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Density of the quotient of non-negative quadratic forms in normal variables with application to the F-statistic

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The density of the quotient of two non-negative quadratic forms in normal variables is considered. The covariance matrix of these variables is arbitrary. The result is useful in the study of the robustness of the F-test with respect to errors of the first and second kind. An explicit expression for this density is given in the form of a proper Riemann-integral on a finite interval, suitable for numerical calculation.

Keywords: Ratio of quotient of quadratic forms in normal variables, F-test, F-statistic, numerical evaluation of probability densities

1. Introduction

Let $Y \sim N_n(\mu, \boldsymbol{\Omega})$ be the n-variate normal distribution with expectation μ and covariance matrix $\boldsymbol{\Omega}$. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two non-negative definite $n \times n$-matrices. Set

$$X = (Y'\boldsymbol{B}Y)/(Y'\boldsymbol{A}Y)$$
$$F = (\mathrm{tr}(\boldsymbol{A})/\mathrm{tr}(\boldsymbol{B}))X \tag{1}$$

A relatively simple expression is given for the density g of X (or equivalently for the density of F). For numerical calculation, some eigenvalues and eigenvectors must be computed once and then a one-dimensional proper Riemann-integral on a finite interval must be evaluated for each point $x \in \mathbb{R}$ to get the value $g(x)$.

A special case with singular $\boldsymbol{A}$ and $\boldsymbol{B}$ arises with quotients of orthogonal projections. Let L and R be two orthogonal linear subspaces of $\mathbb{R}^n$ of dimensions l and r, respectively, ($l \geq 1$, $r \geq 1$, $l + r \leq n$). Set

$$X = |Y_L|^2/|Y_R|^2$$
$$F = (r/l)X \tag{2}$$

with $Y_L = \boldsymbol{P}_L Y$ and $\boldsymbol{P}_L$ the orthogonal projection matrix belonging to L; Y_R and R are similarly defined. Then Equation 1 leads to Equation 2 for $\boldsymbol{A} = \boldsymbol{P}_R$, $\boldsymbol{B} = \boldsymbol{P}_L$.

This result is useful in studying the robustness of the F-test in linear models. Let $Y = \boldsymbol{Z}\beta + \varepsilon$ with $\boldsymbol{Z} \in \mathbb{R}^{n \times k}$ the (non-stochastic) matrix of explanatory variables and $\varepsilon \sim N_n(0, \boldsymbol{\Omega})$. Then $Y \sim N_n(\mu, \boldsymbol{\Omega})$ with $\mu = \boldsymbol{Z}\beta \in \mathbb{R}^n$. An (identifiable) hypothesis H_0 in terms of restrictions on β is equivalent to H_0: $\mu \in L_0$ with L_0 some linear subspace of $\mathscr{R}(\boldsymbol{Z})$. The usual F-statistic F for testing H_0: $\mu \in L_0$ against H_1: $\mu \in \mathscr{R}(\boldsymbol{Z}) - L_0$ is given by F in Equation 2, where L and R are determined by $L \perp L_0$, $L + L_0 = \mathscr{R}(\boldsymbol{Z})$ and $R \perp \mathscr{R}(\boldsymbol{Z})$, $R + \mathscr{R}(\boldsymbol{Z}) = \mathbb{R}^n$.

The usual assumption, $\boldsymbol{\Omega} = \sigma^2 I_n$, gives $F \sim F^l_r(\delta)$, the non-central F-distribution with degrees of freedom l and r and non-centrality parameter $\delta = |\mu_L|^2/\sigma^2$. Equivalently, X follows the distribution with density

$$\exp(-\tfrac{1}{2}\delta) \sum_{k=0}^{\infty} \frac{(\delta/2)^k}{k!} p(x; l/2 + k, r/2) \quad x > 0 \tag{3}$$

where $p(x; \rho_1, \rho_2)$ stands for the density of the beta-distribution of the second kind given by

$$p(x; \rho_1, \rho_2) = x^{\rho_1 - 1}(1 + x)^{-\rho_1 - \rho_2}/B(\rho_1, \rho_2) \quad x > 0 \tag{4}$$

So with an expression for the density g of X for general μ and $\boldsymbol{\Omega}$ it is possible to study the robustness of the F-test for specified probabilities for errors of the first and second kind.

