next Theorem 4.8 gives such conditions. Moreover, it shows that in case there is a unique equilibrium the corresponding actions are obtained by those described in Theorem 4.6.

Theorem 4.8 *Consider the differential game (3,1).*
This game has a unique open-loop Nash equilibrium for every initial state if and only if

1. *The set of coupled algebraic Riccati equations (17) has a LRS solution, and*
2. *the two algebraic Riccati equations (16) have a stabilizing solution.*

Moreover, the unique equilibrium actions are given by (19). □

5 An Example

Consider the next simplified macroeconomic model in which the governments of two symmetric countries aim at stabilizing domestic policies (see e.g. [2] and [3]).

$$\begin{aligned} \dot{s}(t) &= -\phi_2 s(t) - \phi_1 u_1(t) + \phi_1 u_2(t),\ x(0) = x_0 && (21)\\ q_1(t) &= \alpha u_1(t) + \beta u_2(t) + \gamma s(t) && (22)\\ q_2(t) &= \beta u_1(t) + \alpha u_2(t) - \gamma s(t), && (23) \end{aligned}$$

where s is a measure of international competitiveness, u_i is the domestic real money supply (used to control the system) and q_i denotes the deviation of the real output of country $i,\ i=1,2$, from its natural level. All parameters in this model are assumed to be positive. Moreover we assume that output in a country is more affected by its domestic monetary policy than by the monetary policy pursued abroad (i.e. $\alpha > \beta$). The policy makers in each country choose their optimal monetary policy so as to minimize the costs of the output gap (and inflation which might be viewed to be a fraction of the output gap), the loss of international competitiveness due to a revaluation of the currency and the loss incurred due to the fact that the government uses its control instrument. Assuming that the cost functions are additive and quadratic and policy makers plan for an infinite horizon we get the next objective functional for country $i,\ i=1,2:$

$$\int_0^\infty \{as^2(t)+bq_i^2(t)+\bar{r}u_i^2\}dt. \tag{24}$$

Introducing $x(t) := [s(t)\ q_1(t)\ q_2(t)]$ we can rewrite this model into the descriptor form (1,2) with

$$E=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix};\ A=\begin{bmatrix}-\phi_2&0&0\\\gamma&-1&0\\-\gamma&0&-1\end{bmatrix};\ B_1=\begin{bmatrix}-\phi_1\\\alpha\\\beta\end{bmatrix};$$
$$B_2=\begin{bmatrix}\phi_1\\\beta\\\alpha\end{bmatrix};\ \bar{Q}_1=\begin{bmatrix}a&0&0\\0&b&0\\0&0&0\end{bmatrix} \text{ and } \bar{Q}_2=\begin{bmatrix}a&0&0\\0&0&0\\0&0&b\end{bmatrix}.$$

Since $\alpha > \beta$ it is easily verified that Assumption 2.3 is satisfied.

With $Y := \begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$ and $X := \begin{bmatrix}1&0&0\\\gamma&1&0\\-\gamma&0&1\end{bmatrix}$ the matrix pencil (E,A) can be rewritten into its Weierstrass canonical form

$$Y^TEX=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \text{ and } Y^TAX=\begin{bmatrix}-\phi_2&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Therefore, with $X_1^T=[1,\ \gamma,\ -\gamma]$;

$$X_2=\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix};\ Y_1^T=\begin{bmatrix}1\\0\\0\end{bmatrix};\ \text{and } Y_2^T=\begin{bmatrix}0&0\\-1&0\\0&-1\end{bmatrix};$$

M_1 and M_2 are respectively

$$\begin{bmatrix}a+\gamma^2b&\alpha\gamma b&\beta\gamma b\\\alpha\gamma b&\bar{r}+\alpha^2b&\alpha\beta b\\\beta\gamma b&\alpha\beta b&\beta^2b\end{bmatrix} \text{ and } \begin{bmatrix}a+\gamma^2b&-\beta\gamma b&-\alpha\gamma b\\-\beta\gamma b&\beta^2b&\alpha\beta b\\-\alpha\gamma b&\alpha\beta b&\bar{r}+\alpha^2b\end{bmatrix}.$$

