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den Hertog, D.; Roos, C.; Terlaky, T.

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A Polynomial Method of Weighted Centers for Convex Quadratic Programming

D. den Hertog
C. Roos
T. Terlaky*
Faculty of Technical Mathematics and Computer Science
Delft University of Technology
P.O. Box 356, 2600 AJ Delft
The Netherlands

ABSTRACT

A generalization of the weighted central path-following method for convex quadratic programming is presented. This is done by uniting and modifying the main ideas of the weighted central path following method for linear programming and the interior point methods for convex quadratic programming. By means of the linear approximation of the weighted logarithmic barrier function and weighted inscribed ellipsoids, 'weighted' trajectories are defined. Each strictly feasible primal-dual point pair defines such a weighted trajectory. The algorithm can start in any strictly feasible point that defines a weighted trajectory, which is followed through the algorithm.

This algorithm has the nice feature, that it is not necessary to start the algorithm close to the central path and so additional transformations are not needed. In return, the theoretical complexity of our algorithm is dependent on the position of the starting point. Polynomiality is proved under the usual mild conditions.

1. INTRODUCTION

Quadratic Programming (QP) and linear complementarity problems (LCP) are one of the most widely studied problems of mathematical programming. Several effective methods were developed for solving QP problems already in the early period of mathematical programming (see, *e.g.* Cottle-Dantzig [6], Beale [4], Lemke [20], Wolfe [34], Van der Heyden [33]). These methods are based on Dantzig's simplex method [7] for linear programming (LP). An extensive very, good survey of the

*On leave from the Eötvös University, Budapest.

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existing methods can be found in Murty's [27] book. Recently two major directions can be observed concerning linear and quadratic programming.

1. COMBINATORIAL ABSTRACTION, COMBINATORIAL METHODS. Todd [32], and Morris and Todd [26] give a combinatorial generalization of QP and LCP by formulating the QP and LCP problem of oriented matroids. Todd generalized Lemke's [20] method as well. Klafszky-Terlaky [15] and Fukuda-Terlaky [10] gave finite pivoting rules for QP and oriented matroid QP.

2. POLYNOMIAL METHODS. First Khachian [14] gave a polynomial time algorithm, the ellipsoid method, for LP and he and his colleagues [19] generalized the ellipsoid method for QP. Ellipsoid methods turned out to be inefficient in practice, so research was stopped in this field. Karmarkar's [13] interior point method for LP opened a new research area in LP, QP and even for smooth convex programming. These interior point methods are theoretically efficient and their practical performance is promising too.

Since Karmarkar's [13] projective method for LP, several authors published new polynomial time interior methods for LP and QP. These type of methods have already a very extended literature. Without completeness we mention some publications in each of the main directions.

1. *Projective methods* : Karmarkar's [13] original paper opened this class. Later, among others Anstreicher [2] and Gonzaga [11] studied these type of methods. Anstreicher extended the analysis to fractional linear programming and Ye and Tse [37] applied this type of method to QP. Recently Freund [9] introduced weights to the projective method.

2. *Affine scaling methods* : Proposed by Dikin [8] already in 1967 for LP and QP without any attempt to polynomiality proof. This method was rediscovered by Barnes [3]. Studied by Megiddo and Shub [22], further developed and extended to QP by Ye [35].

3. *Path-following methods* : Nice algorithms and elegant proofs are given by Gonzaga [11], Roos [28], Roos and Vial [30]. Kojima, Mizuno and Yoshise [16], and Ye [36] extended this

approach to LCP. Ben Daya and Shetty [5] studied a barrier function approach to convex QP as well. Without polynomiality proof Megiddo [21] and Adler and Monteiro [1] investigated weighted trajectories. Later Monteiro, Adler and Resende [25] investigated a primal-dual affine algorithm, which in fact follows the weighted path of the problem. Also for the LP case, Roos and den Hertog [29] constructed a polynomial algorithm based on weighted trajectories. Our paper utilizes most of the ideas of this paper.

4. *Affine potential reduction methods* : May be this newly discovered algorithm class is the most promising, since the algorithms of this class enjoy the advantages of path-following algorithms without their drawback, namely no initial solution close to the central path is needed. Gonzaga [12] presented an algorithm of this type for linear programming, and Liu and Goldfarb [18] and Kojima, Mizuno and Yoshise [17] generalized this approach for quadratic programming. Recently Ye [38] and Ye and Pardalos [39] gave potential reduction algorithms for a class of linear complementarity problems.