The question of robustness of the F-test is a very old problem. A detailed study for heteroskedasticity and autocorrelation in some special ANOVA-designs can be found in Scheffe (1959). Readers are referred to this book for an overview.

The best references within the context of the general problem are Lugannani and Rice (1984) and Magnus (1986).

2. Statement of results

Let $(\lambda_j, \boldsymbol{h}_j)$, $j = 1, \ldots, n$, be the eigenvalues and orthogonal eigenvectors of $\boldsymbol{\Omega}$. Set $\alpha_j = \boldsymbol{h}_j' \boldsymbol{A} \boldsymbol{h}_j$, $\beta_j = \boldsymbol{h}_j' \boldsymbol{B} \boldsymbol{h}_j$. Throughout this section it is assumed that $\alpha_j \lambda_j > 0$ for some j, $\beta_j \lambda_j > 0$ for some j and that β_j / α_j is not constant in j. Let $I = (\min \beta_j/\alpha_j, \max \beta_j/\alpha_j)$, where min and max extend over $j = 1, \ldots, n$ with $\lambda_j > 0$ and $(\alpha_j, \beta_j) \neq (0, 0)$. The following theorem 1 precedes the main theorem 2 and is interesting in its own right.

Theorem 1. The density g of X defined by Equation 1 is restricted to the interval I and its value at $x \in I$ is given by

$$g(x) = \frac{\mathrm{e}^{-\frac{1}{2}\Sigma\delta_k}}{4\pi\mathrm{i}} \sum_{j=1}^{n} \alpha_j \lambda_j \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \{1 - \delta_j/(1 - c_j z)\} \mathrm{e}^{\frac{1}{2}\Sigma\delta_k/(1 - c_k z)}$$

$$\times \prod (1 - c_k z)^{-\frac{1}{2} - \delta_{kj}}\, \mathrm{d}z \tag{5}$$

where the Σ and Π operators extend over $k = 1, \ldots, n$ with $\lambda_k > 0$ and with

$$\begin{aligned} \delta_j &= (\boldsymbol{h}_j' \mu)^2 / \lambda_j \\ c_j &= \lambda_j (\beta_j - \alpha_j x) \end{aligned} \tag{6}$$

Example 1. ($\boldsymbol{\Omega} = \sigma^2 \boldsymbol{I}_n$, Equation 2, $\mu \in L$): If $\boldsymbol{\Omega} = \sigma^2 \boldsymbol{I}_n$ then $\lambda_j = \sigma^2$ for all j. Hence, without loss of generality, it is possible to take $\boldsymbol{h}_j$ such that $L = \mathscr{R}(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)$, $R = \mathscr{R}(\boldsymbol{h}_{l+1}, \ldots, \boldsymbol{h}_{l+r})$. Then $\delta_j = (\boldsymbol{h}_j' \mu)^2/\sigma^2$ for $j = 1, \ldots, l$ and $\delta_j = 0$ elsewhere. This implies $\delta = \Sigma\, \delta_k = |\mu_L|^2/\sigma^2$. Furthermore, $\alpha_j = 1$ for $j = l + 1, \ldots, l + r$, $\beta_j = 1$ for $j = 1, \ldots, l$; other α- and β-values are equal to 0. This gives $I = (0, \infty)$, $c_j = \sigma^2$ for $j = 1, \ldots, l$, $c_j = -\sigma^2 x$ for $j = l + 1, \ldots, l + r$ and $c_j = 0$ for $j = l + r + 1, \ldots, n$. Substitution into Equation 5 leads, for any $x > 0$, to

$$g(x) = \frac{\mathrm{e}^{-\delta/2}}{4\pi\mathrm{i}} r\sigma^2 \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{\frac{1}{2}\delta/(1-\sigma^2 z)} (1 - \sigma^2 z)^{-l/2}$$

$$\times (1 + \sigma^2 x z)^{-r/2 - 1}\, \mathrm{d}z$$

$$= \mathrm{e}^{-\delta/2} \sum_{k=0}^{\infty} \frac{(\delta/2)^k}{k!} \frac{r}{4\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} (1 - z)^{-(l/2 + k)}$$

$$\times (1 + xz)^{-(r/2+1)}\, \mathrm{d}z \tag{7}$$

The integral in the sum is a variation of Pochhammer's contour integral for the beta-function (see also Lugannani and Rice (1984) p. 487).