From this we get then, with $d :=$determinant$(G)=(\bar{r}+\alpha^2b)^2-\alpha^2\beta^2b^2$ (which obviously differs from zero), $e := \bar{r} + \alpha(\alpha+\beta)b$ and $f := \bar{r} + (\alpha^2-\beta^2)b$, the next expressions (see Appendix B)

$$J = -\phi_2;\ G = \begin{bmatrix} \bar{r}+\alpha^2 b & \alpha\beta b \\ \alpha\beta b & \bar{r}+\alpha^2 b \end{bmatrix};\ \tilde{A} = -\phi_2 + \frac{2\alpha\gamma\phi_1 b e}{d};$$
$$\tilde{B} = \begin{bmatrix} -\phi_1 & 0 \\ 0 & \phi_1 \end{bmatrix};\ Q = (a+\gamma^2 b)\begin{bmatrix} 1 \\ 1 \end{bmatrix};\ Z = \alpha\gamma b \begin{bmatrix} 1 \\ -1 \end{bmatrix};$$
$$B = [-\phi_1\ \phi_1];\ \tilde{A}_2^T = \begin{bmatrix} -\phi_2 & 0 \\ 0 & -\phi_2 \end{bmatrix} - \frac{\gamma\phi_1 b}{d}\begin{bmatrix} -\alpha f & \beta\bar{r} \\ \beta\bar{r} & -\alpha f \end{bmatrix}.$$

Since $(J, Y_1 B_i)$ are stabilizable and $\begin{bmatrix} Q_i & V_i \\ V_i^T & R_{ii} \end{bmatrix} > 0,\ i = 1, 2$, (16) has a stabilizing solution (see e.g. [9, Theorem 9.4]). Next consider, with $m_{12} := \alpha(\alpha-\beta)\gamma^2 b^2 e$; $m_{22} := -\phi_1\alpha\gamma b f$ and $m_{23} := \phi_1\beta\gamma b\bar{r}$, matrix $M =$

$$\begin{bmatrix} -\phi_2 & 0 & 0 \\ -(a+\gamma^2 b) & \phi_2 & 0 \\ -(a+\gamma^2 b) & 0 & \phi_2 \end{bmatrix} + \frac{1}{d}\begin{bmatrix} 2\phi_1\alpha\gamma b e & -\phi_1^2 e & -\phi_1^2 e \\ m_{12} & m_{22} & m_{23} \\ m_{12} & m_{23} & m_{22} \end{bmatrix}. \tag{25}$$

Assuming that both $\phi_2 d - 2\phi_1\alpha\gamma b e > 0$ and $\phi_2 d - \phi_1(\alpha+\beta)\gamma b e > 0$ it is shown in Appendix D that (17) has a LRS solution. So this game has a unique open-loop Nash equilibrium under these parameter conditions. The unique equilibrium actions are

$$\begin{bmatrix} u_1^*(0) \\ u_2^*(0) \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}^{-1} \begin{bmatrix} q_1(0) - \gamma s(0) \\ q_2(0) + \gamma s(0) \end{bmatrix} \text{ and}$$
$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -\frac{\alpha\gamma b - \phi_1 h}{\bar{r} + \alpha(\alpha-\beta)b}\begin{bmatrix} 1 \\ -1 \end{bmatrix} x_1(t),\ t > 0,$$

where $x_1(t) = e^{(-\phi_2 + \frac{2\phi_1(\alpha\gamma b - \phi_1 h)}{\bar{r}+\alpha(\alpha-\beta)b})t}[1\ 0\ 0]x_0$.
The corresponding equilibrium state trajectory is

$$x^*(t) = \begin{bmatrix} 1 \\ \gamma - k \\ -(\gamma - k) \end{bmatrix} x_1(t),\ \text{with } k = \frac{(\alpha-\beta)(\alpha\gamma b - \phi_1 h)}{\bar{r} + \alpha(\alpha-\beta)b}.$$

The cost for both players are the same, i.e: $\bar{m}([1\ 0\ 0]x_0)^2$, where $\bar{m}$ is given by (30).

6 Concluding Remarks

In this note we considered the linear-quadratic differential game for descriptor systems which have an index one. Both necessary conditions and sufficient conditions were derived for the existence of an open-loop Nash equilibrium. Moreover, conditions were presented that are both necessary and sufficient for the existence of a unique equilibrium. Basically, the results were obtained by reformulating the game as an ordinary affine linear quadratic differential game. Following the lines and combining the results documented in [4] and [5] similar conclusions can be deduced.

