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Aggregation Problem in Input Output Analysis - A Survey

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1. INTRODUCTION

The problem of aggregation in the input-output analysis arises in two cases[1] : (a) the construction of the input-output table; (b) application of the model for forecasting sectoral output. We shall confine our attention to the second case. A considerable amount of theoretical and empirical work has been done with regard to this case : given the input-output table, how can we aggregate for a specific purpose ?

Aggregation can be defined as a process or operation by which *detailed* sectors are consolidated into broad sectors, thus reducing the total number of original sectors. The output, input and coefficients of the group represent, in general, the average (weighted) of those of the original detailed sectors belonging to the group. In this *aggregated* model, besides several groups, there may be some original sectors. In any case, all the sectors in the aggregated model are usually denoted as aggregate sectors. "Aggregation is a matter of degree, since the original sectors are never, in practice, as detailed as is *possible* in principle" (McManus, 1956, p.29).

Aggregation of the detailed sectors, no doubt, serves certain purposes.[1] However, the gains obtained from consolidation have to be weighed against the disadvantages (e.g., an increase in errors, loss of information of the original sectors) due to aggregation. Of course, how far the given amount of expected error is undesirable depends on the specific purpose for which the model is constructed.

Both theoretical and empirical research have been carried out to formulate aggregation criteria. The object of the present paper is to make a critical survey of the existing literature on the aggregation problem. The paper will be organized as follows: In Section I, the aggregation problem will be formally presented. Sector II will deal with aggregation bias, the conditions of zero bias (necessary and sufficient) as developed by researchers. In Section III we will report on some of the empirical works along these lines. Section IV will be devoted to the discussion of the aggregation problem with special reference to the dynamic input-output model. In Section V we shall review certain alternative methods to aggregation, as suggested by Leontief (1967) and others. Concluding remarks along with the possibilities of future research will be presented in Section VI.

2. MATHEMATICAL FORMULATION OF THE AGGREGATION PROBLEMS AND DIFFERENT PROPERTIES

In his paper "Recent Developments in the Study of Inter-Industrial Relationships" (Leontief, 1951, pp. 202-216), Professor Leontief points out the essence of the aggregation problem with a simple example. Let us assume that there were one hundred "industrial" sectors in the economy. They form the original sectors. With the help of the input-output model one can determine the influence of a change in the final demand for cars upon the net output of papers. Now, if the sectors other than the car and the paper industries are consolidated in some way, we obtain a new system, called the hybrid system. If the hybrid system shows the same relationship between the final demand for cars and the new output of paper as the original one, then the consolidation introduced is acceptable. Leontief also carefully points out that what may be an acceptable aggregation scheme for a given purpose may no longer be so from different points of view. He states: "There are many alternative ways of aggregating the ninety-eight (basic classification) original industries under some forty-eight broader headings. Each reclassification will lead to a different system of fifty simultaneous equations and most likely also to a different solution. By comparing these alternative short-cut answers with the known correct solution of our problem, on the one hand, and with each other, it is possible to measure the comparative "goodness", i.e., operational efficiency, of alternative aggregating classifications of the ninety-eight basic industries. Considerable theoretical as well as experimental work in the problem of industrial classification is being done along these lines (Leontief, 1951, p. 217).

Following Leontief's example and illuminating ideas, Hatanaka (1952), followed by Ara (1959), Malinvaud (1956), McManus (1956), Theil (1957) and recently Morimoto (1970, 1971) did a considerable amount of theoretical work on formal properties of the problem of aggregation bias and error, conditions for consistent aggregation, etc. In this section, we shall make an attempt to review their works.

Define

X = column vector of output X_i

(i = 1, ...n)

Y = column vector of final demands Y_i

(i = 1, ...n)

A = input-output coefficient matrix of the original system

$= \left[a_{ij}\right]$ (i, j=1, ...n)

A = input-output coefficient matrix of the aggregated system

$= \left[a_{IJ}\right]$ (I, J=1, ...M)
$M \leq N$

T = aggregational operator

$$= \begin{bmatrix} 1....1, & o & o \\ o....o & 1....1 & o....o \\ o & o & 1....1 \end{bmatrix}$$

