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Closedness of Production Sets

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Closedness of production sets is a primitive assumption of economic theory. It is thought to be purely technical, independent of other assumptions that admit an economic interpretation, such as those pertaining to scale effects. The present note, however, demonstrates that at least for some economies closedness of the full production set can be decomposed into the bare assumption that merely underlies the existence of costs, and two economic assumptions already imposed on such economies, namely free output disposal and non-decreasing returns to scale. In particular, in models of public enterprise pricing the closedness is a bonus that facilitates the imposition of market clearance at no cost.

The subject of this note is a *production set*,

$$Y = \{(x, z) \mid x \in R_+^n \text{ can be produced from } z \in R_+^m\}.$$

Note that we have decomposed activities in their outputs, x, and inputs, z. This technique does not infringe the generality of the description of production in economies with finitely many commodities, as commodities may enter both the input list, $1, \ldots, m$, and the output list, $1, \ldots, n$.

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The notation is Sharkey's (1979), who, however, confines himself to the case of a single, nonproduced input, $m=1$. The decomposition admits an illuminating interpretation of the production set, Y, namely as the graph of the production correspondence: $z \mapsto x$. Dual to this production correspondence is a cost function that exists provided that the input possibilities of any output form a closed set, which is sometimes called the closed upper contour set:

$$\{z \mid (x, z) \in Y\} \text{ are closed.} \tag{a0}$$

This, along with convexity and monotonicity, is a basic assumption of duality theory, namely the one underlying the very *existence of costs* (Shephard, 1970, P.7, or Russell, 1985, L.4). In consumption theory one has even more. Then the role of the production correspondence is assumed by a utility function and (a0) stands for closed iso-utility curves or, at a deeper level, continuous preferences. By a theorem of Debreu (1954), however, the utility function can be picked to be continuous. Then its graph is closed, and therefore,

$$Y \text{ is closed.} \tag{a1}$$

Indeed, (a1) is stronger than (a0). But the reasoning leading from basic assumption (a0) towards (a1) hinges on the possibility to pick utility functions from a family that contains monotone transformations of members. This argument, unfortunately, fails in the context of production theory. Therefore, property (a1) is not subsumed by the basic assumption, (a0). Fig. 1 depicts a production set satisfying (a0), but not (a1).

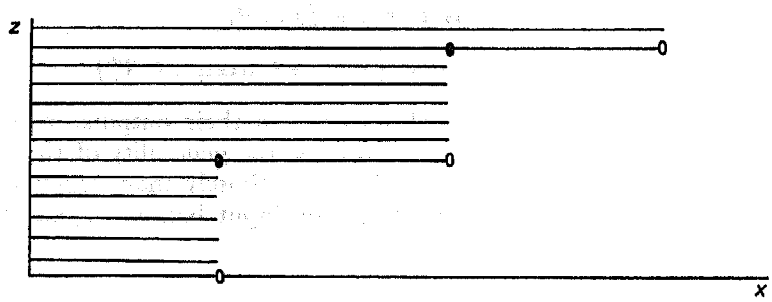

Fig. 1
$Y=\{(z, x) \mid z \geq [x]$, the largest integer less than or equal to $x\}$

Yet (a1) is desirable in production theory and is usually postulated along with nontechnical, economic assumptions. The purpose of the present note is to clarify when one can actually derive (a1) from the more basic assumption, (a0). This turns out to be the case under two clear-cut economic conditions: *free output disposal* and *non-decreasing returns to scale*. Since these conditions are sometimes among the other assumptions made, it follows that the (a0) strengthening of the basic assumption then is redundant. Moreover, in the theory of subsidy-free pricing where these conditions must be fulfilled, the automatic closedness of the production set facilitates the imposition of market clearance at no cost in terms of assumptions (ten Raa, 1983).

Free output disposal and non-decreasing returns, respectively, are formally given by:

$$(x, z) \in Y \text{ implies } (x', z) \in Y \text{ for all } x' \leq x \qquad \text{(a3)}$$

$$\lambda Y \subset Y \text{ for } \lambda \geq 1. \qquad \text{(a6)}$$

The notation and numbering are copied from Sharkey (1979). Now our finding is as follows.

Proposition 1. If a production set admits the existence of costs (a0), has free output disposal (a3), and features non-decreasing returns to scale (a6), then it is closed (a1).

Proof. Let $Y \ni (x^k, z^k) \to (x, z)$. Define λ_k as the maximum of the x_i/x_i^k in sofar they exist and unity. Then λ_k exists, even when some or all x_i are zero (in the latter case x_i/x_i^k may not exist and λ_k becomes unity), and $\lambda_k \downarrow 1$.

For large k, $\lambda_k x_i^k \geq x_i$, $i = 1, \ldots, n$. This follows from λ_k's maximizing property for $x_i > 0$ and is otherwise obvious. In other words, for large k, $\lambda_k x^k \geq x$, where the inequality is defined to hold component by component. By non-decreasing returns to scale (a6), Y contains $(\lambda_k x^k, \lambda_k z^k)$. Since this exceeds $(x, \lambda_k z^k)$ for large k, the latter must also be in Y, using free output disposal (a3). Letting large k go to infinity, Y-member $(x, \lambda_k z^k)$ tends to (x, z).

By closedness of x's input requirement set (a0), this set contains limit point z. Therefore $(x, z) \in Y$.

