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ON THE COST-MINIMIZING NUMBER OF FIRMS

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This note revises the Baumol–Fischer analysis of the cost-minimizing industry structure to the effect that all necessary information is limited to scale assumptions and that bounds on the cost-minimizing number of firms are guaranteed to exist.

1. Introduction

Baumol and Fischer (1978) seek to determine the cost-minimizing industry structure from the assumption that there is a U-shaped average cost profile on any ray in output space. They do so by deriving bounds for the cost-minimizing number of firms. Their analysis has two short-comings. First, Baumol and Fischer invoke the assumption that costs are subadditive in the region of declining average cost in their derivation of the upper bound on the number of firms. This very strong assumption conflicts with their intended objective of limiting information to scale effects. The second shortcoming is that the Baumol and Fischer bounds need not exist. Consider the locus of ray average cost-minimizing outputs and the industry output y^I depicted in fig. 1.
Baumol and Fischer's lower bounding hyperplane to the locus could only be the horizontal axis but then the value of y^I under the homogeneous linear function associated with the hyperplane cannot be unity as is required by Baumol and Fischer (1978, p. 443). For this reason the lower

Output 2

Fig. 1. A locus of minimum ray average costs points for which the Baumol–Fischer lower bounding hyperplane does not exist.

bounding hyperplane and hence the upper bound on the number of firms do not exist. (A rigorous proof can be easily constructed.)

This note offers two revisions of the Baumol–Fischer analysis, each eliminating one of the shortcomings.

2. Doing away with subadditivity

I make the stronger assumption that along any ray in output space *marginal* rather than average cost is initially decreasing. [1] Theorem 1, which is of interest in itself, shows that in a region of declining marginal cost the optimal number of firms cannot exceed the number of commodities. In deriving the upper bound on the cost-minimizing number of firms the latter result can be used instead of Baumol and Fischer's subadditivity assumption which implies that in the region of declining average cost the cost-minimizing number of firms cannot exceed one. However, the result that in the region between the origin and the locus of ray average cost-minimizing outputs the cost-minimizing number of firms cannot exceed the number of commodities (rather than unity) clearly increases the solution value by the number of commodities. This is why Theorem 2 below yields a larger upper bound than Theorem 1A of Baumol and Fischer (1978). In economic terms the discrepancy reflects that the new premises permit anti-complementarity in production whereas this is precluded in Baumol and Fischer (1978, p. 465).

Revision 1.   Assume that C is a continuous cost function, and that for every output vector y of unit length there is a unique number $t^0(y) > 0$, corresponding to the minimum point of the firm's average cost curve

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[1] Declining marginal cost is sufficient but not necessary for ray average costs to decline by proposition 2 of Baumol (1977).

along the ray through vector y, such that $C(ty)$ is strictly concave (marginal cost is strictly decreasing) [2] for $0 < t < t^0(y)$ and $C(ty)/t$ is strictly increasing for $t^0(y) < t < \infty$.

Theorem 1. *C is a function on a region* [3] *in the non-negative orthant of n-dimensional Euclidean space with $C(0) \geq 0$. If $C(tx)$ is (strictly) concave in the scalar t for all x, then for all non-zero $x^1, \ldots, x^m$, $m > n$, there are at most n y^i's, summing up to $x^1 + \ldots + x^m$, such that $\Sigma_i C(y^i)$ is (respectively strictly) less than $C(x^1) + \ldots + C(x^m)$.*

The *proof* is relegated to the appendix.

Theorem 2. *Let revision 1 be satisfied, and let u^0 and l^0 be as in Baumol and Fischer (1978).* [4] *Then the optimal number of firms m^0 for the production of a given industry output vector y^1 is bounded by*

$$\left[\frac{1}{2u^0}\right] + 1 \leq m^0 \leq \max\left(n, \left[\frac{n+1}{l^0}\right] - 1\right),$$

where $[x]$ represents the largest integer less than or equal to x and $\lceil x \rceil$ the smallest integer greater than or equal to x.

We see that the first revision, the strengthening of the assumption on ray cost behavior, truly limits the required information to scale effects, thus overcoming the first shortcoming of Baumol and Fischer's analysis.

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[2] The revision introduces a kink into the cost function at points of minimum ray average costs. However, we can make the more general assumption that instead of a unique $t^0(y)$ there are $t^*(y)$, $t^0_-(y)$ and $t^0_+(y)$, of increasing magnitude, such that marginal cost is strictly decreasing for $0 < t < t^*(y)$ and average cost is strictly decreasing for $t^*_0(y) < t < t^0_-(y)$, constant for $t^0_-(y) < t < t^0_+(y)$ and strictly increasing for $t^0_+(y) < t < \infty$. Such a cost function generalizes that based on the 'nicely convex–concave' production function of Ginsberg (1974); it need not to display a kink and does not invalidate Theorems 2 and 3 below (provided of course that u^0 and l^0 are redefined in the obvious way).

[3] Here a region is the closed rectangle between the origin and some other point, possibly infinity. Theorem 1 holds in the strict sense even if the concavity is *not* strict, provided there is no degeneracy in the sense used in linear programming.

