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SOME INTERREGIONAL INPUT-OUTPUT ANALYSIS

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This paper analyses the structure of interregional input-output models as exemplified by the United Nations' world model of Leontief et al. (1977). The UN model has drawn a great deal of attention., Or, more precisely, conclusions arrived at by means of the model have elicited much interest. Businessmen employ the long-term, world-wide forecasts of major trends. Representatives of the Third World movement make use of the political recommendations. All this attention has not been matched by the interest arised in the economic community. Perhaps this is due to the relative inaccessibility of the UN model to the theorist. One is easily overwhelmed by the fabric of verbal considerata and numerous equations. The purpose of the present paper is to provide a comprehensive account which facilitates discussion and analysis of the interregional input-output model.

We shall try to demonstrate that the basic structure of the world model is simply that of the most elementary Keynesian income equation. The attempt is organised in inductive fashion. First we present the Keynesian income equation, taking into account the regional element by interpreting the variables as spatial distributions. Second, properties of this model are summarised. Third, the necessary but straightforward extension to the vector case is made. Fourth, the UN world model is presented and shown to have the structure of the Keynesian income equation. The properties of the latter carry over. In transforming final demand to regional production levels, the structure transmits spatial integrability and/or smoothness.

This paper is part of a larger project on the application of distribution theory to economics for which I refer to my (1982) article.

Keynes Spatialised

To fix ideas, let us take the simplest model, a baby version of Keynes' model, and then see how it is spatialised. We consider

$$y = cy + x \tag{1}$$

where y is national income which is divided into consumption cy and other expenditures x including investment, government outlays and net exports. Coefficient c is the propensity to consume.

Although it eyes childish, the model and its spatialisation below contain all the essential ingredients of the interregional input-output model.

The solution of the model is

$$y = (1 - c)^{-1}x \tag{2}$$

or

$$y = (1 + c + c^2 + c^3 + \ldots)x. \tag{3}$$

This shows how a raise in exogenous expenditures x have a multiplier effect on national income y. The total effect equals the exogenous expenditure itself, x, plus the direct effect, cx, plus the indirect effects, $c^2x + c^3x + \ldots$. The total effect converges because of the economic fact of life that the propensity to consume lies between zero and one. Of every dollar earned, less than a dollar is spent on consumption. Formally,

$$0 < c < 1. \tag{4}$$

Now let us spatialise. Exogenous expenditures and national income are reinterpreted as spatial distributions x and y. (A formal definition will be given in the next section). In principle, c may remain a scalar. But that would represent the very special case in which people spend their income only at the locations where they earn it. In general, the consumption part of one dollar earned at some location, say the origin, will be allocated according to some spatial pattern. This pattern is the spatial distribution of the propensity to consume. Maintaining notation, this distribution is denoted c. For simplicity we assume that this consumption pattern is the same for all people irrespective of their locations of income. Note that the special case of exclusive local consumption is recaptured by the distribution c which is concentrated in the origin.

How is the model affected ? Let us begin with condition (4). Of course, still less than one dollar is spent on consumption out of every dollar earned. Now how much is spent on consumption (per dollar earned) ? The distribution is c. The total amount is obtained by summing over space : $\int c$. Thus we get

$$0 \leqslant \int c < 1. \tag{4'}$$

Let us consider the equation, (1), next. Consider income at point r : y(r) (heuristically). It is build up of two parts : consumption expenditures at r and other expenditures at r, x(r). How much are the consumption expenditures at r ? Of each dollar earned at some point s, a fraction c(r–s) is spent at our point r. Hence point s contributes c(r–s)y(s). Total consumption expenditures at point r are obtained by summation over all points s : $\int c(r-s)y(s)ds$. This expression is known as the convolution of c and y (at r) and denoted $(c*y)(r)$ (heuristically). In sum (properly dropping ther arguments r) the income equation becomes

$$y = c*y + x. \tag{1'}$$

This completes the spatialisation. Recapitulating, we reinterpreted all scalars as distributions and we replaced the direct product by the convolution.

Keep these two paradigms in mind ! Throughout the article and right now ! Also know that the Dirac distribution or point mass at the origin δ , is the unit distribution, for, heuristically,

$$(x*\delta)(r) = \int x(r-s)\ \delta(s)ds = x(r-0) = x(r).$$

Then the standard solutions (2) and (3) suggest the solution of the spatial model, namely

$$y = (\delta - c)^{*-1} * x \tag{2'}$$

or

$$y = (\delta + c + c^{*2} + c^{*3} + \ldots) * x. \tag{3'}$$

The * symbols are inserted to indicate that products and hence powers are in the sense of convolutions. (2') and (3')will be justified in the next section. Now let us discuss the economics of the solution. We see that a distributed raise of exogenous expenditures has a multiplier effect on national income very much like before. The total effect now equals the exogenous distributed expenditure itself, x, plus the direct effects, the convolution $c*x$, plus the indirect effects, the further convolutions $c^{*2}*x + c^{*3}*x + \ldots$. In other words, an exogenous distributed expenditure spreads through the spatial economy in convolution multiplier fashion. The total effect converges in the sense of distributions precisely because of the spatial economic condition (4'). This will be proved in the next section.