Each of the above mentioned algorithms have some advantages and disadvantages. Computing projections is very time consuming and this is the most "expensive" part of these algorithms. Affine scaling algorithms are believed not to be polynomial [22]. Path-following methods needs an initial solution close enough to the central path. This problem can be solved, *i.e.*, by using the concept of weighted centers. For example Roos and den Hertog's [29] algorithm applies this idea. This algorithm can be initiated from any starting point, since all the interior solutions define a weighted central path. Unfortunately the complexity of these algorithms is not independent from the starting point (from the weight). If a starting point is choosen far from the center, then the theoretical convergence rate could suffer. Algorithms in the fourth group seem to be the most promising, since they unite some of the advantages of the algorithm categories and eliminate some of the drawbacks.

The algorithm proposed in this paper is a joint generalization of Ye's [36] QP method and Roos and den Hertog's [29] concept of weighted trajectories. So our algorithm allows to start from any initial interior solution, it follows the weighted central path defined by the initial solution and this algorithm is designed for solving convex quadratic programming problems.

Ye [36] defined a subproblem for getting a better solution. The objective of this subproblem is the sum of the original objective and the linear approximation of the log barrier function. The "difficult" non-negativity assumptions are replaced by an "ellipsoidal" constraint as it was proposed by Dikin [8]. The approximate solution of this problem gives the next iterate. Adopting the idea of weighted trajectories this subproblem is modified by introducing weights to both of the linear log barrier approximation and to the ellipsoid constraint. This way the algorithm follows the weighted central path defined by the initial solution.

Recently Mizuno [23], [24] presented a new method for linear complementarity problems. His method is quite general, some path-following and potential reduction methods can be derived from Mizuno's algorithm. There is a parameter vector in this algorithm. As pointed out by Kojima (personal communication), if the parameter vector is updated properly, a weighted path-following method for linear complementarity problems is obtained. This provides a method for QP as well, but this is not a direct approach to QP.

In solving QP, our algorithm can start in an arbitrary strictly feasible initial point, weights are defined as it was proposed by Roos and den Hertog [29]. The weighted central path goes through the initial point. So the initial transformations to get close to the central path can be eliminated. This is an obvious advantage of this approach comparing with the existing path-following QP algorithms.

Section 2 contains a description of the QP problem, the "weighted subproblem" and the conceptual algorithm. The solution of this subproblem and properties of the resulted equation system is discussed here. In Section 3 bounds are given for the new solutions obtained by solving the weighted subproblem. These bounds imply that feasibility is preserved and guarantee some fixed reduction of the actual duality gap. Based on these results, the precise algorithm is presented and its polynomial convergence is proved in Section 4. Finally a variant is shortly discussed, where by proper parameter selection our method specializes to Newton's method.

2. QP AND WEIGHTED TRAJECTORIES

We consider the pair of a primal (QP) aad dual (QD) quadratic programming problem in the standard form :

(QP) $\min \{c^T x + \frac{1}{2} x^T Qx : Ax = b, x \geqslant 0\}$,

(QD) $\max \{b^T y - \frac{1}{2} x^T Qx : A^T y - Qx + z = c, z \geqslant 0\}$,

where A is an $m \times n$ matrix, Q is an $n \times n$ positive semidefinite symmetric matrix and b, c, x, y, z are m or n dimensional vectors. The following notations will be used. If x denotes a vector, then the corresponding copital letter X will denote the diagonal matrix with the components of x on the diagonal. Furthermore, e will always denote an all one vector of appropriate length.

It is well known, that primal and dual feasibility together with the so called *complementarity* condition $x^T z = 0$ guarantee optimality. Our algorithm produces a sequence of primal-dual interior feasible solutions that are approximately complementary solutions.