$$\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}z}{(z - a)^{\alpha}(b - z)^{\beta}} = \frac{\Gamma(\alpha + \beta - 1)}{(b - a)^{\alpha + \beta - 1}\Gamma(\alpha)\Gamma(\beta)} \tag{8}$$

where $\mathrm{Re}(\alpha + \beta) > 1$ and $a < 0 < b$. This leads to

$$\frac{r}{4\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} (1 - z)^{-(l/2 + k)} (1 + xz)^{-(r/2 + 1)}\, \mathrm{d}z$$

$$= p(x; l/2 + k, r/2) \tag{9}$$

where p is defined by Equation 4. Hence

$$g(x) = \mathrm{e}^{-\delta/2} \sum_{k=0}^{\infty} \frac{(\delta/2)^2}{k!} p(x; l/2 + k, r/2) \quad x > 0 \tag{10}$$

in agreement with Equation 3.

The following theorem shows that Equation 6 can be written as a proper Riemann-integral on a finite interval.

Theorem 2. (Conditions of theorem 1):

$$g(x) = \frac{1}{4\pi}(a^{-1} + b^{-1}) \exp\left(-\frac{1}{2}\sum \delta_k\right)$$

$$\times \sum_{j=1}^{n} \alpha_j \lambda_j I_j \prod (f_k)^{-\frac{1}{2} - \delta_{kj}} \tag{11}$$

with

$$I_j = \int_0^{\pi/2} B_j(t) \prod (A_k(t))^{-\frac{1}{4} - \frac{3}{2}\delta_{kj}} \exp\left\{\frac{1}{2}\sum \delta_k f_k \cos^2 t / A_k(t)\right\}$$

$$\times \cos^{\frac{1}{2}n - 1} t \cos\left[\sum \left\{\left(\frac{1}{2} + \delta_{kj}\right) \arcsin(\gamma_k \sin t / A_k(t)) - S_k(t)\right\}\right.$$

$$\left. + \arcsin(S_j(t)/C_j(t)) \right] \mathrm{d}t \tag{12}$$

where

$$\begin{aligned} a &= \max(\lambda_j \beta_j) \\ b &= x \max(\lambda_j \alpha_j) \\ f_j &= 1 - \tfrac{1}{2} c_j (a^{-1} - b^{-1}) \\ \gamma_j &= \tfrac{1}{2} c_j (a^{-1} + b^{-1}) / f_j \end{aligned} \tag{13}$$

$$\begin{aligned} A_j(t) &= \cos^2 t + \gamma_j^2 \sin^2 t \\ C_j(t) &= (1 - \delta_j f_j) \cos^2 t + \gamma_j^2 \sin^2 t \\ S_j(t) &= \delta_j f_j \gamma_j \sin t \cos t \\ B_j(t) &= \{C_j^2(t) + S_j^2(t)\}^{\frac{1}{2}} \end{aligned} \tag{14}$$

Remark. Since $a \geq \max c_j$, $b \geq -\min c_j$ it follows that $f_j > 0$ and $|\gamma_j| \leq 1$.

Corollary. If $\mu = 0$ then $\delta_j = 0$ for all j. Then $C_j(t) = A_j(t) = B_j(t)$ and $S_j(t) = 0$ and so Equation 12 reduces to

$$I_j = \int_0^{\pi/2} \prod (A_k(t))^{-\frac{1}{4} - \frac{1}{2}\delta_{kj}} \cos^{\frac{1}{2}n - 1} t$$

$$\times \cos\left[\sum \left(\frac{1}{2} + \delta_{kj}\right) \arcsin(\gamma_k \sin t / A_k(t))\right] \mathrm{d}t \tag{15}$$

Example 2 ($\boldsymbol{\Omega} = \sigma^2 I_n$, Equation 2, $\mu = 0$): Using the results in example 1 it is seen that $\delta_j = 0$ for all j and $a = \sigma^2$, $b = \sigma^2 x$. This leads to $f_j = \frac{1}{2}(1 + 1/x)$, $\gamma_j = 1$ for $j = 1, \ldots, l$; $f_j = \frac{1}{2}(1 + x)$, $\gamma_j = -1$ for $j = l + 1, \ldots, l + r$ and $f_j = 1$, $\gamma_j = 0$ for $j = l + r + 1, \ldots, n$. Substitution into Equations 13–15 leads for any $x \in I = (0, \infty)$ to

$$g(x) = x^{l/2-1}(1 + x)^{-(l+r)/2} 2^{(l+r)/2} \frac{r}{2\pi} \int_0^{\pi/2} \cos^{(l+r)/2-1} t \times \cos\{(l - r)/2 - 1\} \, \mathrm{d}t \tag{16}$$

