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Constraints in perfect-foresight models: The case of old-age savings and public pensions

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Abstract

The solution of discrete-time perfect-foresight models with constrained state variables differs considerably from the solution of models without constraints. In this paper this is worked out for a simple model of decision making on public pensions.

Keywords: Perfect foresight; Constraints; Dynamics; Public pensions

JEL classification: C62; D78; D91; H55; J14

1. The unconstrained model

To shed light on the differences between unconstrained and constrained perfect-foresight models, a simplified version of the model described in Verbon and Verhoeven (1992) and Verhoeven and Verbon (1991) is used. The basic framework is a two-overlapping generations model in which all individuals are identical except for age differences. Each agent lives for two periods. As the model contains no production sector it can be interpreted as a representation of a small open economy or as a model of a pure exchange economy. An individual born at time t receives an endowment (or income) which is normalized at one. A part¹ τ_t of this endowment is taxed away by the government and transferred to the old of that period (i.e. each old individual receives a pension benefit of size $n\tau_t$, where n is the exogenously-determined number of young individuals per old individual, or, equivalently, one plus the rate of population growth). The remainder is used for old-age savings (s_t), which earn no interest, and immediate consumption (c_t^y). So

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¹ The notational rules of this paper are as follows. A subscript indicates the time period to which the variable refers. If the subscript is omitted, the variable is a constant. A superscript denotes the age of the individual concerned (y for young and o for old). Finally, we define $f'(x) \equiv \partial f(x) / \partial x$.

$$c_t^y = 1 - \tau_t - s_t. \quad (1)$$

When old, the individual born at t consumes his savings and the transfer payment from the government:

$$c_{t+1}^o = s_t + n\tau_{t+1}. \quad (2)$$

Lifetime utility of an individual born at time t is the sum of his instantaneous utilities when young and when old:

$$U_t = u(c_t^y) + u(c_{t+1}^o), \quad (3)$$

where $u: \mathbb{R}_{++} \rightarrow \mathbb{R}$ is strictly increasing in its argument, twice differentiable, strictly quasi-concave and satisfies $\lim_{c \downarrow 0} u'(c) = \infty$ and $\lim_{c \rightarrow \infty} u'(c) = 0$.

We assume decentralized and uncoordinated savings behavior. That is, the young maximize their lifetime utility using the savings rate as an instrumental variable and taking the current and next period's tax rate as given. The first-order condition for this optimization problem reads $u'(c_t^y) = u'(c_{t+1}^o)$, which reduces to

$$s_t = \frac{1}{2}(1 - \tau_t - n\tau_{t+1}). \quad (4)$$

Note that savings in period t depend on the anticipated tax rate in period $t + 1$. Expectations are assumed to be rational in the sense of Muth (1961). In the absence of uncertainty, this boils down to perfect foresight.

The government, i.e. the politicians, decide on the tax-transfer scheme. They are under the influence of both the young and the old and maximize the following interest function:²

$$D_t = \lambda u(c_t^o) + n(u(c_t^y) + u(c_{t+1}^o)), \quad (5)$$

where λ measures the effective political influence of an old relative to a young individual. To measure the total political power of the old versus the young generation, the relative numerical size of the young generation n also has to be taken into account. Maximization of the political goal function yields the first-order condition for taxes, $u'(c_t^y) = \lambda u'(c_t^o)$. Notice from this first-order condition and the condition for optimal savings, $u'(c_t^y) = u'(c_{t+1}^o)$, that the model has no stationary state, unless $\lambda = 1$. For other values of the power parameter λ the model will converge to a state in which per capita consumption of the young and old is zero (if $\lambda > 1$) or their per capita consumption is infinitely large (if $\lambda < 1$). In both instances either savings or taxes will be negative. The absence of a stationary state for most parameter values precludes the use of the standard methods to solve perfect-foresight models.