We assume, that x, y and z are given such that :

$$Ax = b, \; x > 0, \tag{1}$$

$$A^T y - Qx + z = c, \; z > 0. \tag{2}$$

Defining

$$\beta := \frac{x^T z}{n} = \frac{c^T x + x^T Qx - y^T Ax}{n} = \frac{(c^T x + \frac{1}{2} x^T Qx) - (y^T b - \frac{1}{2} x^T Qx)}{n} \quad \ldots(3),$$

β is a fraction of the actual duality gap. Further, let $W > 0$ be defined by

$$W^2 := \frac{XZ}{\beta}. \tag{4}$$

Then $w^T w = n$ and $\|Ww\| \leqslant n$. The notation $w^2 = w^T w$ will be frequently used later. Further we assume, that for some $0 < \alpha < 1$ and $\pi = \min_i w_i \leqslant 1$

$$\|W^{-1}(Xz - \beta Ww)\| \leqslant \alpha\beta\pi. \tag{5}$$

At the beginning, the left hand side of inequality (5) is zero by the definition of W, but later, the modified new solutions will only satisfy

(5) as an inequality. Now we are ready to formulate the subproblem that is solved successively during the algorithm. The subproblem is a relaxation and modification of the original QP problem. The nonnegativity conditions are replaced by a "weighted ellipsoidal constraint" (weighted version of the ellipsoidal constraint from the affine scaling methods), and the objective is modified by introducing a linear approximation of the weighted logbarrier function

$$\left(-\lambda \sum_i (w_i)^2 \log x_i\right),$$

where λ is the barrier parameter and $(w_i)^2$ are the weights. So this subproblem is a combination of the subproblems used in affine scaling methods and subproblems used in logbarrier methods. On the other hand this subproblem is the "double weighted" version of Ye's [36] subproblem, where the weights are introduced the same way as Roos and den Hertog [29] introduced weights into the logbarrier function and their $\delta(x, \mu)$ function.

The primal subproblem :

$$\min \tfrac{1}{2}x^{*\mathrm{T}}Qx^* + c^{\mathrm{T}}x^* - \lambda w^{\mathrm{T}} WX^{-1}(x^* - x)$$

$$\text{s.t.} \quad Ax^* = b$$

$$\|WX^{-1}(x^* - x)\| \leqslant \rho < 1,$$

where λ is the barrier parameter.

Since Q is a positive semidefinite matrix, Cholesky factorization can be applied, so it can be written in the form $Q = DD^{\mathrm{T}}$. The dual of this problem can be easily formulated (see *e.g.* Terlaky [31]).

The dual subproblem :

$$\max y^{*\mathrm{T}}b - \nu^{*\mathrm{T}}e - \vartheta\frac{\rho^2}{2} - \tfrac{1}{2}t^{*2} - \frac{1}{2\vartheta}\nu^{*2}$$

$$\text{s.t.} \quad Dt^* - A^{\mathrm{T}}y^* + X^{-1}W\nu^* = -c + \lambda X^{-1}Ww$$

$$\vartheta = 0 \Rightarrow \nu^* = 0.$$

The equilibrium (optimality) conditions are as follows :

$$t^* = Dx^*$$

$$v^* = \vartheta\,[WX^{-1}\,(x^* - x)]$$

$$\vartheta(\|WX^{-1}\,(x^* - x)\| - \rho) = 0.$$

In solving the subproblem, the dual variables t^*, v^* can be eliminated. Let us introduce the notations $\triangle x = x^* - x$, $\triangle y = y^* - y$ and $\triangle z = z^* - z = Q\triangle x - A^T \triangle y$. So the optimality conditions for the subproblem can be expressed as follows :

$$A\triangle x = 0 \tag{6}$$

$$X\triangle z + \vartheta W^2 X^{-1}\,\triangle x = \lambda Ww - Xz \tag{7}$$

$$\rho = \|WX^{-1}\,\triangle x\| < 1. \tag{8}$$

Our algorithm is based on the above equations and on their solutions. If λ and ϑ are given, then $\triangle x$, $\triangle y$ and $\triangle z$ can be computed easily (parameter ρ can be computed as well). The reader can easily verify that the solutions are as follows :

$$\triangle z = \lambda X^{-1} Ww - z - \vartheta W^2 X^{-2} \triangle x$$

$$\triangle x = (Q + \vartheta W^2 X^{-2})^{-1} A^T \triangle y + (Q + \vartheta W^2 X^{-2})^{-1} (\lambda X^{-1} Ww - z)$$

$$\triangle y = -[A(Q + \vartheta W^2 X^{-2})^{-1} A^T]^{-1} A (Q + \vartheta W^2 X^{-2})^{-1} (\lambda X^{-1} Ww - z).$$

As we will show, by appropriately choosing parameters λ and ϑ, the steps $\triangle x$, $\triangle y$, $\triangle z$ are small enough to remain inside the feasible region.