Hence, the original static Leontief system (before aggregation) is

$$X = (I-A)^{-1}Y \quad .... \quad (1)$$

and the aggregated system

$$X^* = (I-A^*)^{-1}Y^* \quad .... \quad (2)$$

Where

$$Y^* = TY$$

The "*aggregation bias*" can be defined as the difference between the outputs which are derived from the aggregated system and those which are derived by aggregating the results of the original system, that is, the aggregation bias:

$$= X^* - TX$$
$$= \left[I-A^*\right]^{-1} Y^* - T\left[I-A\right]^{-1}Y$$
$$= \left[I - A^*\right]^{-1} TY - T\left[I-A\right]^{-1}Y$$
$$= VY$$

where $V = [I-A^*]^{-1}T - T[I-A]^{-1}$ .... (3)

Expanding the inverse matrices in (3), we obtain

$$VY = [(I-A^* + A^{*2} + \ldots.)T$$
$$- T(I + A + A^2 \ldots.)]Y$$
$$= [(A^*T - TA) + (A^{*2} - TA^2) + \ldots.]Y$$

The "first order aggregation bias"[2] can be defined as

$$= (A^*T - TA)Y = \bar{V}Y$$

where $\bar{V} = A^*T - TA$

Conditions for the absence of aggregation bias (and of first order aggregation bias) can be divided broadly into two groups: (1) those concerning the coefficient matrix and (2) those concerning final demand. The following important theorems have been developed:

Theorem I. The aggregation bias vanishes for any final demand if and only if the coefficient matrix satisfies the condition $TA = A^*T$.

This condition was first formulated by Hatanaka[3] (1952) and its interpretation that was elaborated by McManus (1956, p.41), Theil (1957, pp.118-119) and Ara (1959, pp.259-260).

Theorem II. If the final demands are proportioned to those of the base period, the aggregation bias is zero.

This was shown by Baldersten and Whiten[4] (1954, p.108), Malinvaud[5] (1954 , p.200) and Theil (1957, p.120).

Theorem III. If the structure of final demands within each hybrid sector is the same as that of the corresponding outputs in the base period, then the first-order aggregation bias vanishes.

Theorem IV. If some sectors are not aggregated and the change in final demand occurs only in the unaggregated sectors, then the first-order aggregation bias always vanishes regardless of the way of aggregating the rest of the sectors.

Theorems II and IV have been developed by Morimoto recently (1970).

The implications of Theorem IV is that, as far as the effect of a change in final demands of some particular sector(s) on the output of some sectors(s) is concerned, all the other sectors than those which are in question may be aggregated into one exclusive sector.

Weighting

So far we have discussed the problem of aggregation in terms of simple aggregation which is a special case of weighted aggregation. In simple aggregation we aggregate those quantities of different commodities, each of which costs S 1.00 (or quantities of any other *common unit*). Apparently, the unit of each commodity is nothing but the amount which costs S 1.00 (or any other common unit). This choice of *units* is *arbitrary*. In weighted aggregation, we allow ourselves to choose units for all commodities. Thus, we aggregate a particular quantity of one commodity with a particular quantity of other commodities. The respective commodities are the new units or *weights*. They can be *chosen* so that the aggregation bias is minimized. This choice of weights is based on the definition of the bias, i.e., which bias the researcher finds relevant for his study.

We still have the possibility of aggregating the same quantities as we did in simple aggregation. Consequently, simple aggregation is a special case of weighted aggregation and the latter will yield a bias which is at least as small. It should also be mentioned that weighted aggregation can be viewed as simple aggregation under a *well known* price regime, viz., those prices for which our *chosen* aggregation units cost S 1.00. Hence, the economic meaning of the choice of weights is the choice of base year prices.

McManus (1956) and Morimoto (1971) discussed in detail the particular sets of weights for which aggregation bias vanishes in special cases. Morimoto establishes a theoretical relationship between the weights in consistent aggregation and the prices derivable from the dual price system in the input-output model. It is shown that (1) "the 'macro' case where the coefficient matrix is indecomposable and all sectors are aggregated into one all-inclusive sector, (2) the case where the coefficient matrix is completely decomposable and (3) the case where the coefficient matrix is cyclic, are the three special cases in weighted aggregation where there always exists a certain unique set of weights, g, ...gn, such that the aggregation bias vanishes for any final demands". (pp.141-142).

3. SOME EMPIRICAL STUDIES

All the above theoretical discussions are based on the basic idea - how to achieve "perfect aggregation" - implied in Hatanaka's condition (Theorem I, Section II) of homogeneity of input structure. That is, the aggregated coefficients of a macro sector are not affected by changes in the production pattern within the macro sector.