Let us now consider the dynamics of the model in more detail. In order to simplify the analysis we assume instantaneous utility to be logarithmic, ($u(c) = \ln(c)$). This implies that the first-order condition for taxes τ_t can be written as

² For a justification of this function, see Verbon (1988). If $n < \lambda$ the model outcomes coincide with the optimal policy of a government that tries to maximize a social welfare function of the form $W = \sum_{t=0}^{\infty} (n/\lambda)^t U_t$ (see the first-order condition for taxes given below).

$$\tau_t = \frac{\lambda(1 - s_t) - s_{t-1}}{n + \lambda} \tag{6}$$

Eqs. (4) and (6) can be used to generate the following dynamical system:

$$\begin{bmatrix} s_t \\ \tau_{t+1} \end{bmatrix} = M_t \cdot \begin{bmatrix} s_{t-1} \\ \tau_t \end{bmatrix} + N_t, \tag{7}$$

where

$$M_t = \begin{bmatrix} -\frac{1}{\lambda} & -\frac{n}{\lambda} - 1 \\ \frac{2}{\lambda n} & \frac{2}{\lambda} + \frac{1}{n} \end{bmatrix}, \quad N_t = \begin{bmatrix} 1 \\ -\frac{1}{n} \end{bmatrix}.$$

Note that one of the state variables (s_{t-1}) is predetermined, while the other (τ_t) is non-predetermined.

Now we can derive the *locus* on which the tax rate is constant (in the following we will refer to this as the $\Delta\tau_t = 0$ locus):

$$\tau_t = -\frac{1}{2n + \lambda(1 - n)} \cdot s_{t-1} + \frac{\lambda}{2n + \lambda(1 - n)}. \tag{8}$$

The analogously-defined $\Delta s_{t-1} = 0$ locus is given by

$$\tau_t = -\frac{1 + \lambda}{n + \lambda} \cdot s_{t-1} + \frac{\lambda}{n + \lambda}. \tag{9}$$

Figs. 1 and 2 indicate the typical dynamical development in the current model. In the first

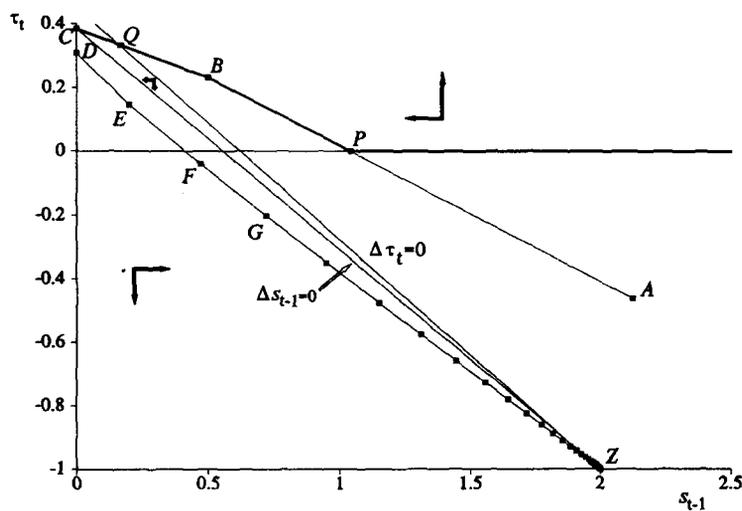


Fig. 1. The dynamics of public pensions and old-age savings for $\lambda = 1.25$ and $n = 2$.