Before presenting the selected parameter values and the derived estimations, another feature of the above subproblem is discussed below.

Equations (7, 6) can be expressed as follows if $\triangle z$ is eliminated :

$$\begin{pmatrix} Q + \vartheta W^2 X^{-2} & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \triangle x \\ \triangle y \end{pmatrix} = \begin{pmatrix} \lambda X^{-1} Ww - z \\ 0 \end{pmatrix}. \tag{9}$$

This equation system is the Kuhn-Tucker system of the weighted primal subproblem. From this system the following remarks are straightforward :

1. If $W = E$ then Ye's [36] method follows.
2. In the LP case, when $\lambda = \vartheta$ this search direction problem reduces to Roos and den Hertog's [29] search direction problem of their weighted centers method.

3. In the LP case, when $\lambda=\vartheta$ and $W=E$, the solution of this system is the projected Newton direction of the "logbarrier" problem (see Roos and Vial [30]).

4. If ϑX^{-2} stands instead of $\vartheta W^2 X^{-2}$ then the obtained direction corresponds to that subproblem where weights are introduced only to the logbarrier approximation. Unfortunately we could not prove polynomiality of the algorithm based on this search direction.

Based on different selections of the values λ, ϑ and weight W, different directions and so different algorithms (not necessarily polynomial) can be constructed. The algorithms based on the not studied directions can be a subject of further research. In the coming sections the following algorithm is analyzed with a special parametar set. This algorithm corresponds to the "double weighted" subproblem and the derived search direction.

Algorithm (*QPW*)

Input:

A pair (x^0, z^0) such that equations (1, 2, 3, 4) hold. An accuracy parameter $\epsilon>0$ is given.

begin

while $\beta>\epsilon$ **do**

begin

Let parameters α, ϑ and λ be defined appropriately.

Let Δx, Δy and Δz be solutions of equations (7, 6).

Let $x=x+\gamma\Delta x$, $z=z+\gamma\Delta z$, $y=y+\gamma\Delta y$,

where γ is such that strict feasibility is preserved.

end

end

It will be shown that for a careful choice of the parameters ϑ, λ and α, the steplength γ can always be equal to one and the solutions follow the weighted path of the problem (inequality (5) is fulfilled in each step), whereas the parameter selection enables us to prove polynomial convergence. It will also be shown, that another parameter selection coincides with a Newton method for the weighted logarithmic barrier problem.

3. ESTIMATIONS, PRELIMINARY LEMMAS

Some estimations will be proved in this section. These estimations will justify the validity of the proposed algorithm. Its polynomiality will be proved in Section 4 by utilizing these estimations. As it was mentioned in the previous section, an appropriate choice of parameters λ and ϑ is essential. Let

$$\lambda=\left(1-\frac{\alpha\pi}{\sqrt{n}}\right)\beta \quad \text{and} \quad \vartheta=\beta. \tag{10}$$

Based on this special selection of the parameters, a series of lemmas are proved.

LEMMA 1. $\|W^{-1}(\lambda Ww-Xz)\|\leqslant\sqrt{2}\,\alpha\beta\pi.$

PROOF. Let us use first some trivial transformations.

$$\|W^{-1}(\lambda Ww-Xz)\|^2=\|W^{-1}(\beta Ww-Xz)-\frac{\alpha\beta\pi}{\sqrt{n}}w\|^2$$
$$=\|W^{-1}(\beta Ww-Xz)\|^2+\frac{\alpha^2\beta^2\pi^2w^2}{n}-\frac{2\alpha\beta\pi}{\sqrt{n}}[\beta w^2-e^{\mathrm{T}}Xz].$$

It follows from equation (4) that $w^2=n$, and from (3) that $e^{\mathrm{T}}Xz=\beta n$. These two together imply that the last expression in square brackets is zero. So by assumption (5) we have :

$$\|W^{-1}(\lambda Ww-Xz)\|^2\leqslant\alpha^2\pi^2\beta^2+\alpha^2\pi^2\beta^2=2\alpha^2\beta^2\pi^2. \quad \square$$

Remark. Since Q is positive semidefinite and $A\triangle x=0$ by equation (6), it is obvious, that

$$\triangle x^{\mathrm{T}}\triangle z=\triangle x^{\mathrm{T}}Q\triangle x-\triangle y^{\mathrm{T}}A\triangle x\geqslant 0. \tag{11}$$

Now we are ready to prove the following estimation.

