

## Tilburg University

### Closedness of production sets

Ten Raa, M.H.

*Published in:*  
Zeitschrift für Nationalökonomie

*Publication date:*  
1985

[Link to publication in Tilburg University Research Portal](#)

*Citation for published version (APA):*  
Ten Raa, M. H. (1985). Closedness of production sets. *Zeitschrift für Nationalökonomie*, 45(2), 155-160.

#### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

#### Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

## Miszellen — Miscellanea

### Closedness of Production Sets

By

Thijs ten Raa, Rotterdam, The Netherlands\*

(Received November 29, 1984; revised version received February 7, 1985)

Closedness of production sets is a primitive assumption of economic theory. It is thought to be purely technical, independent of other assumptions that admit an economic interpretation, such as those pertaining to scale effects. The present note, however, demonstrates that at least for some economies closedness of the full production set can be decomposed into the bare assumption that merely underlies the existence of costs, and two economic assumptions already imposed on such economies, namely free output disposal and non-decreasing returns to scale. In particular, in models of public enterprise pricing the closedness is a bonus that facilitates the imposition of market clearance at no cost.

The subject of this note is a *production set*,

$$Y = \{(x, z) | x \in R_+^n \text{ can be produced from } z \in R_+^m\}.$$

Note that we have decomposed activities in their outputs,  $x$ , and inputs,  $z$ . This technique does not infringe the generality of the description of production in economies with finitely many commodities, as commodities may enter both the input list,  $1, \dots, m$ , and the output list,  $1, \dots, n$ .

---

\* I would like to thank R. Robert Russell for incisive comments on the first draft and the referee for a suggestion to complete the note in a very natural way. Sloan Foundation support through New York University is gratefully acknowledged. The Netherlands Organization for the advancement of Pure Research (Z. W. O.) and the Universiteitsfonds Rotterdam kindly provided travel funding.

The notation is Sharkey's (1979), who, however, confines himself to the case of a single, nonproduced input,  $m=1$ . The decomposition admits an illuminating interpretation of the production set,  $Y$ , namely as the graph of the production correspondence:  $z \mapsto x$ . Dual to this production correspondence is a cost function that exists provided that the input possibilities of any output form a closed set, which is sometimes called the closed upper contour set:

$$\{z \mid (x, z) \in Y\} \text{ are closed.} \quad (\text{a0})$$

This, along with convexity and monotonicity, is a basic assumption of duality theory, namely the one underlying the very *existence of costs* (Shephard, 1970, P. 7, or Russell, 1985, L. 4). In consumption theory one has even more. Then the role of the production correspondence is assumed by a utility function and (a0) stands for closed iso-utility curves or, at a deeper level, continuous preferences. By a theorem of Debreu (1954), however, the utility function can be picked to be continuous. Then its graph is closed, and therefore,

$$Y \text{ is closed.} \quad (\text{a1})$$

Indeed, (a1) is stronger than (a0). But the reasoning leading from basic assumption (a0) towards (a1) hinges on the possibility to pick utility functions from a family that contains monotone transformations of members. This argument, unfortunately, fails in the context of production theory. Therefore, property (a1) is not subsumed by the basic assumption, (a0). Fig. 1 depicts a production set satisfying (a0), but not (a1).

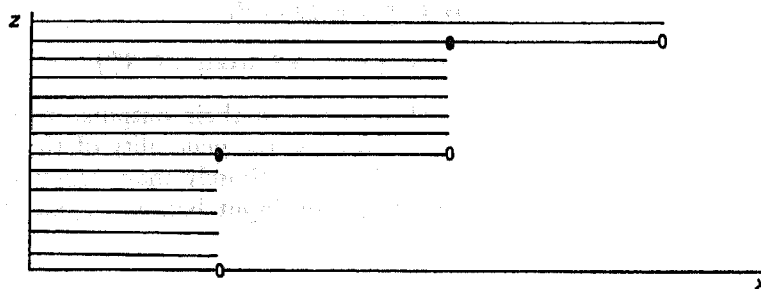


Fig. 1

$$Y = \{(z, x) \mid z \geq [x], \text{ the largest integer less than or equal to } x\}$$

Yet (a1) is desirable in production theory and is usually postulated along with nontechnical, economic assumptions. The purpose of

the present note is to clarify when one can actually derive (a1) from the more basic assumption, (a0). This turns out to be the case under two clear-cut economic conditions: *free output disposal* and *non-decreasing returns to scale*. Since these conditions are sometimes among the other assumptions made, it follows that the (a0) strengthening of the basic assumption then is redundant. Moreover, in the theory of subsidy-free pricing where these conditions must be fulfilled, the automatic closedness of the production set facilitates the imposition of market clearance at no cost in terms of assumptions (ten Raa, 1983).