The above results can be generalized straightforwardly to the N-player case. Furthermore, since Q_i are assumed to be indefinite, the obtained results can be directly used to (re)derive properties for the zero-sum game. If players discount their future loss, similar to [4, Chapter 3.6], it follows from Theorem 4.8 that if the discount factor δ is "large enough" the game has generically a unique open-loop Nash equilibrium (all that changes is that matrix A has to be replaced by $A - \frac{1}{2}\delta I$ everywhere). Finally we conclude from (18) that the conclusion in [8], that if the game has an open-loop Nash equilibrium for every initial state either there is a unique equilibrium or an infinite number of equilibria, applies here too.

Obviously there are still many open problems to be solved. For instance, problems that were not dealt with here are how to proceed in case the system is of a higher order index. Furthermore, the approach taken here is not motivated from a numerical point of view. Stated differently, there may be other ways to obtain the equilibrium actions advertized here which are from a numerical point of view much more preferable (like for the linear quadratic control problem (see [10])). Also the question emerges whether it is possible to solve this problem without making a preliminary "state decomposition". Furthermore, all of these problems can be analyzed also under different information structures.

Appendix A: shorthand notation M_i

The matrices defined in (11) are given by:

$$\begin{aligned}
Q_i &:= X_1^T \bar{Q}_i X_1,\ V_i := -X_1^T \bar{Q}_i X_2 Y_2 B_1,\\
W_i &:= -X_1^T \bar{Q}_i X_2 Y_2 B_2,\ N_i := B_1^T Y_2^T X_2^T \bar{Q}_i X_2 Y_2 B_2,\\
R_{11} &:= B_1^T Y_2^T X_2^T \bar{Q}_1 X_2 Y_2 B_1 + \bar{R}_1,\\
R_{21} &:= B_2^T Y_2^T X_2^T \bar{Q}_1 X_2 Y_2 B_2,\\
R_{12} &:= B_1^T Y_2^T X_2^T \bar{Q}_2 X_2 Y_2 B_1,\\
R_{22} &:= B_2^T Y_2^T X_2^T \bar{Q}_2 X_2 Y_2 B_2 + \bar{R}_2.
\end{aligned}$$

Appendix B: Notation

The next shorthand notation will be used.

$$A_2 = \mathrm{diag}\{J, J\};\ B = Y_1[B_1,\ B_2];$$

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix};\ G = \begin{bmatrix} [0\ I\ 0]\, M_1 \\ [0\ 0\ I]\, M_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} R_{11} & N_1 \\ N_2^T & R_{22} \end{bmatrix};$$

where we assume throughout that this matrix G is invertible,

$$\widetilde{B}^T := \mathrm{diag}\{B_1^T Y_1^T, B_2^T Y_1^T\};\ \widetilde{B}_1^T := \begin{bmatrix} B_1^T Y_1^T \\ 0 \end{bmatrix};$$

$$\widetilde{B}_2^T := \begin{bmatrix} 0 \\ B_2^T Y_1^T \end{bmatrix};\ Z := \begin{bmatrix} [0\ I\ 0]\, M_1 \\ [0\ 0\ I]\, M_2 \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V_1^T \\ W_2^T \end{bmatrix};$$

$$Z_i := [I\ 0\ 0] M_i \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} = [V_i,\ W_i],\ \widetilde{Q}_i := Q_i - Z_i G^{-1} Z;$$

$$\widetilde{Q} := \begin{bmatrix} \widetilde{Q}_1 \\ \widetilde{Q}_2 \end{bmatrix};\ \widetilde{S}_i := B G^{-1} \widetilde{B}_i^T;\ \widetilde{A}_2^T := A_2^T - \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} G^{-1} \widetilde{B}^T$$

$$\widetilde{S} := [\widetilde{S}_1,\ \widetilde{S}_2], \widetilde{A} := J - B G^{-1} Z;\ \text{and } M := \begin{bmatrix} \widetilde{A} & -\widetilde{S} \\ -\widetilde{Q} & -\widetilde{A}_2^T \end{bmatrix}.$$

Notice that M also equals

$$\begin{bmatrix} J & 0 & 0 \\ -Q_1 & -J^T & 0 \\ -Q_2 & 0 & -J^T \end{bmatrix} + \begin{bmatrix} -B \\ Z_1 \\ Z_2 \end{bmatrix} G^{-1} \begin{bmatrix} Z, & \tilde{B}_1^T, & \tilde{B}_2^T \end{bmatrix}.$$