LEMMA 2. $\|W^{-1}(\lambda Ww-Xz)\|^2\geqslant\|W^{-1}X\triangle z\|^2+\|\vartheta WX^{-1}\triangle x\|^2.$

PROOF. Using equation (7) we have :

$$\|W^{-1}(\lambda Ww-Xz)\|^2=\|W^{-1}X\triangle z+\vartheta WX^{-1}\triangle x\|^2$$
$$=\|W^{-1}X\triangle z)\|^2+\|\vartheta WX^{-1}\triangle x\|^2+2\vartheta\triangle x\triangle z$$
$$\geqslant\|W^{-1}X\triangle z\|^2+\|\vartheta WX^{-1}\triangle x\|^2.$$

The last inequality follows from (11). □

LEMMA 3. $\|W^{-1}X\Delta z\| \leqslant \sqrt{2}\,\alpha\beta\pi$, and

$\|WX^{-1}\Delta x\| \leqslant \sqrt{2}\alpha\pi$.

PROOF. Obvious from Lemma 1 and Lemma 2. □

LEMMA 4. $\|Z^{-1}\Delta z\| \leqslant \frac{\sqrt{2}\alpha}{1-\alpha}$.

PROOF. Using Lemma 3 we have :

$$\|Z^{-1}\Delta z\| = \|(Z^{-1}X^{-1}W)(W^{-1}X\Delta z)\| \leqslant \frac{\sqrt{2}\alpha\pi\beta}{\beta\pi(1-\alpha)} = \frac{\sqrt{2}\alpha}{1-\alpha},$$

where the following estimations were used for $(Z^{-1}X^{-1}W)$. From (5) we have

$$-\frac{x_i z_i}{w_i} + \beta w_i \leqslant \alpha\beta\pi$$

$$\text{i e.} \qquad -\frac{x_i z_i}{w_i} \leqslant \beta(\alpha\pi - w_i).$$

Using the definition of π we have :

$$\frac{w_i}{x_i z_i} \leqslant \frac{1}{\beta(w_i - \alpha\pi)} \leqslant \frac{1}{\beta\pi(1-\alpha)}.$$

The last inequality provides an upper bound for each of the coordinates of $(Z^{-1}X^{-1}W)$. This estimation is used above. The lemma is proved. □

LEMMA 5. $\|\Delta X\Delta z\| \leqslant \beta\alpha^2\pi^2$.

PROOF. The following equations, inequalities are obvious :

$$\|\Delta X\Delta z\| = \|(\Delta XX^{-1}W)(W^{-1}X\Delta z)\| \leqslant \frac{1}{\vartheta}\|\vartheta WX^{-1}\Delta x\|\|W^{-1}X\Delta z\|$$

$$\leqslant \frac{1}{\vartheta}\,\frac{\|\vartheta WX^{-1}\Delta x\|^2 + \|W^{-1}X\Delta z\|^2}{2}$$

$$\leqslant \frac{1}{2\vartheta}\|W^{-1}(\lambda Ww - Xz)\|^2 \leqslant \frac{1}{2\beta}\,2\alpha^2\beta^2\pi^2 = \alpha^2\pi^2\beta.$$

The arithmetical-geometrical mean inequality, Lemma 2, Lemma 1 and equation (10) was used. □

Denote $x^*=x+\triangle x$ and $z^*=z+\Delta z$, as it was introduced in the previous section. These definitions allows us to formulate the following equations :

$$X^*z^*=(X+\triangle X)(z+\Delta z)=Xz+X\Delta z+\triangle Xz+\triangle X\Delta z$$

$$=Xz+X\triangle z+\vartheta W^2X^{-1}\Delta x-\vartheta W^2X^{-1}\triangle x+\triangle Xz+\triangle X\Delta z.$$