This requirement, especially, cannot always be fulfilled for practical application. Researchers have tried to find ways for "best aggregation". Several attempts Kossov (1972), Blin and Cohen (1977), Kymn (1977), Fisher (1958), Ghosh and Sarkar (1970), Theil and Uribe (1967) have been made in this direction. In this section we shall make a review of these studies.

Two approaches were adopted: (a) to determine the size of the matrix, N, and then to find best A* which would minimize the aggregation bias, (b) Alternative, to choose an upper bound, say, δ, on the basis of admissible aggregation bias and then to derive the conditions on disaggregated sector coefficients to accept or reject any form of sector aggregation.

In the first case, Hatanaka's criterior is obtained when it reaches its lower bound size. Kossov follows the second rule. He shows that if one standardizes all coefficients for each sector (setting $\sum_i a_{ij} = o$ and $\sum_i a_{ij}^2 = 1$) the condition implies simply that the correlation coefficient between any two column vector of A be at least as large as the value of a function in E and A. Using the determinant of correlation coefficient matrix as overall measure of goodness of fit, one can find the acceptable aggregation scheme which would result in a minimal increase in the value of this determinant.

But implementation of Kossov's scheme is very complicated, as the order of grouping of industries remain to be determined. Blin and Cohen (1977), however, have made some attempts empirically in this direction. They have proposed a notion of technological similarity between industries as a basis for grouping and a class of *cluster analytic methods* for algorithmic implementation.

They have used *hierarchical fusion algorithms* - starting with n clusters and reducing them into a single one.

They have applied their technique to the U.S. 1967 (83 x 83) direct coefficient table. To know how sensitive the results are to inclusion or exclusion of the value added component, they have used several specifications of two clustering algorithms: (1) Ward's method with the Euclidian distance between sectors as a measure of technological similarity; (2) the centroid method with the Euclidian distance. Additionally, each of these runs has been performed both with and without the value added coefficient. The results reveal that (1) there is, in general, much, less difference between algorithms if they treat value added similarly, e.g., both include, or exclude it - than if they treat it differently; (2) the centroid method leads to very close dendrograms whether a distance or a correlation measure of similarity is used.

Kymn (1977) has presented an aggregation system using the 1963 U.S.A. 83-order table in which non-energy sectors have been merged while maintaining the energy sector. Instead of the correlation method, Kymn has used the Ijiri Coefficient[6] and has quantified and analyzed the patterns of trade-off between aggregation bias and reduction in the order of the input-output table.

He examines five[7] different aggregation methods to reach for the optimal level of aggregation. Energy users are separated into three "big" groups and seven possible models of aggregation are generated. Each of the five different aggregation methods is assumed with regard to the seven possible modes of aggregation by using the Ijiri coefficients. After selecting the best aggregation method (method 5 is found to be performing best), Kymn quantitatively evaluates the trade-off between aggregation bias and reduced tables for that method. He concludes that with the lowest order of aggregation, 34 x 34 and the seocnd highest value in the Ijiri coefficient, 0.977230, its performance is better than other aggregation schemes. Kymn's study suggests a practical method of minimising an aggregation bias if one is interested in a group of sectors. Especially the use of Ijiri coefficient to find the optimal aggregation method and to assess quantitatively the trade-off between aggregation bias and reduction in the size of the input-output table is new. However, Kymn has not shown clearly why this is a superior measure of aggregation bias. One may possibly use correlation coefficient as a degree of interdependence among the industries, as Kossov has suggested.

W.D. Fisher (1958) has suggested a "good" aggregation procedure by using *Squared Error Measures*. His idea can be summarized below:

Let b_{ij} be elements of $(I-A)^{-1}$

and b_{IJ} be elements of $(I-A^*)^{-1}$

and $b_{Ij} = \sum_{i \varepsilon I} b_{ij}$

where $i \varepsilon I$ means the summation is for all sectors that have been grouped into given aggregated sector I.

Two measures are suggested: (a) the measure defined for the special purpose problem is

$$C'_I = \sum_{j=1}^{n} (\bar{b}_{IJ} - b_{Ij})^2 \, \bar{Y}_j \qquad \ldots.(4)$$

where Y_j is the final demand of the jth sector and the bar indicates that the variables are based on data in the base year.