Appendix C: Proof of Theorem 3.1

The proof of this theorem can be given along the lines of [4, Exercise 7.5 and Theorem 7.1]. Suppose that $(u_1^*(.), u_2^*(.))$ is a Nash equilibrium. Then, according the maximum principle, the Hamiltonian

$$H_i = [x_1^T\ u_1^T\ u_2^T] M_i \begin{bmatrix} x_1 \\ u_1 \\ u_2 \end{bmatrix} + \tilde{\psi}_i^T (J x_1 + Y_1 B_1 u_1 + Y_1 B_2 u_2),$$

is minimized by player i with respect to u_i. This yields the necessary conditions

$$G \begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = - \begin{bmatrix} V_1^T x_1(t) + \frac{1}{2} B_1^T Y_1^T \tilde{\psi}_1(t) \\ W_2^T x_1(t) + \frac{1}{2} B_2^T Y_1^T \tilde{\psi}_2(t) \end{bmatrix}.$$

Due to the invertibility assumption on matrix G we can rewrite this as

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1} \begin{bmatrix} V_1^T x_1(t) + \frac{1}{2} B_1^T Y_1^T \tilde{\psi}_1(t) \\ W_2^T x_1(t) + \frac{1}{2} B_2^T Y_1^T \tilde{\psi}_2(t) \end{bmatrix}.$$

Here the n-dimensional vectors $\tilde{\psi}_1(t)$ and $\tilde{\psi}_2(t)$ satisfy

$$\begin{aligned} \dot{\tilde{\psi}}_i(t) = -(2Q_i x_1(t) + 2V_i u_1^*(t) + 2W_i u_2^*(t) + J^T \tilde{\psi}_i(t)), \\ \text{with } \tilde{\psi}_i(t_f) = 2Q_{it_f} x(t_f). \end{aligned} \qquad (26)$$

and

$$\dot{x}_1(t) = Jx(t) - BG^{-1} \begin{bmatrix} V_1^T x_1(t) + \frac{1}{2} B_1^T Y_1^T \tilde{\psi}_1(t) \\ W_2^T x_1(t) + \frac{1}{2} B_2^T Y_2^T \tilde{\psi}_2(t) \end{bmatrix};$$
$$x_1(0) = [I_n \ 0] X^{-1} x_0.$$

In other words, if the problem has an open-loop Nash equilibrium then the differential equation (with $\psi_i(t) := \frac{1}{2}\tilde{\psi}_i(t)$)

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ \psi_1(t) \\ \psi_2(t) \end{bmatrix} = M \begin{bmatrix} x_1(t) \\ \psi_1(t) \\ \psi_2(t) \end{bmatrix} \tag{27}$$

with boundary conditions $x_1(0) = [I_n \ 0]X^{-1}x_0,\ \psi_1(t_f) - Q_{1t_f}x_1(t_f) = 0$ and $\psi_2(t_f) - Q_{2t_f}x(t_f) = 0$, has a solution.

Let $y(t) := [x_1^T(t),\ \psi_1^T(t),\ \psi_2^T(t)]^T$. Then the above reasoning shows that, if there is a Nash equilibrium, then for every $x_1(0)$ the next linear two-point boundary value problem has a solution.

$$\dot{y}(t) = My(t), \text{ with } Py(0) + Qy(T) = [x_1^T 0\ 0\ 0]^T. \tag{28}$$

Here

$$P = \begin{bmatrix} I\ 0\ 0 \\ 0\ 0\ 0 \\ 0\ 0\ 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 & 0 \\ -Q_{1T} & I & 0 \\ -Q_{2T} & 0 & I \end{bmatrix}.$$

Following the lines of [4, Theorem 7.1, p.341], the first statement of the theorem follows then. To derive the other statements recall that $x_2(t) = -Y_2(B_1u_1(t) + B_2u_2(t))$ and $x(t) = X \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

