A LOWER BOUND FOR THE LAPLACIAN EIGENVALUES OF A GRAPH-PROOF OF A CONJECTURE BY GUO

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March 2008
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Abstract
We show that if $\mu_j$ is the $j$-th largest Laplacian eigenvalue, and $d_j$ is the $j$-th largest degree ($1 \leq j \leq n$) of a connected graph $\Gamma$ on $n$ vertices, then $\mu_j \geq d_j - j + 2$ ($1 \leq j \leq n - 1$). This settles a conjecture due to Guo.

Keywords: Graphs, Laplacian eigenvalues. JEL-code C0.

1 Introduction

Let $\Gamma$ be a finite simple (undirected, without loops) graph on $n$ vertices. Let $X = V\Gamma$ be the vertex set of $\Gamma$. Write $x \sim y$ to denote that the vertices $x$ and $y$ are adjacent. Let $d_x$ be the degree (number of neighbors) of $x$.

The adjacency matrix $A$ of $\Gamma$ is the 0-1 matrix indexed by $X$ with $A_{xy} = 1$ when $x \sim y$ and $A_{xy} = 0$ otherwise. The Laplacian matrix of $\Gamma$ is $L = D - A$, where $D$ is the diagonal matrix given by $D_{xx} = d_x$, so that $L$ has zero row and column sums.

The eigenvalues of $A$ are called eigenvalues of $\Gamma$. The eigenvalues of $L$ are called Laplacian eigenvalues of $\Gamma$. Since $A$ and $L$ are symmetric, these eigenvalues are real. Since $L$ is positive semidefinite (indeed, for any vector $u$ indexed by $X$ one has $u^\top Lu = \sum (u_x - u_y)^2$ where the sum is over all edges $xy$), it follows that the Laplacian eigenvalues are nonnegative. Since $L$ has zero row sums, 0 is a Laplacian eigenvalue. In fact the multiplicity of 0 as eigenvalue of $L$ equals the number of connected components of $\Gamma$.

Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$ be the Laplacian eigenvalues. Let $d_1 \geq d_2 \geq \ldots \geq d_n$ be the degrees, ordered nonincreasingly. We will prove that $\mu_i \geq d_i - i + 2$ with basically one exception.

2 Exception

Suppose $\mu_m = 0 < d_m - m + 2$. Then $d_m \geq m - 1$, and we find a connected component with at least $m$ vertices, hence with at least $m - 1$ nonzero Laplacian eigenvalues. It follows that this component has size precisely $m$, and hence $d_1 = \ldots = d_m = m - 1$, and the component is $K_m$. Now $\Gamma = K_m + (n - m)K_1$ is the disjoint union of a complete graph on $m$ vertices and $n - m$ isolated points. We’ll see that this is the only exception.
3 Interlacing

Suppose $M$ and $N$ are real symmetric matrices of order $n$ and $m$ with eigenvalues $\lambda_1(M) \geq \ldots \geq \lambda_m(M)$ and $\lambda_1(N) \geq \ldots \geq \lambda_m(N)$, respectively. If $M$ is a principal submatrix of $N$, then it is well known that the eigenvalues of $M$ interlace those of $N$, that is,

$$\lambda_i(N) \geq \lambda_i(M) \geq \lambda_{n-m+i}(N) \text{ for } i = 1, \ldots, m.$$ 

Less well-known, (see for example [3]) is that the interlacing inequalities also hold if $M$ is the quotient matrix of $N$ with respect to some partition $X_1, \ldots, X_m$ of $\{1, \ldots, n\}$. This means that $(M_{i,j})$ equals the average row sum of the block of $N$ with rows indexed by $X_i$ and columns indexed by $X_j$.

4 The lower bound

**Theorem 1** Let $\Gamma$ be a finite simple graph on $n$ vertices, with vertex degrees $d_1 \geq d_2 \geq \ldots \geq d_n$, and Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. If $\Gamma$ is not $K_m + (n - m)K_1$, then $\mu_m \geq d_m - m + 2$.