For the general purpose prediction[8], the formula is:

$$C' = \sum_{j=1}^{M} C'_I = \sum_{I=I}^{M} \sum_{j=1}^{n} (\bar{b}_{IJ} - b_{Ij})^2 \, \bar{Y}_j \qquad \ldots.(5)$$

(b) A somewhat cruder measure is suggested as follows:

$$C_I'' = \sum_{j=1}^{n} (\bar{a}_{IJ} - a_{Ij})^2 \qquad \ldots.(6)$$

for the special problem, where $a_{IJ} = \sum_{i\epsilon I} a_{ij}$ and a_{ij} and $\bar{a}_{KJ}$ are elements in A and A*, respectively. Similarly, for the general purpose prediction, $C'' = \Sigma C_I''$ is suggested.

The practical advantage of this criterion is that this criterion employs the input-output coefficient without inverting the I - A*.

Fisher has made experiments in which an 18 x 18 matrix representing the United States economy in 1939 has been aggregated in thirteen different ways into 8 x 8 matrices and the various criteria for each of the different aggregation partitions were computed and compared and his results indicate that "given the prediction objective, substantially lower errors may be obtained by deliverate aggregation procedures based on minimal distance ideas than on haphazard procedures" (p.359).

Recently, Neudecker (1970) has reformulated Fisher's approach in matrix algebra and has also suggested an alternative approach which turns out to be not only compatible with Fisher's, but even better, accordingly, to the criteria adopted by Fisher.

Not completely analogous to Fisher, but using some concept of distance as in a transportation model, Ghosh and Sarkar (1970) have developed certain rules for ordering input-output matrices so that one can proceed to aggregate the neighbouring section.

The flows in an input-output matrix have been looked at similar to the flows from origin to destination as in a transportation model. One can look at the sector ordering as analogous to finding the location of both the centres of origin and the ultimate destination of a group of commodities. The basic difference, of course, is that unlike a transportation problem, in an input-output matrix the origin and destination can be moved about, subject to certain rules. In this sense, the flow matrices under different ordering can, therefore, be viewed as alternative spatial configurations under certain sets of rules. The ordering problem is then, to some extent, reduced to a spatial location problem and the ordering index becomes closely linked with notions of distance.

Consider the following two matrices:

Flow Matrix X					*Ordering Indicator Matrix D Associated with X*				
Row No.	*Column No.* 1	2	3	4	*Row No.*	*Column No.* 1	2	3	4
1	X_{11}	X_{12}	X_{13}	X_{14}	1	0	1	2	3
2	X_{21}	X_{22}	X_{23}	X_{23}	2	1	0	1	2
3	X_{31}	X_{32}	X_{33}	X_{34}	3	2	1	0	1
4	X_{41}	X_{42}	X_{43}	X_{44}	4	3	2	1	0

The matrix X represents the flow matrix and matrix D represents the associated ordering indicator matrix which defines the continuity of a source of origin and destination as determined by a specific ordering. The problem is now to minimize $\sum_i\sum_j d_{ij}\, x_{ij}$ for all possible ordering, subject to the input-output rule regarding the change of position of the rows and column.[9]

The optimal ordering for sectors will be such that for any other ordering:

$$\Sigma\Sigma d_{ij}\, x_{ij} \text{ (optimal)} < \Sigma\Sigma\, d_{ij}\, x_{ij}$$

The implication is that in such a sector ordering all industries that are heavily dependent on each other through supply or through demand will be placed as close as possible. Then the neighbouring sectors can be aggregated.

The main drawback of this method seems to be that it is not clear how best the aggregation would be, neither is the size of the aggregation bias generated, if any, clear. Another problem may arise due to the large size of the table. Ghosh and Sarkar have suggested a heuristic approach for a larger matrix which may not always give the desired result.

Theil and Uribe (1967) have applied *information approach* to the problem of aggregation in input-output analysis. They have calculated the information value of the original table, and that of the aggregated table, the latter always being less than or equal to that of the original table. The information loss has then been decomposed into several parts — portion due to the hetergenuity of the input structure, the portion due to the heterogenuity of the output structure and the portion due to the overall effect. Interestingly enough, Theil and Uribe have found these effects to be significantly stable in the period 1949-1960.