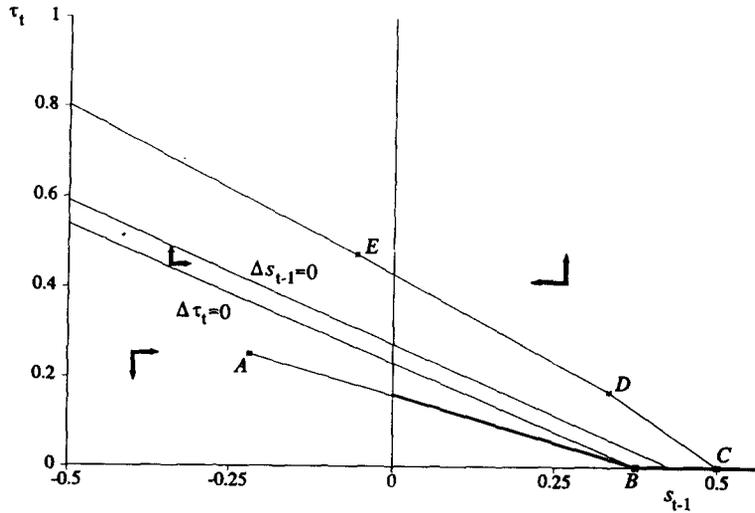


Fig. 2. The dynamics of public pensions and old-age savings for $\lambda = 0.75$ and $n = 2$.

figure the relative power of an old individual $\lambda = 1.25$ and the relative population size $n = 2$. The direction of motion is indicated by dark arrows. Suppose, for example, that at time $t = 1$ the economy starts in point A . The following period the system is in B . Then, the economy goes to C , where the previous period's savings are exactly zero, after which the economy slowly converges to the zero-consumption state Z via D, E, F , etc. Fig. 2 is based on the assumption that politicians attach a relatively large weight to the welfare of a young individual, i.e. $\lambda = 0.75$, while $n = 2$. In this case, the model converges to a state where the old and the young have an infinite amount of consumption. This implies that taxes must be infinitely large, while savings are minus infinity. Thus, starting in point A , the system explodes via B, C, D , etc.

2. The constrained model

Let us contrast the model without constraints of the previous section with a model in which constraints on the savings and tax rate have a decisive impact on the outcomes. Therefore, we introduce two constraints that are more or less standard in the literature on the subject (see, for example, Hansson and Stuart, 1989; and Verbon and Verhoeven, 1992). Firstly, we rule out negative private savings (on the aggregate) as this implies that resources are transferred from as yet unborn to current generations, which creates an enforcement problem.³ Another reason for imposing this condition is that it is physically impossible to have negative aggregate savings in a (closed) exchange economy:

$$s_t \geq 0, \text{ for all } t. \tag{10}$$

³ We abstain here from government debt.

Secondly, we rule out negative transfers by assuming that property rights solely permit gifts to the preceding generation during old age but not taking from them:

$$\tau_t \geq 0, \quad \text{for all } t. \quad (11)$$

If requirements (10) and (11) are combined with conditions (4) and (6) the first-order conditions for savings and taxes in the constrained model are obtained:

$$s_t = \max\left\{0, \frac{1}{2}(1 - \tau_t - n\tau_{t+1})\right\}, \quad (12)$$

$$\tau_t = \max\left\{0, \frac{\lambda(1 - s_t) - s_{t-1}}{n + \lambda}\right\}. \quad (13)$$

When the non-negativity constraints (10) and (11) are not binding these first-order conditions can be summarized by the dynamic system described by Eqs. (7).