Free output disposal and non-decreasing returns, respectively, are formally given by:

$$(x, z) \in Y \text{ implies } (x', z) \in Y \text{ for all } x' \leq x \quad (\text{a3})$$

$$\lambda Y \subset Y \text{ for } \lambda \geq 1. \quad (\text{a6})$$

The notation and numbering are copied from Sharkey (1979). Now our finding is as follows.

*Proposition 1.* If a production set admits the existence of costs (a0), has free output disposal (a3), and features non-decreasing returns to scale (a6), then it is closed (a1).

*Proof.* Let  $Y \ni (x^k, z^k) \rightarrow (x, z)$ . Define  $\lambda_k$  as the maximum of the  $x_i/x_i^k$  insofar they exist and unity. Then  $\lambda_k$  exists, even when some or all  $x_i$  are zero (in the latter case  $x_i/x_i^k$  may not exist and  $\lambda_k$  becomes unity), and  $\lambda_k \downarrow 1$ .

For large  $k$ ,  $\lambda_k x_i^k \geq x_i$ ,  $i=1, \dots, n$ . This follows from  $\lambda_k$ 's maximizing property for  $x_i > 0$  and is otherwise obvious. In other words, for large  $k$ ,  $\lambda_k x^k \geq x$ , where the inequality is defined to hold component by component. By non-decreasing returns to scale (a6),  $Y$  contains  $(\lambda_k x^k, \lambda_k z^k)$ . Since this exceeds  $(x, \lambda_k z^k)$  for large  $k$ , the latter must also be in  $Y$ , using free output disposal (a3). Letting large  $k$  go to infinity,  $Y$ -member  $(x, \lambda_k z^k)$  tends to  $(x, z)$ .

By closedness of  $x$ 's input requirement set (a0), this set contains limit point  $z$ . Therefore  $(x, z) \in Y$ .

Q. E. D.

*Remarks.* 1. Figure 1, 2 and 3, provide counterexamples when precisely one of the conditions, (a6), (a0), or (a3), respectively, is not satisfied.

2. Sharkey's (1979) assumption (a1), can be reduced to the basic assumption, (a0), since he assumes (a3) and (a6).

3. Shephard (1970, p. 90) reduces closedness of the production set to nine properties that include the closedness of input sets, the continuity of input sets with respect to output levels, and free output disposal. The continuity property (Shephard, 1970, p. 5 where  $u \leq u_0$  is a misprint as it should be  $u < u_0$ ) enforces the possibility to pick a continuous production function by mathematical assumption.

4. ten Raa (1983, Lemma 1), provides a dual result for cost functions: they are shown to be continuous from below on the assumptions of non-decreasingness and subhomogeneity. It is hereby pushed back to the level of production sets.

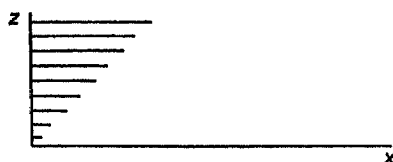


Fig. 2  
 $Y = \{(x, z) | 0 \leq x < z\}$

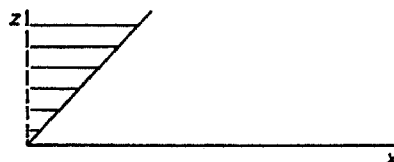


Fig. 3  
 $Y = \{(x, z) | 0 < x \leq z\}$

Proposition 1 reduces closedness of the production set to the existence of costs, provided free output disposal and non-decreasing returns. Since free output disposal is an assumption that is relatively weak and well localized in economic theory, the drawback of Proposition 1 is its limitation to the case of non-decreasing returns to scale (a6). The latter assumption rules out, for example, strictly convex production sets as encountered in neoclassical economics. For this purpose, it remains to complete the analysis by considering the case of non-increasing returns to scale. Much, though not all, can be salvaged by the following mind construct. Given