5. AGGREGATION PROBLEM IN DYNAMIC INPUT-OUTPUT MODELS

Not much research (both theoretical and empirical) has been doene dealing with the aggregation problem in dynamic input-output analysis. Ara (1959), McManus (1956), Morimoto (1970) and Ven (1974) have paid attention to the theoretical aspect of the problem. In this section we would make a brief review of this. A dynamic system may be derived from a static system, either through the introduction of (1) time lags or (2) stock-flow relationships (or a combination of both) (Leontief, 1956, pp.81-83).

We shall discuss the problem of aggregation in both the cases.

In the input-output analysis of the first case (which may be called a lagged system), the lagged relation is usually expressed in different equations such as:

$$X_t = AX_{t-1} + Y \qquad \ldots.(9)$$

Let the aggregation system of (1) be

$$X_t^* = A^* X^*_{t-1} + Y^* \qquad \ldots.(10)$$

Where,

$$Y^* = TY$$

and

$$X_o^* = TXo$$

Here the aggregation problem may arise in the following two issues: (a) the stability property of the system before and after the aggregation and (b) the aggregation bias along the time path.

While Ara (1959) has dealt with only the problem of the stability property, Morimoto (1970) has been concerned with the aggregation bias along the time path. They have proved the following two theorems:

Theorem I : (Ara)

If the aggregation of sectors in an indecomposable system is acceptable (namely TA + A*T), the A and A* system are equivalent with respect to the dynamic stability condition.

Theorem II : (Morimoto)

The aggregation bias in the lagged Leontief system vanishes for any final demands and for any initial conditions if and only if TA = A*T.

The aggregation problem, in the second type (the dynamic Leontief system) is studied by McManus (1956, pp.30-39). He formulated the condition for the absence of aggregation bias as (a) $A^*T = TA$ and (b) $B^*T = TB$.[10] However, Morimoto argues that McManus's proof gives the necessity, but not the sufficiency of the condition and he restated the theorem as follows:

Theorem : (Morimoto)

In the open dynamic systems (11) and (12) where $Y^* = TY$ and $X_o^* = TX_o$, the aggregation bias vanishes for any final demands and for any initial conditions if and only if $A^*T = TA$ and $B^*T = TB$. Ven (1974) has also derived some conditions for linear aggregation of dynamic input-output models and imposes on the productive capacity the condition of invariance of structure within technological industries. Though very few in number, the above studies have at least shed some light on the theoretical aspect of the aggregation problem in dynamic model. No empirical work (at least to the knowledge of the present authors) has been done to provide some guideline to practical work.

6. ALTERNATIVES TO AGGREGATION

In this section we shall present some alternative procedures as developed by Ghosh (1960) and Leontief (1967) which seem to be better than aggregation in certain cases.

In the input-output analysis, as we know, the productive system is presented as a system of general interdependence in which, in principle, every industry has a direct link with ever other industry. In practice, however, one may find on looking at the table that industries tend to form groups with a great deal of buying and selling within groups but relatively small between groups. If there were no transactions between groups, the production system would take the form of a set of independent sub-systems and these sub-systems could for various purposes be treated separately and the computing work in input-output analysis would be reduced. But observation shows a certain amount of transactions, however small, takes place between blocks and this cannot be ignored. Ghosh (1960) has made two assumptions to take into account the minor cells. In the first place, he has assumed that the amount available for absorption by industries outside the group to which a given industry belongs, is governed by supply considerations. In this case, the output of an industry depends on final demand and the demand expected within its group and is then raised by an appropriate small proportion to take into account the intermediate demand outside that group.

The second assumption is that absorptions are always related to demand - by averaging the demands for j's product over industries of type K, (if industries j and k are in different groups), what is important in considering the demand for j's product by k-type industries is the aggregate output of these industries and not its composition.

If the sectors are divided into two blocks, denoted respectively, by r and s the basic equation of the model can be written in partitioned form as:

$$\begin{bmatrix} X_r \\ \\ X_r \end{bmatrix} = \begin{bmatrix} A_{rr} & A_{rs} \\ \\ A_{sr} & A_{ss} \end{bmatrix} \begin{bmatrix} X_r \\ \\ X_s \end{bmatrix} + \begin{bmatrix} Y_r \\ \\ Y_s \end{bmatrix} \quad \ldots.(16)$$

If there are no transactions between groups, i.e. $A_{rs} = A_{sr} = 0$ then to obtain matrix multiplier it is necessary to only invert the submatrices $(I - A_{rr})$ and $(I - A_{ss})$.