2.1. The stationary state of the constrained model

Let us derive the stationary state of the constrained system by using Fig. 1. Suppose we start at time $t = 1$ in point B . Then, as in the unconstrained system, we are in C at time $t = 2$. That is $(s_1, \tau_2) = (0, \tau(C))$ with $\tau(C) = \lambda/(n + \lambda)$ (this follows from Eq. (6) with $s_t = s_{t-1} = 0$). In the unconstrained system, the economy would go to point D in the next period. However, now not only the anticipated value for τ_3 corresponding to point $D(\tau(D))$ is consistent with the first-order conditions at time $t = 2$, but also all points on the vertical axis above D ; that is, all points (s_2, τ_3) with $\tau_3 \geq \tau(D)$. To see this, note that $1 - \tau_2 - n\tau(D) = 0$, implying that the optimal savings rate $s_2 = 0$ for all $\tau_3 \geq \tau(D)$. So, at the time the path hits an axis (or, put differently, when a constraint becomes binding) a point can be mapped on a half-line. This is an important feature, as it allows for an extra degree of freedom. But which of the anticipated values for τ_3 on the half-line are consistent with perfect foresight? In order to be an expectation formed under perfect foresight, it does not suffice that the anticipated value of τ_3 is consistent with the first-order conditions in period 2. In addition, the time path of savings rates s_t and tax rates τ_t must be consistent with the first-order conditions for savings and taxes for all periods $t \geq 3$. This allows us to select $\tau(C)$ as the unique anticipated value for τ_3 by the following reasoning. All choices for τ_3 on the vertical axis above C ($\tau_3 > \tau(C), s_2 = 0$) lead to an inconsistency: given that $s_2 = 0$, substitution of the constraint $s_3 \geq 0$ in Eq. (6) implies that $\tau_3 \leq \tau(C)$, which contradicts the assumption that $\tau_3 > \tau(C)$. The possibility $\tau_3 = \tau(D)$ can also be excluded as this would imply that the economy would go from C to D, E, F etc. and consequently the non-negativity constraint on taxes would be violated after two periods. The expectation $\tau(D) < \tau_3 < \tau(C)$ leads to the same inconsistency: from a point on the line CD the system will go to a point on DE and so on, sooner or later crossing the horizontal axis. So, every other anticipation than $\tau_3 = \tau(C)$ is not consistent with perfect foresight. To check that the anticipation $\tau_3 = \tau(C)$ is consistent with perfect foresight it suffices to show that s_2 is constrained to zero. This is true as long as $\lambda > 1$: then $(1/2)(1 - \tau - n\tau) < 0$ for $\tau = \lambda/(n + \lambda)$. Obviously, $\lambda = 1.25$ satisfies this condition and $\tau_3 = \tau_2 = \tau(C)$ is the unique anticipation consistent with perfect foresight. In other words, there exists a unique stationary state that lies at the intersection of the $\Delta s_t = 0$ locus and the vertical axis. Following the same reasoning,

$(s, \tau) = (0, \lambda/(n + \lambda))$ can be shown to be the stationary state for the constrained model for all $\lambda > 1$. In a similar way it can be shown that in Fig. 2, where $\lambda < 1$, the stationary-state outcome for the constrained model lies at point C with $(s(C), \tau(C)) = (1/2, 0)$. The system is taken to this outcome via point B , i.e. the intersection of the $\Delta\tau_t = 0$ locus and the horizontal axis.⁴