Repeatedly using equation (7) we have :

$$X^*z^*=\lambda Ww+\triangle XX^{-1}(Xz+X\triangle z-\vartheta Ww) \tag{12}$$

$$X^*z^*=\lambda Ww+\triangle XX^{-1}(\lambda Ww-\vartheta W^2X^{-1}\Delta x-\vartheta Ww). \tag{13}$$

LEMMA 6. *Let* β *and* β^* *be defined by the corresponding solutions as in* (3). *Then*

$$\left[1-\frac{\alpha\pi}{\sqrt{n}}-\frac{\sqrt{2}\alpha^2\pi^2}{n}\right]\beta\leqslant\beta^*.$$

PROOF. Multiplying equation (12) by e and using the definition of β^* we have :

$$n\beta^*=x^{*\mathrm{T}}z^*=e^{\mathrm{T}}X^*z^*=\lambda w^2+\triangle x^{\mathrm{T}}X^{-1}(Xz+X\Delta z-\vartheta Ww)$$

$$=\lambda n+(\triangle x^{\mathrm{T}}X^{-1}W)[W^{-1}(Xz-\beta Ww)]+\Delta x^{\mathrm{T}}\triangle z.$$

Here equations (4) and (10) were used. Applying inequality (11) we have

$$n\beta^*\geqslant\lambda n-\|WX^{-1}\triangle x\|\;\|W^{-1}(Xz-\beta Ww)\|\geqslant\lambda n-\sqrt{2}\alpha^2\pi^2\beta.$$

Equation (5) and Lemma 3 are used in the last estimation. So by the definition of λ (see eq. (10)) we have :

$$\beta^*\geqslant\left[1-\frac{\alpha\pi}{\sqrt{n}}-\frac{\sqrt{2}\alpha^2\pi^2}{n}\right]\beta.$$

The proof is complete. □

LEMMA 7. $$\beta^*\leqslant\left[1-\frac{\alpha\pi}{\sqrt{n}}+\frac{\alpha^2\pi^2}{4n}\right]\beta=\left(1-\frac{\alpha\pi}{2\sqrt{n}}\right)^2\beta.$$

PROOF. Multiplying equation (13) by e and using the definition of β^* we have :

$$n\beta^*=x^{*\mathrm{T}}z^*=e^{\mathrm{T}}X^*z^*=\lambda w^2+\triangle x^{\mathrm{T}}X^{-1}(\lambda Ww-\vartheta W^2X^{-1}\triangle x-\vartheta Ww)$$

$$=\lambda n+(\triangle x^{\mathrm{T}}X^{-1}W)(\lambda-\beta)w-\beta\|WX^{-1}\Delta x\|^2.$$

Here equations (4) and (10) were used. Applying again (4) and (10) we have

$$n\beta^* \leqslant \lambda n + \|WX^{-1}\triangle x\|\alpha\pi\beta - \beta\|WX^{-1}\triangle x\|^2.$$

The right hand side quadratic expression can be replaced by its maximal value, so we have :

$$n\beta^* \leqslant \lambda n + \frac{\alpha^2\pi^2\beta}{4}.$$

By substituting the chosen value of λ as given in (10) we have the desired expression

$$\beta^* \leqslant \left[1 - \frac{\alpha\pi}{\sqrt{n}} + \frac{\alpha^2\pi^2}{4n}\right]\beta.$$

The proof is complete. □

LEMMA 8. $\|W^{-1}(X^*z^* - \beta^* Ww)\| \leqslant \left(\frac{5}{4} + 2\sqrt{2}\right)\alpha^2\beta\pi.$

PROOF. Using the definition of π (see (4)) we have :

$$\|W^{-1}(X^*z^* - \beta^* Ww)\| \leqslant \frac{1}{\pi}\|X^*z^* - \beta^* Ww)\|$$

$$= \frac{1}{\pi}\|(X^*z^* - \lambda Ww) + (\lambda - \beta^*)Ww\|.$$

By a further estimation we have :

$$\|W^{-1}(X^*z^* - \beta^* Ww)\| \leqslant \frac{1}{\pi}\left[\|X^*z^* - \lambda Ww\| + |\lambda - \beta^*|\,\|Ww\|\right]. \quad (14)$$

From Lemma 6 and Lemma 7 we have, that :

$$|\lambda - \beta^*| \leqslant \left(\frac{\alpha^2\pi^2}{4n} + \frac{\sqrt{2}\alpha^2\pi^2}{n}\right)\beta.$$