The integral is a variant for the integral expression for the beta-function (see Gradshteyn and Ryzhik (1965) p. 375)

$$\int_0^{\pi/2} \cos^{\alpha+\beta-1} t \cos(\alpha - \beta - 1) \, \mathrm{d}t = \pi / \{2^{\alpha+\beta}(\alpha + \beta) B(\alpha, \beta + 1)\} \tag{17}$$

where $\operatorname{Re} \alpha > 0$ and $\operatorname{Re} \beta > -1$. This leads to $g(x) = p(x; l/2, r/2)$, where p is defined by Equation 4.

3. Proof of theorems

Lemma 1. Let (X_1, X_2) have an absolutely continuous distribution with joint characteristic function φ. If $X_2 \geq 0$ a.s. and $E\{X_2\} < \infty$ then $Y = X_1/X_2$ has a density g given by

$$g(y) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \left(\frac{\partial \varphi(u_1, u_2)}{\partial u_2} \right) \Bigg|_{u_2 = -yu_1} \mathrm{d}u_1 \tag{18}$$

Proof. See Cramer (1946), exercise 6, p. 317 or Geary (1944) and for the multivariate generalization Phillips (1985).

Lemma 2. Let $X \sim N_n(\mu, \boldsymbol{\Omega})$, $\boldsymbol{\Omega} = \boldsymbol{T}\boldsymbol{T}' > \boldsymbol{0}$ with $\boldsymbol{T} \in \mathbb{R}^{n \times n}$. Let $X_1 = X'\boldsymbol{A}_1 X$, $X_2 = X'\boldsymbol{A}_2 X$ with symmetric $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{n \times n}$. Then the joint characteristic function φ of X_1 and X_2 is given by

$$\varphi(u_1, u_2) = |\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C}|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\eta'\eta\} \exp\{\tfrac{1}{2}\eta'(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\eta\} \tag{19}$$

where

$$\begin{aligned} \eta &= \boldsymbol{T}^{-1}\mu \\ \boldsymbol{C} &= u_1 \boldsymbol{T}'\boldsymbol{A}_1\boldsymbol{T} + u_2 \boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T} \end{aligned} \tag{20}$$

Proof. See Magnus (1986), lemma 5, p. 102.

Lemma 3. (Conditions of lemma 2 with $\boldsymbol{A}_2 \geq \boldsymbol{0}$): If $\mathrm{vec}(\boldsymbol{A}_1)$ and $\mathrm{vec}(\boldsymbol{A}_2)$ are linearly independent, then the density g of $Y = X_1/X_2$ is given by

$$\begin{aligned} g(y) = \frac{\mathrm{e}^{-\frac{1}{2}\eta'\eta}}{4\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} & \mathrm{e}^{\frac{1}{2}\eta' S^{-1}(y,z)\eta} |S(y, z)|^{-\frac{1}{2}} \\ & \times [\mathrm{tr}(S^{-1}(y, z)\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}) \\ & + \eta' S^{-1}(y, z)\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}S^{-1}(y, z)\eta] \, \mathrm{d}z \end{aligned} \tag{21}$$

where

$$S(y, z) := \boldsymbol{I}_n - z(\boldsymbol{T}'\boldsymbol{A}_1\boldsymbol{T} - y\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}) \tag{22}$$

Proof. Note that (X_1, X_2) has an absolutely continuous distribution iff $\mathrm{vec}(A_1)$ and $\mathrm{vec}(A_2)$ are linearly independent. Lemmas 1 and 2 and the following formula are used:

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{A}^{-1}}{\mathrm{d}x} &= -\boldsymbol{A}^{-1} \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}x} \boldsymbol{A}^{-1} \\ \frac{\mathrm{d}|\boldsymbol{A}|}{\mathrm{d}x} &= |\boldsymbol{A}| \,\mathrm{tr}\left(\boldsymbol{A}^{-1} \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}x}\right) \quad |\boldsymbol{A}| \neq 0 \end{aligned} \tag{23}$$