Thus the theory of distributions enables us to handle the seemingly complex problem of spatialisation without loss of operational results by reframing the economic variables and their algebraic relationships in a distribution space.

Analytical Results

Those who merely want to grasp the main line of this article should skip this technical section. The purpose of this section is to post the solution of equation (1') as well as its analytical properties.

c, x and y are scalar distributions over space in the sense of Laurent Schwartz, i.e. continuous linear functionals from the test space of infinitely differentiable functions on geographical space with compact supports to the reals. c is assumed to be nonnegative. By Schwartz (1957), c is a Radon measure and can thus be extended to the larger test space of continuous and bounded functions. (Infinite values of the measure will be ruled out by an economic assumption on c which is seen to imply boundedness.) The enlarged test space contains the constants. $\langle c, 1 \rangle$ is denoted by $\int c$ $\int$ properly generalises the Lebesgue-Stieltjes integral. c is assumed to fulfil the economic condition $\int c < 1$.

Invoking the Dirac distribution, δ , the equation, (1'), reads

$$(\delta - c) * y = x. \tag{1''}$$

Consider the left hand side operator. By convoluting through one sees that if $\delta + c + c^{*2} + c^{*3} + \ldots$ exists, then it is the inverse of $\delta - c$. Thus, if $\Sigma_0^\infty c^{*k}$ exists, then, convoluting through (1''), one obtains the solution for y, namely (3'). In fact, $\Sigma_0^\infty c^{*k}$ converges and is continuous on the enlarged test space equipped with the sup-norm. (This holds a fortiori on the standard test space of Laurent Schwartz.)

Proposition 1. c is nonnegative and fulfils the economic condition $\int c < 1$. Then $\Sigma_0^\infty c^{*k}$ exists and is continuous (on the enlarged test space).

Proof. See ten Raa (1982) .

The inverse operator is very convenient. The unknown y is as regular as the known vector distribution x in terms of integrability and differentiability.

Proposition 2. c is as in proposition 1. Then $\Sigma_0^\infty c^{*k}$ preserves the combination of nonnegativity and p-integrability, $1 \leqslant p \leqslant \infty$ (i.e., if $x \geqslant 0$ and $\|x\|_p \leqslant \infty$, then the same holds for y.).

Proof. See ten Raa (1982).

Proposition 3. c is as inproposition 1. Then $\Sigma_0^\infty c^{*k}$ preserves p times bouded and continous differentiability.

Proof. See ten Raa (1982).

Extension to the Vector Case

We must mention the extension of the analysis to vector-valued distributions. As before, followers of just the main line of the article should skip this material. Notwithstanding, the extension is straightforward.

The roles of coefficient c, known x and unknown y, are now assumed by nxn-dimensional nonnegative coefficients matrix distribution A and known and unknown n-dimensional vector distributions f and q, respectievely. The equation to be analysed, so far (1'), becomes

$$q = A*q + f. \tag{5}$$

The convolution product is defined by $(A*q)_i = \Sigma_{j=1}^n a_{ij}*q_j$ where the latter * stands for familiar scalar convolution.

The crux is the generalisation of the economic condition $\int c < 1$. $\int A$ is now rapidly defined by $(\int A)_{ij} = \int a_{ij}$. Note that $\int A$ is an ordinary non-matrix. What about the bound ? $\int A$ is assumed to fulfil the Hawkins-Simon conditions. Indeed, these conditions properly generalise the economic condition $\int c < 1$.

Precisely as in the scalar case one sees that if $\Sigma_0^\infty A*^k$ exists, then it is the inverse distribution and one obtains the solution of (5), namely

$$q = \Sigma_0^\infty A^{*k}*f. \tag{6}$$

Here the following observations are pertinent. Propositions 1, 2 and 3 - with the economic condition generalised by the Hawkins-Simon conditions - apply to A.