On the first assumption, $A_{rs}\ X_s$ is replaced by $\hat{a}_r\ x_r$, and $A_{sr}\ X_r$ is replaced by $\hat{a}_s X_x$, $\hat{a}_r$ being a diagonal matrix whose elements denote the production of the output of one of the r-type industries available for s-type industries, and where $\hat{a}_s$ has a corresponding meaning. With this notation, (16) can be written:

$$\begin{bmatrix} X_r \\ X_s \end{bmatrix} \begin{bmatrix} A_{rr} + a_r & 0 \\ 0 & A_{ss} + a_s \end{bmatrix} \begin{bmatrix} X_r \\ X_s \end{bmatrix} \quad \begin{bmatrix} Y_r \\ Y_s \end{bmatrix} \quad \ldots.(17)$$

This leads to:

$$X_r = (I - A_{rr} - a_r)^{-1} Y_r$$

$$X_s = (I - A_{ss} - a_s)^{-1} Y_s$$

On the basis of the second assumption, the matrix A would be reduced twice in the separate aggregation. In one case, A_{rs} can be replaced by a column sector a_r whose elements are the average input coefficient and A_{sr} by $a_{s'}$, which is a row vector of the column sums of A_{sr}. Vectors X_x and Y_s are replaced by their aggregated element N_s and M_s respectively. We obtain the following form:

$$\begin{bmatrix} X_r \\ \\ N_s \end{bmatrix} = \begin{bmatrix} A_{rr} & a_r \\ \\ A_s & 0 \end{bmatrix} \quad \begin{bmatrix} X_r \\ \\ N_s \end{bmatrix} + \begin{bmatrix} Y_r \\ \\ M_s \end{bmatrix}$$

or

$$\begin{bmatrix} X_r \\ \\ N_s \end{bmatrix} = \begin{bmatrix} I - A_{rr} & -a_r \\ \\ -a_{s'} & 1 \end{bmatrix}^{-1} \quad \begin{bmatrix} Y_r \\ \\ M_s \end{bmatrix}$$

In the second case, A_{rs} is replaced by the row vector a_r' and A_{sr} by the column sector a_s and N_r and M_r will replace sector X_r and Y_r. The following form is obtained:

$$\begin{bmatrix} N_r \\ X_s \end{bmatrix} = \begin{bmatrix} 0 & a_r' \\ a_{sr} & A_{ss} \end{bmatrix} \begin{bmatrix} N_r \\ M_s \end{bmatrix} + \begin{bmatrix} M_r \\ Y_s \end{bmatrix}$$

which gives:

$$\begin{bmatrix} N_r \\ X_s \end{bmatrix} = \begin{bmatrix} 1 & -a_r' \\ -a_s & I-A_{ss} \end{bmatrix}^{-1} \begin{bmatrix} M_r \\ Y_s \end{bmatrix}$$

Thus it is not necessary to invert matrices of an order larger than r+1 or s+1, whichever is greater.

Ghosh has made some experiments with the input-output matrix of the United Kingdom for 1948 using the above procedures and has found good results. He concludes that "whenever an input-output table shows a strong clustering tendency, such clusters may be picked out and used for forecasting of variables within the cluster. We also find that by correcting the cluster for secular changes, we may obtain quite useful projections without the work involved in correcting the whole matrix for such changes. ....Further, this approach simplifies the vast details of the input-output matrix and reduces it to manageable proportions." (p.96).

Leontief (1967) has also proposed a double inversion *method, an inversion method* by which the input-output system for a subset of sectors in the original system may be prepared. Namely, if the sectors are divided into groups, as in equation (16), the original system (16) can be written as:

$$\begin{bmatrix} X_r \\ X_s \end{bmatrix} = \begin{bmatrix} B_{rr} & B_{rs} \\ B_{sr} & B_{ss} \end{bmatrix} \begin{bmatrix} Y_r \\ Y_s \end{bmatrix} \qquad \text{....(18)}$$