Differentiation of Equation 19 leads with Equation 20 and

$$\begin{aligned} \frac{\partial}{\partial u_2} |\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C}|^{-\frac{1}{2}} &= \mathrm{i}|\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C}|^{-\frac{1}{2}} \mathrm{tr}\{(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}\} \\ \frac{\partial}{\partial u_2} (\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1} &= 2\mathrm{i}(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1} \end{aligned} \tag{24}$$

to

$$\begin{aligned} \frac{\partial \varphi(u_1, u_2)}{\partial u_2} = \mathrm{i}\varphi(u_1, u_2)[&\mathrm{tr}\{(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}\} \\ &+ \eta'(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\eta] \end{aligned} \tag{25}$$

So with Equation 22

$$\begin{aligned} \varphi(u_1, -yu_1) = |S(y, 2\mathrm{i}u_1)|^{-\frac{1}{2}} &\exp\{-\tfrac{1}{2}\eta'\eta\} \\ &\times \exp\{\tfrac{1}{2}\eta' S^{-1}(y, 2\mathrm{i}u_1)\eta\} \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{\partial \varphi(u_1, u_2)}{\partial u_2}\bigg|_{u_2 = -yu_1} = \mathrm{i}\varphi(u_1, -yu_1)[&\mathrm{tr}\{S^{-1}(y, 2\mathrm{i}u_1)\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}\} \\ &+ \eta' S^{-1}(y, 2\mathrm{i}u_1)\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}S^{-1}(y, 2\mathrm{i}u_1)\eta] \end{aligned} \tag{27}$$

Substitution of these expressions into Equation 18 together with $z = 2\mathrm{i}u_1$ leads to Equation 21.

Proof of theorem 1. Suppose $\boldsymbol{\Omega} > \boldsymbol{0}$ or, equivalently, $\lambda_j > 0$ for all j. Use Equations 21 and 22 with $\boldsymbol{A}_1 = \boldsymbol{B}$ and $\boldsymbol{A}_2 = \boldsymbol{A}$. Since $\boldsymbol{\Omega} = \Sigma \lambda_j h_j h_j'$ it is possible to take $\boldsymbol{T} = \Sigma \lambda_j^{\frac{1}{2}} h_j h_j'$. This gives

$$\begin{aligned} \boldsymbol{T}'\boldsymbol{B}\boldsymbol{T} &= \sum \beta_j \lambda_j \boldsymbol{h}_j \boldsymbol{h}_j' \\ \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T} &= \sum \alpha_j \lambda_j h_j h_j' \\ S = S(y, z) &= \sum (1 - c_j z) \boldsymbol{h}_j \boldsymbol{h}_j' \\ S^{-1} &= \sum (1 - c_j z)^{-1} \boldsymbol{h}_j \boldsymbol{h}_j' \\ |S|^{-\frac{1}{2}} &= \prod (1 - c_j z)^{-\frac{1}{2}} \\ \mathrm{tr}(S^{-1}\boldsymbol{T}'\boldsymbol{A}\boldsymbol{T}) &= \sum (1 - c_j z)^{-1} \alpha_j \lambda_j \\ \eta &= \sum \delta_j^{\frac{1}{2}} \boldsymbol{h}_j \\ \eta'\eta &= \sum \delta_j \\ \eta' S^{-1} \eta &= \sum \delta_j (1 - c_j z)^{-1} \\ \eta' S^{-1} \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T} S^{-1} \eta &= \sum \alpha_j \lambda_j \delta_j (1 - c_j z)^{-2} \end{aligned} \tag{28}$$

Substitution into Equation 21 with the Kronecker symbol $\delta_{kj} = 1$ if $k = j$, $\delta_{kj} = 0$ if $k \neq j$ leads to Equation 5.

For fixed j with $\alpha_j \lambda_j > 0$ the integrand in Equation 5 is $O(|z|^{-\frac{3}{2}})$ for $|z| \to \infty$; furthermore it has singular points in the half plane $\mathrm{Re}\, z > 0$ iff $x < \beta_j / \alpha_j$ for some j and singular points in $\mathrm{Re}\, z < 0$ iff $x > \beta_j / \alpha_j$ for some j. So $g(x) = 0$ if $x > \max(\beta_j/\alpha_j)$ or $x < \min(\beta_j/\alpha_j)$. This concludes the proof of the theorem for $\boldsymbol{\Omega} > \mathbf{0}$. The general case follows by continuity arguments with respect to the eigenvalues λ_j of $\boldsymbol{\Omega}$.