2.2. The uniqueness of the convergent path

Now we have derived the stationary state of the constrained model, the question arises whether, as in the familiar unconstrained perfect-foresight models, for any value of the predetermined variable (here: s_{t-1}) a unique value for the non-predetermined variable (here: τ_t) can be found that puts the system on a trajectory to the stationary state. The answer for the current example is 'yes', as can be shown using Fig. 1. In the unconstrained system, all points on line BC (the straight line that links points B and C) are mapped on line CD . This follows from the linearity of the unconstrained system. However, as explained above, in the constrained system, C is the stationary-state outcome and does not lead to D . By the same token, in the constrained system, all points on line BC lead to C in the next period. So when initial savings are zero, the stationary state can be reached immediately by setting the tax rate equal to $\tau(C)$. For $0 < s_{t-1} \leq s(B)$ the tax rate can be chosen on line BC , bringing the system to the stationary state after one period. For $s(B) < s_{t-1} < s(P)$ a point on line PB can be chosen. This leads the system to a point on line QC in the next period. As line QC is part of line BC , this implies that the system stabilizes in C after two periods. For initial savings larger than $s(P)$ (e.g. $s_{t-1} = s(A)$) one would like to choose a negative tax rate (e.g. $\tau_t = \tau(A)$) so as to reach the stationary state in two or more periods. This is not possible, however. That is, the non-negativity constraint on the tax rate is initially binding. In the unconstrained system, the economy moves from point P to Q in the following period. As can be checked by comparing Eqs. (4), (6), (10) and (11), this implies that the economy is in point Q in the constrained system if the economy was in point P , or in any point on the horizontal axis to the right of P , in the previous period. From Q the system goes to the stationary state C . Therefore, we can conclude that for any value of the (predetermined) savings rate s_{t-1} a value of the (non-predetermined) tax rate τ_t exists that eventually leads to the stationary state. These values of the jump variable are depicted as a thick line in Fig. 1. As can be deduced from the dynamic properties of the system, no other choice for the initial τ_t can lead to the stationary state. More importantly, if the system is not put on the trajectory to the stationary state, the non-negativity constraints on the variables will be violated within finite time. This leaves rational agents no choice but to opt for this trajectory; only if the system is on this time path can the first-order conditions be satisfied in the current and all future time periods. Similarly, it can be shown that the thick line in Fig. 2 gives the values of the non-predetermined variable τ_t , given s_{t-1} , that bring the system on the unique consistent perfect-foresight path that eventually takes it to the stationary state in point C .⁵

⁴ If the path of tax and savings rates hits the horizontal axis (i.e. $\tau_t = 0$), an extra degree of freedom ensues in the decision-making process concerning s_t . This is reflected one period later in Fig. 2 as it is drawn in (s_{t-1}, τ_t) -space.

⁵ These results can rather easily be extended to the case of non-stationarity of the model's parameters (or shocks) and a general form of the utility function U_t (see Verhoeven, 1993).

2.3. Comparison with unconstrained perfect-foresight models

In contrast with the unconstrained model of Section 1 there is always a stationary state in the constrained model of this section. There are, however, more general and fundamental differences between a constrained model as presented here and standard unconstrained perfect-foresight models. For instance, it should be noted that the result that there is only one consistent choice for the initial non-predetermined variable in the constrained model does *not* depend on the assumption of saddlepoint stability as in familiar unconstrained perfect-foresight models. Furthermore, note that in the constrained model, no matter where you start, the stationary state is reached within a *limited* number of periods. In our example it takes a maximum of three periods to reach the stationary state. In a familiar unconstrained perfect-foresight model, on the other hand, this state can be approached arbitrarily close, but the system will not really enter the stationary state in finite time. Moreover, the unique value of the jump variable is solely based on rationality requirements. That is, we do not have to *assume* that the non-predetermined variable jumps so that the economy is placed on a converging time path, as in familiar perfect-foresight models exhibiting saddlepoint stability. In those models there is, in principle, an infinite number of trajectories that are divergent but nevertheless fully rational; that is, consistent with the first-order conditions. In general, these exploding trajectories can only be excluded by assumption. Sometimes it is possible to exclude the divergent time paths by imposing transversality conditions (see, for example, Blanchard and Fischer, 1989, Appendix 2A). However, transversality conditions are, in general, sufficient, but not necessary, requirements for optimal decision-making behavior (see, for example, Seierstad and Sydsæter, 1977).

In the familiar saddlepoint-stable perfect-foresight model the effect of shocks, whether anticipated or not, can be analyzed in a simple way. The same is true for a constrained model as presented here. In the case of an *unanticipated* shock the non-predetermined variable τ_t immediately jumps to the convergent path we just derived. However, contrary to an unconstrained model, in some cases the jump does *not* conflict with the first-order conditions of the predetermined variable. Suppose, for example, that the economy is in the stationary state (point *C*) in Fig. 1 when an unanticipated, once-and-for-all parameter change occurs that causes the stationary state to shift a little upward on the vertical axis. As we have seen, such a point may rationally be expected for a future time period if the economy is in point *C*. This implies that even if the agents *anticipate* the shock, they will not change their behavior before the shock. They just adjust their expectation of the tax rate that will hold at the time of the shock. Consequently, when the shock occurs the new stationary state immediately results. The reason for this is that, confronted with the higher future pension benefits, they would have liked to decrease savings. But because saving is already equal to zero in the stationary state, this is not possible.