Expression (4) implies that $\|Ww\| \leqslant n$ so

$$|\lambda - \beta^*|\,\|Ww\| \leqslant \left(\frac{1}{4} + \sqrt{2}\right)\alpha^2\beta\pi^2. \quad (15)$$

Furthermore using formulas (12) and (10) the following relations hold :

$$\|X^*z^* - \lambda Ww\| = \|WX^{-1}(Xz - \vartheta Ww) + (\triangle XX^{-1})(X\triangle z)\|$$

$$\leqslant \|WX^{-1}\triangle x\|\,\|W^{-1}(Xz - \beta Ww)\| + \|\triangle X\triangle z\|.$$

From Lemma 3, Lemma 5 and the basic assumption (5) we have :

$$\|X^*z^* - \lambda Ww\| \leqslant \sqrt{2}\alpha^2\beta\pi^2 + \alpha^2\beta\pi^2. \quad (16)$$

Combining the above derived inequalities the desired expression follows :

$$\|W^{-1}(X^*z^*-\beta^* Ww)\| \leqslant \frac{1}{\pi}\left[\sqrt{2}\,\alpha^2\pi^2\beta+\alpha^2\pi^2\beta+\left(\frac{1}{4}+\sqrt{2}\right)\alpha^2\pi^2\beta\right]$$

$$=\left(\frac{5}{4}+2\sqrt{2}\right)\alpha^2\beta\pi. \quad \square$$

4. CONVERGENCE ANALYSIS OF ALGORITHM (QPW) WITH A PROPER PARAMETER SELECTION

First a theorem is proved in this section, which guarantees that the solutions generated by the algorithm will remain inside the feasibility region, while a reduction factor (less than 1) can be given for the new β value, and the solutions remain close to the weighted central path, *i.e.*, inequality (5) remains valid. These properties can be proved by a careful selection of parameter α.

THEOREM 1. *Let* $\alpha=\frac{1}{10}$, *then the following four inequalities hold* :

(*a*) $\|X^{-1}\Delta x\|<1$

(*b*) $\|Z^{-1}\Delta z\|<1$

(*c*) $\beta^* \leqslant \left(1-\frac{\pi}{20\sqrt{n}}\right)^2\beta.$

(*d*) $\|W^{-1}(X^*z^*-\beta^* Ww)\| \leqslant \alpha\pi\beta^*.$

PROOF. Part (*a*) follows from Lemma 3 :

$$\|X^{-1}\Delta x\|=\|W^{-1}(WX^{-1}\Delta x)\| \leqslant \frac{1}{\pi}\|WX^{-1}\Delta x\|$$

$$\leqslant \frac{\sqrt{2}\,\alpha\pi}{\pi}=\frac{\sqrt{2}}{10}<1.$$

Part (*b*) follows from Lemma 4.

$$\|Z^{-1}\Delta z\| \leqslant \frac{\sqrt{2}\,\alpha}{1-\alpha}=\frac{\sqrt{2}}{9}<1.$$

Lemma 7 implies part (*c*).

$$\beta^* \leqslant \left(1-\frac{\alpha\pi}{2\sqrt{n}}\right)^2\beta=\left(1-\frac{\pi}{20\sqrt{n}}\right)^2\beta.$$

Finally, part (*d*) follows from Lemma 8 :

$$\|W^{-1}(X^*z^*-\beta^* Ww)\| \leq \left[\left(\frac{5}{4}+2\sqrt{2}\right)\alpha\right]\alpha\pi\beta$$

$$\leq \left[\left(\frac{5}{4}+2\sqrt{2}\right)\alpha\frac{1}{1-\frac{\alpha\pi}{\sqrt{n}}-\frac{\sqrt{2\alpha^2\pi^2}}{n}}\right]\alpha\pi\beta^*$$

$$\leq \left[\left(\frac{5}{4}+2\sqrt{2}\right)\frac{1}{10}\right]\frac{1}{1-\frac{1}{10}-\frac{\sqrt{2}}{100}}\alpha\pi\beta^* < \alpha\pi\beta^*.$$

The theorem is proved. □

As we mentioned, parts (*a*) and (*b*) ensure that $x^* = x + \triangle x$ and $z^* = z + \triangle z$ are strictly positive, part (*c*) says that $\beta^* < q\beta$, where $q < 1$ and part (*d*) guarantees that inequality (5) holds for x^*, z^*. Now we are ready to present an appropriate selection of the parameters for Algorithm (QPW).