Proof of theorem 2. The substitution $s = (b - a - 2abz)/(b + a)$ and $c = (b - a)/(b + a)$ can be made into Equation 5. Then $1 - c_k z = (1 + \gamma_k s)/f_k$ and so

$$g(x) = \frac{\mathrm{e}^{-\frac{1}{2}\Sigma\delta_k}}{8\pi \mathrm{i}} (a^{-1} + b^{-1}) \sum_{j=1}^{n} \alpha_j \lambda_j \prod (f_k)^{\frac{1}{2} + \delta_{kj}} I_j(c) \quad (29)$$

with

$$I_j(c) = \int_{-\mathrm{i}\infty + c}^{\mathrm{i}\infty + c} \left\{ \prod (1 + \gamma_k s)^{-\frac{1}{2} - \delta_{kj}} \right\} \times \{1 - \delta_j f_j / (1 + \gamma_j s)\} \mathrm{e}^{\frac{1}{2}\Sigma\delta_k f_k/(1 + \gamma_k s)} \, \mathrm{d}s \quad (30)$$

The integrand has singular points at $s = -1/\gamma_k$. Since $a \geq \max c_k$ and $b \geq -\min c_k$ then $|\gamma_k| \leq 1$ and so all singular points are outside $\{s: |\mathrm{Re}\, s| < 1\}$. Therefore $I_j(c)$ does not depend on c provided that $|c| < 1$. Since $|b - a|/(b + a) < 1$ it is possible to replace the particular value $c = (b - a)/(b + a)$ by $c = 0$. This gives the intermediate result

$$g(x) = \frac{\mathrm{e}^{-\frac{1}{2}\Sigma\delta_k}}{8\pi} (a^{-1} + b^{-1}) \sum_{j=1}^{n} \alpha_j \lambda_j \prod (f_k)^{\frac{1}{2} + \delta_{kj}} I_j \quad (31)$$

with

$$I_j = I_j(0) = \int_{-\infty}^{\infty} \left\{ \prod (1 + \mathrm{i}\gamma_k u)^{-\frac{1}{2} - \delta_{k_j}} \right\} \times \{1 - \delta_j f_j / (1 + \mathrm{i}\gamma_j u)\} \mathrm{e}^{\frac{1}{2}\Sigma\delta_k f_k/(1 + \mathrm{i}\gamma_k u)} \, \mathrm{d}u \quad (32)$$

This expression can be rewritten in the form of a Riemann integral on a finite interval. Substitution of $u = \mathrm{tg}\, t$ and $\mathrm{d}u = \cos^2 t \, \mathrm{d}t$ together with

$$\begin{aligned} 1 + \mathrm{i}\gamma_k u &= A_k(t) \cos t \exp\{\mathrm{i} \arcsin(\gamma_k \sin t / A_k(t))\} \\ 1 - \delta_j f_j / (1 + \mathrm{i}\gamma_j u) &= A_j^{-1}(t) B_j(t) \\ &\quad \times \exp\{\mathrm{i} \arcsin(S_j(t)/A_j(t))\} \qquad (33) \\ \exp\{\tfrac{1}{2}\delta_k f_k / (1 + \mathrm{i}\gamma_k u)\} &= \exp\{\tfrac{1}{2}\delta_k f_k \cos^2 t / A_k(t)\} \\ &\quad \times \exp\{-\mathrm{i} S_k(t)\} \end{aligned}$$

leads to Equations 11 and 12.

References

Cramer, H. (1963) *Mathematical Methods of Statistics*, Princeton University Press, Princeton.

Geary, R. C. (1944) Extension of a theorem by Harold Cramer. *Journal of the Royal Statistical Society*, **17**, 56–57.

Gradshteyn, I. S. and Ryzhik, I. M. (1965) *Tables of Integrals, Series and Products*, 4th edn, Academic Press, New York.

Lugannani, R. and Rice, S. O. (1984) Distribution of the ratio of quadratic forms in normal variables: numerical methods. *SIAM Journal of Statistics and Computing*, **5**, 476–488.

Magnus, J. R. (1986) The exact moments of a ratio of quadratic forms in normal variables. *Annales d'Economie et de Statistique*, **4**, 95–109.

Scheffé, H. (1959) *The Analysis of Variance*, Wiley, New York.

Seber, G. A. F. (1977) *Linear Regression Analysis*, Wiley, New York.