[4]

$$u^0 = \min_{hy^1 = 1} \max_{|y| = 1} ht^0(y)y \quad \text{and} \quad l^0 = \max_{hy^1 = 1} \min_{|y| = 1} ht^0(y)y.$$

3. Guaranteeing existence

In the situation of fig. 1 the upper bound on the number of firms does not exist. However, we nevertheless know that one firm is the optimal solution. Since the industry output y^I requires production of output 1 only, the situation is essentially a single-product case and then average cost is minimum at the industry output y^I. In fact, the Baumol–Fischer bounds would exist if they were defined in terms of output 1 only and then they would yield the optimum number of firms – unity. The immediate generalization of the latter observation constitutes the second revision. I define the lower and upper binding hyperplanes with respect to the *reduced* output space of produced goods. Within the reduced space industry output is strictly positive and Theorem 2 then shows that the bounds exist. [5]

Revision 2. Define u^0 and l^0 with respect to the space of commodities with positive entries in the industry output vector, y^I.

Theorem 3. Let revisions 1 [6] *and 2 be satisfied. Then the bounds in Theorem 2 exist.*

4. Conclusions

The analysis of Baumol and Fischer (1978) has two shortcomings. Their derivation of the bounds on the cost-minimizing number of firms hinges upon an assumption which conflicts with their intended objective, limiting information to scale effects. And the bounds need not exist. Two revisions described in this note eliminate both the shortcomings.

Appendix: Proofs

Proof of Theorem 1. Let $x^1, \ldots, x^m$ be as in the theorem. Since $m > n$,

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[5] Another consequence of the second revision is that the bounds on the number of firms are sharper.

[6] Revision 1 is inessential in the sense that only *average* cost must initially be decreasing, as is the case in Baumol and Fischer (1978).

they must be linearly dependent:

$$\sum_{i=1}^{m} c_i x^i = 0, \qquad c_i \text{ not all zero.} \tag{1}$$

Without loss of generality,

$$c_1 \le c_i \le c_m, \quad \text{all } i. \tag{2}$$

Note that $c_1 < 0 < c_m$ since otherwise $c_1 \ge 0$ or $c_m \le 0$ and by (2) all $c_i \ge 0$ or ≤ 0 which by (1) and the assumption all x^i non-zero would imply $c_i = 0$ violating (1). Consequently, $c_1^{-1} < 0 < c_m^{-1}$. Let $c_1^{-1} \le \lambda \le c_m^{-1}$. Then $\lambda c_1 \le 1$ and $\lambda c_m \le 1$. But by (2), λc_i lies between λc_1 and λc_m. It follows that $\lambda c_i \le 1$. Consequently, for all i, $0 \le (1 - \lambda c_i)x^i \le \sum_{i=1}^{m}(1 - \lambda c_i)x^i = \sum_{i=1}^{m} x^i \le c$ using (1) and the assumption on $x^1, \ldots, x^m$. Consequently, $C[(1 - \lambda c_i)x^i]$ are well defined, and, by assumption on C, (strictly) concave in λ. Consequently, their sum is (respectively strictly) concave in λ. Consequently, the sum is (respectively strictly) minimal for a corner value of λ, i.e., $\lambda = c_1^{-1}$ (case 1) or $\lambda = c_m^{-1}$ (case 2). The minimum is (respectively strictly) less than the value for $\lambda = 0$:

Case 1. $\sum_{i=1}^{m} C[(1 - c_1^{-1}c_i)x^i]$ is (respectively strictly) less than

$$\sum_{i=1}^{m} C\big[(1 - 0c_i)x^i\big] = C(x^1) + \ldots + C(x^m).$$

In this case, let

$$y^1 = \big(1 - c_1^{-1}c_2\big)x^2, \ldots, y^{m-1} = \big(1 - c_1^{-1}c_m\big)x^m. \quad \text{Then}$$

$$y^1 + \ldots + y^{m-1} = \sum_{i=2}^{m} \big(1 - c_1^{-1}c_i\big)x^i = \sum_{i=1}^{m} \big(1 - c_1^{-1}c_i\big)x^i$$

$$= \sum_{i=1}^{m} x^i - c_1^{-1} \sum_{i=1}^{m} c_i x^i = x^1 + \ldots + x^m \quad \text{by (1).}$$

Also, by the assumption $C(0) \ge 0$, $C(y^1) + \ldots + C(y^{m-1}) = \sum_{i=2}^{m} C[(1 - c_1^{-1}c_i)x^i] \le \sum_{i=1}^{m} C[(1 - c_1^{-1}c_i)x^i]$ which in this case is (respectively strictly) less than $C(x^1) + \ldots + C(x^m)$.

Case 2. $\sum_{i=1}^{m} C[(1 - c_m^{-1}c_i)x^i]$ is (respectively strictly) less than $\sum_{i=1}^{m} C[(1 - 0c_i)x^i] = C(x^1) + \ldots + C(x^m)$. In this case, let $y^1 = (1 - c_m^{-1}c_1)x^1, \ldots,$ $y^{m-1} = (1 - c_m^{-1}c_{m-1})x^{m-1}$. Then $y^1 + \ldots + y^{m-1} = x^1 + \ldots + x^m$ and $C(y^1) + \ldots + C(y^{m-1})$ is (respectively strictly) less than $C(x^1) + \ldots + C(x^m)$ just as in Case 1. If at most ny^i's are non-zero, then, by the assumption $C(0) \geq 0$, they are as desired. Otherwise the y^i's are linearly dependent and we can again reduce the number of vectors by one. This can be repeated until there are at most ny^i's, as was desired. Q.E.D.

References

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