Q. E. D.

Remarks. 1. Figure 1, 2 and 3, provide counterexamples when precisely one of the conditions, (a6), (a0), or (a3), respectively, is not satisfied.

2. Sharkey's (1979) assumption (a1), can be reduced to the basic assumption, (a0), since he assumes (a3) and (a6).

3. Shephard (1970, p. 90) reduces closedness of the production set to nine properties that include the closedness of input sets, the continuity of input sets with respect to output levels, and free output disposal. The continuity property (Shephard, 1970, p. 5 where $u \leq u_0$ is a misprint as it should be $u < u_0$) enforces the possibility to pick a continuous production function by mathematical assumption.

4. ten Raa (1983, Lemma 1), provides a dual result for cost functions: they are shown to be continuous from below on the assumptions of non-decreasingness and subhomogeneity. It is hereby pushed back to the level of production sets.

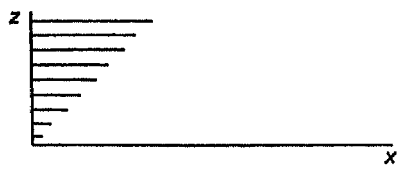

Fig. 2
$Y = \{(x, z) | 0 \leq x < z\}$

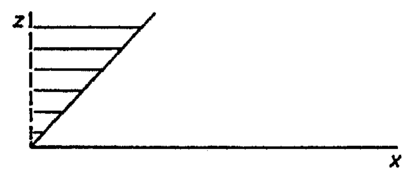

Fig. 3
$Y = \{(x, z) | 0 < x \leq z\}$

Proposition 1 reduces closedness of the production set to the existence of costs, provided free output disposal and non-decreasing returns. Since free output disposal is an assumption that is relatively weak and well localized in economic theory, the drawback of Proposition 1 is its limitation to the case of non-decreasing returns to scale (a6). The latter assumption rules out, for example, strictly convex production sets as encountered in neoclassical economics. For this purpose, it remains to complete the analysis by considering the case of non-increasing returns to scale. Much, though not all, can be salvaged by the following mind construct. Given the production set of the preceding analysis, define the perverse production set that "produces" inputs from outputs. Existence of costs, free output disposal, and non-decreasing returns to scale, turn *existence of revenues* (b0), *free input disposal* (b3), and *non-increasing returns to scale* (b6), which are defined formally by:

$$\{x \mid (x, z) \in Y\} \text{ are closed} \tag{b0}$$

$$(x, z) \in Y \text{ implies } (x, z') \in Y \text{ for all } z' \geq z. \tag{b3}$$

$$\lambda Y \subset Y \text{ for } \lambda \leq 1. \tag{b6}$$

The analogous result for the production sets considered here is as follows.

Proposition 2. If a production set admits the existence of revenues (b0), has free input disposal (b3), and features non-increasing returns to scale (b6), then it is closed relative to the space with strictly positive inputs.

Proof. Let $Y \in (x^k, z^k) \to (x, z)$. Define μ_k as the minimum of the z_i/z_i^k in sofar they exist and unity. Then μ_k exists and $\mu_k \uparrow 1$, *provided* that $z_i > 0$ for all i, that is $z \gg 0$.

Always, $\mu_k z_i^k \leq z_i$, $i = 1, \ldots, m$. This follows from μ_k's minimizing property for $z_i^k > 0$ and is otherwise obvious. In other words, always $\mu_k z^k \leq z$. By non-increasing returns to scale (b6), Y contains $(\mu_k x^k, \mu_k z^k)$. Since this is exceeded by $(\mu_k x^k, z)$, the latter must also be in Y, using free input disposed (b3). Letting k go to infinity, Y-member $(\mu_k x^k, z)$ tends to (x, z) provided that $z \gg 0$. By closedness of z's output possibility set (b0), this set contains limit point $z \gg 0$. Therefore $(x, z) \in Y$ if $z \gg 0$.

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Remarks. 1. Figure 4 displays an example. Note that the production set is not closed at zero input.

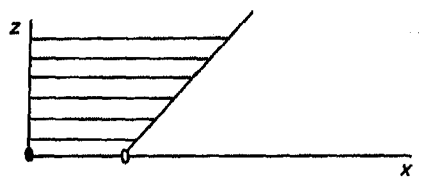

Fig. 4
$Y = \{(x, z) \mid 0 \leq x \leq z + 1, z > 0\} \cup \{(0, 0)\}$

2. A non-decreasing returns production set with an analogous boundary defect is given in Fig. 5.

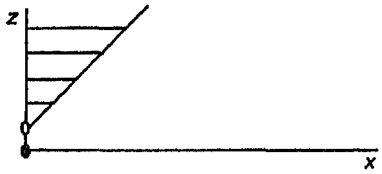

Fig. 5
$Y = \{(x, z) | 0 \leq x \leq z - 1, z > 1\} \cup \{(0, 0)\}$

Proposition 1 rules this out by the assumption of closed input requirement sets, (a 0). Unfortunately, the assumption of closed output possibility sets, (b 0), does not do the same job in Proposition 2, as Fig. 4 clearly illustrates.

3. The assumption underlying the existence of revenues, (b 0), has been suggested by the referee. It corresponds to Shephard (1970, A. 5, p. 185).

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