Even more surprisingly, such a lethargic reaction can also *temporarily* occur when the shock is anticipated. This can easily be understood. Suppose that the parameters are constant and that the system is initially in the stationary state. Suppose, furthermore, that the jump-variable τ_t is changed in anticipation of a future shock on the parameter set. In the case depicted in Fig. 1, after the stationary-state outcome is left, the system will transgress the non-negativity constraint on taxes after, at most, three periods. It follows from the preceding analysis that this is a general result (unless $\lambda = 1$): starting from a stationary state, Eqs. (10) or (11) will be

violated within a finite number of periods after the tax rate is changed. Consequently, rational savers and politicians will not immediately react to an anticipated change in one of the parameters if this shock will occur in a future period that is sufficiently distant in time. So, if the system is in the stationary state and it becomes known that the parameter set will change in some future time period, the typical pattern that emerges is that for a number of periods the stationary state is sustained, after which the tax and savings rates are changed in anticipation of the shock. Assuming the shock is temporary, the system will return to the stationary state after a finite number of periods (see Figs. 1, 2 and 3 in Verbon and Verhoeven, 1992). If the shock is permanent, the economy will converge to a new stationary state. It is important to note that such a pattern could not arise in an unconstrained perfect-foresight model. In these models, rational agents can never expect that an existing stationary state is disturbed for the first time at some future period; that would contradict the first-order conditions of the predetermined variable at the time the stationary state is left. In a constrained perfect-foresight model, agents can expect the stationary state to be disturbed at some future time period since, as is demonstrated above, the deviation from the stationary state can be consistent with the first-order conditions of the predetermined variable.

3. Conclusions

The introduction of constraints in perfect-foresight models can profoundly change its long- and short-run properties. In this paper, this is shown for a simple model of decision making on old-age savings and public pensions. The underlying ideas can, however, be applied to other models by simply transplanting the methods used in Sub-sections 2.1 and 2.2. Consider, for example, the extension of the Samuelson (1958) overlapping-generations model with money, analyzed by Blanchard and Fischer (1989) in Chapter 5. They conclude that if the monetary equilibrium is unstable, all paths should be excluded on which initial savings of young individuals (or the demand for money) is larger than in the monetary equilibrium. They base this conclusion on the observation that on these paths savings will be ever-increasing without converging to some upper bound. But, as Blanchard and Fischer argue, savings can never exceed available endowments, so these explosive paths must be excluded. In other words, paths with an initial level of savings that is higher than in the monetary equilibrium are excluded by referring to some upper bound on savings. However, if this upper bound is introduced explicitly rather than implicitly in the model, it can readily be established that there emerges a new stationary state in which the savings rate equals this maximum level. In the vein of the analysis of the previous sections, it can then also be shown that this stationary state is reached within a finite number of periods as long as savings in the initial period are larger than in the monetary equilibrium.

Another example where constraints on the parameters have an important influence on the outcomes is Hansson and Stuart (1989). They develop a model of decision making on old-age savings and intergenerational transfers in which political decision making is assumed to be constitutional. Both savings and transfers are required to be non-negative. As in the model of this paper, the stationary-state outcomes are determined by these constraints. Moreover, the stationary state is reached in finite time (see, for example, Hansson and Stuart, 1989, note 11).

Contrary to this note, however, Hansson and Stuart (1989) (as well as Verbon and Verhoeven, 1992; and Verhoeven and Verbon, 1991) fail to give insight in a general and practical analytical toolbox that can be used to handle discrete-time perfect-foresight models with constrained state variables.

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