Interregional Input-Output

The United Nations world model of Leontief et al. (1977) is a multiregional input-output model, specified as follows. q, f, m, e, A_0 and μ denote, respectively, a supply and a final demand vector, an import and an export vector, an input-output coefficients matrix and an import coefficients vector. All these entities are functions of the regions r, r = 1, ..., 15, the number of regions in the world model. Furthermore, $\theta(r,s)$ denotes region r's export shares vector in market region s. For region r the equations are

$$q(r) = A_0[(r)\ q(r) - m(r)] + f(r) + e(r),\ m(r) = \hat{\mu}(r)q(r) \text{ and}$$
$$e(r) = \Sigma_s \hat{\theta}(r,s)m(s) \tag{7}$$

where

$$0 \leqslant \mu(r) \leqslant i = (1 \ldots 1),\ \Sigma_r \mu(r) < 15i,\ \theta(r,s) \geqslant 0 \text{ and}$$
$$\Sigma_r \theta(r,s) = i. \tag{7a}$$

Here we subscribe to the "analytically satisfactory" model as opposed to the "computationally convenient" one of Leontief et al. (1977, p. 22). (In fact, the "analytically satisfactory" model turns out to be more tractable !) The strict inequality excludes from consideration the banal possibility that some good is completely imported everywhere, i.e. produced nowhere. (In fact, for indecomposable technologies $A_0(r)$ this would be the case of all supply and final demand equal to zero). It should be noted that Leontief et al. (1977) assume that export shares are the same for all markets s : $\theta(r,s) = \theta(r)$. But we shall maintain the refined picture of trade, $\theta(r,s)$. (In Paelinck's terminology, we use full information input-output, whereas Leontief et al. work with limited information input-output).

To highlight the basic structure of the model, consider the case in which technology and import structure are uniform and export patterns are also basically the same in that only the relative location of a market s matters : $A_0(r) = A_0, \mu(r) = \mu$ and $\theta(r,s) = \theta(\| r-s \|)$. Here $\|r-s\|$ is a symbol for the distance between regions r and s, e.g. in the contiguity sense of Paelinck (1982, p. 44). (It should be mentioned that Paelinck's contiguity distance is only a pseudo-distance between points, for distinct points may be on zero distance from each other, namely when they belong to a common region. This complication may be overcome by taking a Minkowski distance instead. In fact, when the regional classification becomes finer and finer, the contiguity pseudo-distance tends to the sup distance. However, I would prefer a more realistic Minkowski distance. For these metric matters I refer to my (1983) paper). By substitution and simplification, (7) and (7a) reduce to

$$q(r) = A_0 \widehat{i-\mu} q(r) + \Sigma_s \hat{\theta}(\|r-s\|)\hat{\mu} q(s) + f(r) \tag{8}$$

where

$$0 \leqslant \mu < i,\ \theta(\|r-s\|) \geqslant 0 \text{ and } \Sigma_r \theta(\|r-s\|) = i. \tag{8a}$$

A region r is an element of space. Space can remain an index set as in the world model or can now be structured, e.g. into the Euclidian plane. Both interpretations are consistent with the subsequent argument.

We redefine export shares as a nonnegative vector distribution θ across space. Of one unit impulse of imports at the origin, region r supplies, heuristically, $\theta(r)$. Since summing over r we must recapture the unit of imports, we assume $\int \theta = i$. Then the middle term on the right hand side of (8) becomes, heuristically, $\hat{\theta} * \hat{\mu}\, q(r)$. Dropping the arguments r we capture the basics of the world model in a nutshell :

$$q = A^* q + f \tag{5}$$

with

$$A = A_0 \widehat{i-\mu}\, \delta + \hat{\theta}\hat{\mu},\ 0 \leqslant \mu < i,\ \theta \geqslant 0, \text{ and } \int \theta = i. \tag{5a}$$

Note that $A \geqslant$ and $\int A = A_0 \widehat{i-\mu} + \hat{i}\hat{\mu}$. As always, A_0 is assumed to fulfil the Hawkins-Simon conditions. It follows that $\int A$ is a convex combination of a matrix (A_0) with spectral radius less than one and a matrix ($\hat{1}$) with spectral radius one where the weight of the latter (μ) is strictly less than $\hat{1}$. Intuitively, then, the spectral radius of $\int A$ itself must also be less than one. This fact has been established in Ten Raa (1982). The result means that $\int A$ fulfils the Hawkins-Simon conditions.

We thus have seen that the Hawkins-Simon conditions on a local scale (A_0) carry over to the global operator (A) ;this enables us to apply the existence and regularity propositions as extended for the vector case. By proposition 1, the solution q of (5) with (5a) is given by (6) and q is as regular as f in the sense of propositions 2 and 3.

Conclusion

The structure of interregional input-output models, exemplified by the United Nations' world model, is that of the Keynesian income equation, properly spatialised. Spatial integrability and/or smoothness are transmitted from the exogenous variables (final demand) to the endogenous ones (regional production levels). By the linearity of the model, the dependence of the exogenous on the endogenous variables is equally pleasant in terms of integral norms and smoothness. In short, interregional input-output is economically transparent and mathematically well behaved.

Références

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Abstract

This paper analyses the structure of interregional input-output models, exemplified by the United Nations' world model. The structure is shown to be that of the Keynesian income equation, properly spatialised. Pleasant properties of the latter model hold for the interregional input-output model, such as the transmission of spatial integrability and/or smoothness from the exogenous to the endogenous variables.