the production set of the preceding analysis, define the perverse production set that “produces” inputs from outputs. Existence of costs, free output disposal, and non-decreasing returns to scale, turn *existence of revenues* (b0), *free input disposal* (b3), and *non-increasing returns to scale* (b6), which are defined formally by:

$$\{x | (x, z) \in Y\} \text{ are closed} \tag{b0}$$

$$(x, z) \in Y \text{ implies } (x, z') \in Y \text{ for all } z' \geq z. \tag{b3}$$

$$\lambda Y \subset Y \text{ for } \lambda \leq 1. \tag{b6}$$

The analogous result for the production sets considered here is as follows.

*Proposition 2.* If a production set admits the existence of revenues (b0), has free input disposal (b3), and features non-increasing returns to scale (b6), then it is closed relative to the space with strictly positive inputs.

*Proof.* Let  $Y \in (x^k, z^k) \rightarrow (x, z)$ . Define  $\mu_k$  as the minimum of the  $z_i/z_i^k$  insofar they exist and unity. Then  $\mu_k$  exists and  $\mu_k \uparrow 1$ , provided that  $z_i > 0$  for all  $i$ , that is  $z \gg 0$ .

Always,  $\mu_k z_i^k \leq z_i, i=1, \dots, m$ . This follows from  $\mu_k$ 's minimizing property for  $z_i^k > 0$  and is otherwise obvious. In other words, always  $\mu_k z^k \leq z$ . By non-increasing returns to scale (b6),  $Y$  contains  $(\mu_k x^k, \mu_k z^k)$ . Since this is exceeded by  $(\mu_k x^k, z)$ , the latter must also be in  $Y$ , using free input disposal (b3). Letting  $k$  go to infinity,  $Y$ -member  $(\mu_k x^k, z)$  tends to  $(x, z)$  provided that  $z \gg 0$ . By closedness of  $z$ 's output possibility set (b0), this set contains limit point  $z \gg 0$ . Therefore  $(x, z) \in Y$  if  $z \gg 0$ .

Q. E. D.

*Remarks.* 1. Figure 4 displays an example. Note that the production set is not closed at zero input.

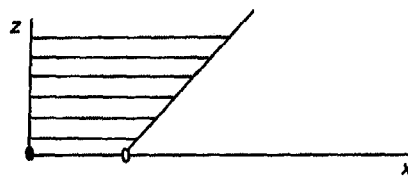


Fig. 4

$$Y = \{(x, z) | 0 \leq x \leq z + 1, z > 0\} \cup \{(0, 0)\}$$

2. A non-decreasing returns production set with an analogous boundary defect is given in Fig. 5.

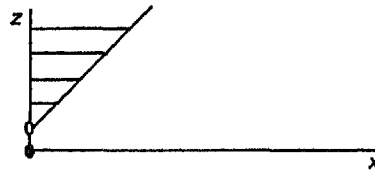


Fig. 5

$$Y = \{(x, z) | 0 \leq x \leq z - 1, z > 1\} \cup \{(0, 0)\}$$

Proposition 1 rules this out by the assumption of closed input requirement sets, (a0). Unfortunately, the assumption of closed output possibility sets, (b0), does not do the same job in Proposition 2, as Fig. 4 clearly illustrates.

3. The assumption underlying the existence of revenues, (b0), has been suggested by the referee. It corresponds to Shephard (1970, A. 5, p. 185).

#### References

G. Debreu (1954): *Representation of a Preference Ordering by a Numerical Function*, in: R. M. Thrall, C. H. Coombs, R. L. Davis (eds.): *Decision Processes*, New York, pp. 159–165.

Th. ten Raa (1983): *Supportability and Anonymous Equity*, *Journal of Economic Theory* 31, pp. 176–181.

R. R. Russell (1985): *Measures of Technical Efficiency*, *Journal of Economic Theory*, forthcoming.

W. W. Sharkey (1979): *Existence of a Core When There are Increasing Returns*, *Econometrica* 47, pp. 869–876.

R. W. Shephard (1970): *Theory of Cost and Production Functions*, Princeton.

Address of author: Dr. Thijs ten Raa, Erasmus University, P. O. Box 1738, 3000 DR Rotterdam, The Netherlands.