"$\Leftarrow$ part" This part is left to the reader. $\square$

Appendix D: some worked details of the Example

Consider matrix M from (25). Notice that this matrix has the following structure:

$$M = \frac{1}{d} \begin{bmatrix} -p & -q & -q \\ -r & s & t \\ -r & t & s \end{bmatrix}, \tag{29}$$

where $q = \phi_1^2 e > 0$, $r = ad + \gamma^2 eb\bar{r} > 0$ and $t = \phi_1\beta\gamma b\bar{r} > 0$. Under the assumption that $\phi_2 d - 2\phi_1\alpha\gamma be > 0$ it follows then that also $p > 0$ and $s > 0$. Moreover, the assumption that $\phi_2 d - \phi_1(\alpha+\beta)\gamma be > 0$ implies that $s - t > 0$ too.
The eigenvalues of $d*M$ are: $\{s-t, \frac{1}{2}(s+t-p \pm \sqrt{(s+t+p)^2 + 8qr})\}$. So under the above parameter conditions, matrix M has precisely one negative eigenvalue, i.e. $\lambda = \frac{1}{2}(s+t-p-\sqrt{(s+t+p)^2+8qr})$.
An eigenvector corresponding with this eigenvalue is:

$$[p_0, p_1, p_2]^T = [\frac{1}{2}(s+t+p+\sqrt{(s+t+p)^2+8qr}), 1, 1]^T.$$

So, by Theorem 4.8, the game has a unique equilibrium. From Lemma 4.2 we have that the solution of (17) is

$$P = \begin{bmatrix} \frac{2}{s+t+p+\sqrt{(s+t+p)^2+8qr}} \\ \frac{2}{s+t+p+\sqrt{(s+t+p)^2+8qr}} \end{bmatrix}.$$

Denoting $h := \frac{2}{s+t+p+\sqrt{(s+t+p)^2+8qr}}$, the equilibrium actions are then according (19)

$$\begin{aligned} \begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} &= -G^{-1}(Z + \tilde{B}^T P)x_1(t) \\ &= -\frac{1}{d} \begin{bmatrix} \bar{r} + \alpha^2 b & -\alpha\beta b \\ -\alpha\beta b & \bar{r} + \alpha^2 b \end{bmatrix} \begin{bmatrix} \alpha\gamma b - \phi_1 h \\ -(\alpha\gamma b - \phi_1 h) \end{bmatrix} x_1(t) \\ &= -\frac{(\alpha\gamma b - \phi_1 h)e}{d} \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_1(t) \\ &= -\frac{\alpha\gamma b - \phi_1 h}{\bar{r} + \alpha(\alpha - \beta)b} \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_1(t). \end{aligned}$$

Here $x_1(t) = e^{(-\phi_2 + \frac{2\phi_1(\alpha\gamma b - \phi_1 h)}{\bar{r}+\alpha(\alpha-\beta)b})t}[1\ 0\ 0]x_0$.
The corresponding equilibrium state trajectory is then (see Theorem 4.6)

$$x^*(t) = X \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where with $k := \frac{(\alpha-\beta)(\alpha\gamma b - \phi_1 h)}{\bar{r}+\alpha(\alpha-\beta)b}$

$$\begin{aligned} x_2(t) &= Y_2[B_1\ B_2]G^{-1}(Z + \tilde{B}^T P)x_1(t) \\ &= k \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_1(t). \end{aligned}$$

So,

$$x^*(t) = \begin{bmatrix} 1 \\ \gamma - k \\ -(\gamma - k) \end{bmatrix} x_1(t).$$

From (20)we have that the corresponding costs for the players are $x_0^T \bar{M}_i x_0,\ i = 1, 2$, where, with

$$A_{cl} = J - BG^{-1}(Z + \tilde{B}^T P) = -\phi_2 + \frac{2\phi_1(\alpha\gamma b - \phi_1 h)}{\bar{r} + \alpha(\alpha - \beta)b},$$

$\bar{M}_i$ is the unique solution of the Lyapunov equation

$$\begin{aligned} [I,\ (-G^{-1}(Z + \tilde{B}^T P))^T]M_i[I,\ (-G^{-1}(Z + \tilde{B}^T P))^T]^T + \\ A_{cl}^T \bar{M}_i + \bar{M}_i A_{cl} = 0. \end{aligned}$$

With $\delta := \frac{\alpha\gamma b - \phi_1 h}{\bar{r}+\alpha(\alpha-\beta)b}$, simple calculations show that $\bar{M}_i$ equals

$$\bar{m} := \frac{a + \gamma^2 b + 2(\beta - \alpha)\gamma b\delta + (\bar{r} + (\alpha - \beta)^2 b)\delta^2}{2(\phi_2 - 2\phi_1\delta)}. \tag{30}$$

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