Parameter selection for Algorithm (QPW)

A pair of initial solutions (x^0, z^0) is given and W is defined by (4), such that assumptions (1, 2, 3, 4, 5) hold. Let $\alpha = 0.9$. An accuracy parameter $\epsilon > 0$ is given. Let ϑ and λ be defined by (10) and β defined by (3) in each iterational cycle. Let $\gamma = 1$.

The validity of Algorithm (QPW) with the above parameters is justified by Theorem 1. Its complexity is presented in the next theorem.

THEOREM 2. *Using the above presented parameter selection, Algorithm (QPW) stops after at most* $\frac{10\sqrt{n}}{\pi}\log\frac{\beta^0}{\epsilon}$ *iterations.*

PROOF. The algorithm stops when the actual $\beta < \epsilon$. After N steps, when the algorithm stops, for the actual value of β we have the following upper bound, that follows from part (*c*) of the previous theorem,

$$\beta \leq \beta^0\left(1-\frac{\pi}{20\sqrt{n}}\right)^{2N} < \epsilon.$$

From this, for step number N we have the following relation :

$$2N\log\left(1-\frac{\pi}{20\sqrt{n}}\right) < \log\frac{\epsilon}{\beta^0}.$$

Since $\log(1-x) < -x$, it is enough to require, that

$$2N\left(-\frac{\pi}{20\sqrt{n}}\right) < \log\frac{\epsilon}{\beta^0}.$$

By easy computation we have the desired bound.

$$2N\frac{\pi}{20\sqrt{n}} > \log\frac{\beta^0}{\epsilon}$$

$$N > \frac{10\sqrt{n}}{\pi}\log\frac{\beta^0}{\epsilon}.$$

The theorem is proved. □

It is obvious that for the last generated points x, y and z the relation $Xz < \epsilon e$ holds and for the duality gap we have :

$$(c^T x + \tfrac{1}{2}x^T Q x) - (y^T b - \tfrac{1}{2}x^T Q x) \leqslant n\epsilon.$$

If all the data are integers and ϵ is small enough (*i.e.* $\epsilon = 2^{-L}$ where L is the length of input data), then — under nondegeneracy assumption — the optimal solutions can easily be identified from the final solutions.

5. THE ALGORITHM AS NEWTON'S METHOD

The parameter selection in the previous section (10) is analogous to. Ye [36]. Other parameter selections provide other algorithm variants One interesting selection is :

$$\lambda = \left(1 - \frac{\alpha\pi}{\sqrt{n}}\right)\beta \quad \text{and let} \quad \vartheta = \lambda. \tag{17}$$

In this case our method reduces to Newton's method according to the weighted logarithmic barrier function

$$c^T x + \tfrac{1}{2}x^T Q x - \lambda \sum_{i=1}^{n} w_i^2 \log x_i.$$

Considering equation (9) this is obvious. The analysis of this algorithm is identical. The lemmas and theorems hold as follows :

LEMMA 1. Same.

LEMMA 2. Same.

LEMMA 3.
$$\|W^{-1}X\Delta z\| \leqslant \sqrt{2}\alpha\beta\pi,$$
$$\|WX^{-1}\Delta x\| \leqslant \frac{\sqrt{2}\alpha\pi}{1-\alpha}.$$

LEMMA 4. Same.

LEMMA 5. $$\|\Delta X \Delta z\| \leqslant \frac{\beta \alpha^2 \pi^2}{1-\alpha}.$$

LEMMA 6. $$\beta^* \geqslant \beta \left(1 - \frac{\alpha \pi}{\sqrt{n}} - \frac{2\alpha^2 \pi^2}{n(1-\alpha)} \right).$$

LEMMA 7. $$\beta^* \leqslant \lambda.$$

LEMMA 8. $$\|W^{-1}(X^* z^* - \beta^* W w) \leqslant \frac{5\beta\alpha^2\pi}{1-\alpha}.$$

THEOREM 1. Same.

THEOREM 2. Same.

Thus it becomes even more evident that our algorithm is a small step path-following method.

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