Where:

$$B = (I - A)^{-1} \qquad \text{....(19)}$$

So,

$$X_r = B_{rr}Y_r + B_{rs} Y_s \qquad \text{....(20)}$$

$$X_s = B_{sr}Y_r + B_{ss} Y_s \qquad \text{....(21)}$$

$$B_{rr}^{-1}X_r = Y_r + B_{rr}^{-1} B_{rs} Y_s \qquad \text{....(22)}$$

Define:

$$A_{rr*} = I - B_{rr}^{-1} \qquad \text{....(23)}$$

$$Y_r* = Y_r + B_{rr}^{-1} B_{rs} Y_s \qquad \text{....(24)}$$

Then (22) can be written as:

$$(I - A*)\ X_r = Y_r* \text{ or } X_r + (I - A_{rr}*)^{-1}\ Y_r*$$

$$= \left[I - A_{rr} - A_{rs}\ (I - A_{ss})^{-1}\ A_{sr}\right]^{-1} Y_r* \qquad \text{....(25)}$$

Eqn. (25) is an exact reduction of (18). For a $\begin{bmatrix} Y_r \\ Y_s \end{bmatrix}$

X_r is the solution using (18) and is exactly equal to X_r obtained from (25). Ghosh's approach is an approximation to Leontief's method. In Ghosh's method, the effects of sectors not in the group are reflected in B_{rr} and not in Y_r*.

Compared with a normal aggregation procedure, Leontief's method focusses on a few selected sectors and eliminates the remaining sectors. Aggregation errors are then avoided by redefining variables in a simplified system (in the above case, final demand) in such a way that the aggregation structure is totally consistent under the new definition of the variables. "In contrast to conventional aggregation, such analytical reduction is achieved without distortion of any of the basic structural relationships"[11] However, there is not much computational advantage in this approach, since we have to invert the matrix twice, which we do find in Ghosh's approximation method.

Disaggregation Approach

So far, we have discussed the problem of aggregation from detailed sectors to the aggregated form. But there may be another approach to the problem. As our knowledge about empirical production structure is growing, it is probably more true that we do not get the micro structure first in detail, from which a macro structure is derived by aggregation. Rather what is available to us first is the macro-structure which we gradually refine to obtain a more detailed micro-framework, one of the reasons being that collection and processing of the data are time consuming and costly, too. For this reason, Fei (1956) has distinguished the more practical problem of disaggregation from the aggregation problem. Fei derives a fundamental thesis whose economic significance can be briefly described: Under the strict assumption of input-output analysis (i.e., the absence of factor substitutability, the absence of joint product, and the linearity of the production function), one can imagine the matrix A is the unique ideal coefficient matrix and that the matrix A* is the *realistic empirical coefficient matrix.*

The matrix A* and Leontief inverse $(I - A^*)^{-1}$ are assumed to be available. How to find ideal matrix A ? To obtain the ideal matrix involves the following stages of work:

(i) decision with respect to the particular way in which a mixed sector of A* is to be split up into a number of more homogeneous sectors;

(ii) the collection of statistical data for the construction of a more homoegeneous coefficient matrix A with larger dimensions;

(iii) the inversion of $(I - A)$ to obtain $(I - A)^{-1}$.

Since (ii) and (iii) are costly and time consuming, Fei suggests a "test" of the significance of the choice of (i) on the bearing of the partial knowledge of (ii). The first step in the test is to approximate the ideal coefficient matrix A and its Leontief inverse $(I\text{-}A)^{-1}$ by augmenting the aggregated coefficient matrix A* and its Leontief inverse $(I\text{-}A^*)^{-1}$, respectively, and the second step is the estimation of the true Leontief inverse $(I\text{-}A)^{-1}$ from the approximation.

Fei's idea seems to be novel and interesting and may be found to be useful, though ideas have not been carried out later. Empirical investigation along with theoretical research of the existence of different types of augmentation operators and the solution to the resulting adjustment problems should be carried out.

In this connection we should note that Malinvaud (1956) has suggested a very simple rule to predict total output of different industries from an aggregated model — "just a proportional sub-division." (p.201.)

Conclusion

The above review of the literature on the aggregation problem in the input-output analysis have revealed the importance of the issue in the use of the analytical tool for different purposes. We have found that theorists like Hatanako, Theil, McManus, Morimoto, Kossov were preoccupied with perfect aggregation (simple or weighted) i.e. determination of A matrices for which bias vanishes.

But in practice it is hard to find such class of matrices. Given a matrix A the problem is : what is the best aggregation to minimize the error? Researchers (i.e. Balderston and Whitin, Fisher, Balin and Cohen, Kymn), as we have noticed, attempted to find good aggregation. Even then, we have seen, the criterion suggested or used depends very much on the specific purpose of the study. Of the empirical studies we have reviewed, two seem to be more useful for practical work. If the researcher is interested in some particular group of sectors, for example, resource sector, Kymn's work is a good starting point — how to aggregate other sectors in an optimal way. One may attempt using correlation coefficient or Kessov's method instead of Ijiro coefficient. In the case of a large input-output table with a substantial number of minor or zero cells, Ghosh's approximation method may be useful and it has computational advantage also.

However, most of the empirical literature suggests that the preference of the researchers is reflected on the choice of the aggregation method and the choice of bias is arbitrary. Instead all subjective considerations should be put in the choice of a bias and possibly a prior grouping of industries. Once this has been settled, the rest is the technical question — the method of aggregation is determined, at least in principle, Kossov's paper is developed along this line. Yet his ideas, it may be noted, do not include the weights — importance of which has been indicated in the earlier section. Once a particular bias is defined, the researcher has to choose the weight which would minimize it. To facilitate this step further analytical work has to be done to produce certain rules. Once certain rules are developed then these should be applied for practical work.

Foot-Notes

1. For example, some minimum levels of aggregation may be required even before the data collection for the model starts as (i) the detailed data may not be obtainable and (ii) the cost of collection, sorting, processing and tabulating the data may be reduced. When the model is ready, further aggregation may be suggested depending on the specific objective of the model construction. Aggregation may also save some computing time.

2. The definition to first order aggregation bias is due to Theil (1957, p.117).

3. Hatanaka actually derived two conditions : (a) the microsectors aggregated into the same macro sector must not have mutual transactions and (b) they must also have the same cost structure vis-a-vis all other macro sectors when prices of original commodities are used as weights of aggregation. However, McManus (1956) later on proved the first condition to be unnecessary when the gross method (without eliminating intra-sector transfers) are used or when the net outputs of aggregated sectors are defined properly to eliminate such mutual transactions. Ara (1959) has further shown that for the acceptability of aggregation, it is not necessary (but sufficient) that the input coefficient of the industrial sectors to be aggregated should be completely the same.

4. The Bladersten-Whiten paper is interesting because of the instructive examples, though it was criticised by McManus since their formulae, as they stand, "require (1-A) to be inverted before one can see whether or not any proposed aggregation will give consistent results. But once the inverse is known, all the outputs are easily found— and in the original detail — and so the aggregation is then unnecessary".

5. Malinvaud is concerned with the formal characteristics of the aggregation process, and with sufficient conditions under which their process does involve any contradictions.

6. Ijiri coefficient is:

$$A_c^2 \; \frac{\mathrm{tr}\; T\,(I-A)\; T^gT\; (I-A)'\; T'}{\mathrm{tr}\; T\,(I-A)\;(I-A)'\; T'} \qquad 0 \leq A_c^{\,2} \leq 1$$

where A_c^2 represents the Ijiri linear aggregation coefficient, T^g = the generalized inverse of T, I = an identity matrix, A = the technology matrix, tr = trace.

7. Five different methods in the search for the optimal level of aggregation are (a) relative shares in final demand absorbed by aggregated output, (b) the appropriate sums of the components of the Leontief inverse matrix, (c) the component of the Leontief inverse matrix, (d) the sensitivity coefficient and (e) the repercussion coefficient.

8. Where the prediction error of more than one industry or all industries is concerned.

9. For example, any column of the matrix can only be shifted to a new position if the corresponding row is shifted to the new position.

10. The original and the aggregated systems expressed in differential equations by McManus are:

$$(I-A)\ X - BX = Y \qquad \ldots.\ (11)$$

$$(I-A^*)\ X^* - B^*\ X^* = Y^* \qquad \ldots.\ (12)$$

where B is the capital input coefficient. The aggregation bias vanishes when:

$$X^* = TX \qquad \ldots.\ (13)$$

$$Y^* = TY \qquad \ldots.\ (14)$$

so that

$$X^* = TX \qquad \ldots.\ (15)$$

On substitution of (13), (14) and (15) into (12) and manipulating he obtains:

$$(TA - A^*T)\ X + (TB - B^*T)\ X = 0 \qquad \ldots.\ (16)$$

(16) is satisfied by any X and X, if and only if (a) $A^*T = TA$ and (b) $B^*T = TB$.

11. Leontief develops this approach to make "a comparison of the structural properties of five economies — or of the two structural characteristics of the same economy at two different points of time."

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