The case $m = 1$ of this theorem ($\mu_1 \geq d_1 + 1$ if there is an edge) is due to Grone & Merris [1]. The case $m = 2$ ($\mu_2 \geq d_2$ if the number of edges is not 1) is due to Li & Pan [4]. The case $m = 3$ is due to Guo [2], and he also conjectured the general result.

Let us separate out part of the proof as a lemma.

**Lemma 2** Let $S$ be a set of vertices in the graph $\Gamma$ such that each vertex in $S$ has at least $e$ neighbors outside $S$. Let $m = |S|$, $m > 0$. Then $\mu_m \geq e$. If $S$ contains a vertex adjacent to all other vertices of $S$, and $e > 0$, then $\mu_m \geq e + 1$.

**Proof** Consider the principal submatrix $L_S$ of $L$ with rows and columns indexed by $S$. Let $L(S)$ be the Laplacian of the subgraph induced on $S$. Then $L_S = L(S) + D$ where $D$ is the diagonal matrix such that $D_{ss}$ is the number of neighbors of $s$ outside $S$. Since $L(S)$ is positive semidefinite and $D \geq eI$, all eigenvalues of $L_S$ are not smaller than $e$, and by interlacing $\mu_m \geq e$.

Now suppose that $S = \{s_0\} \cup T$, where $s_0$ is adjacent to all vertices of $T$. Throw away all edges entirely outside $S$, and possibly also some edges leaving $S$, so that each vertex of $S$ has precisely $e$ neighbors outside $S$. Also throw away all vertices outside $S$ that now are isolated. Since these operations do not increase $\mu_m$, it suffices to prove the claim for the resulting situation.

Consider the quotient matrix $Q$ of $L$ for the partition of the vertex set $X$ into the $m + 1$ parts $\{s\}$ for $s \in S$ and $X \setminus S$. We find, with $r = |X \setminus S|,

$$Q = \begin{pmatrix}
    e + m - 1 & -1 & \ldots & -e \\
    -1 & -e \\
    \vdots & L_T & \vdots \\
    -1 & -e \\
    -e/r & -e/r & \ldots & -e/r & em/r
\end{pmatrix}$$
Consider the quotient matrix $R$ of $Q$ for the partition of the vertex set $X$ into the 3 parts $\{s_0\}, T, X \setminus S$. Then
\[
R = \begin{pmatrix}
e + m - 1 & 1 - m & -e \\
-1 & e + 1 & -e \\
-e/r & -e(m - 1)/r & em/r
\end{pmatrix}.
\]

The eigenvalues of $R$ are 0, $e + m$, and $e + me/r$, and these three numbers are also the eigenvalues of $Q$ for (right) eigenvectors that are constant on the three sets $\{s_0\}, T, X \setminus S$. The remaining eigenvalues $\theta$ of $Q$ belong to (left) eigenvectors perpendicular to these, so of the form $(0, u^\top, 0)$ with $\sum u = 0$. Now $L_T u = \theta u$, but $L_T = L(T) + (e + 1)I$ and $L(T)$ is positive semidefinite, so $\theta \geq e + 1$.

Since $me/r \geq 1$ (each vertex in $S$ has $e$ neighbors outside $S$ and $|S| = m$, so at most $me$ vertices in $X \setminus S$ have a neighbor in $S$), it follows that all eigenvalues of $Q$ except for the smallest are not less than $e + 1$. By interlacing, $\mu_m \geq e + 1$. \hfill $\Box$

**Proof** (of the theorem). Since $\mu_m \geq 0$ we are done if $d_m \leq m - 2$. So, suppose that $d_m \geq m - 1$.

Let $\Gamma$ have vertex set $X$, and let $x_i$ have degree $d_i$ ($1 \leq i \leq n$). Put $S = \{x_1, \ldots, x_m\}$. Put $e = d_m - m + 1$, then we have to show $\mu_m \geq e + 1$.

Each point of $S$ has at least $e$ neighbours outside. If each point of $S$ has at least $e + 1$ neighbours outside, then we are done by the lemma. And if not, then a point in $S$ with only $e$ neighbours outside is adjacent to all other vertices in $S$, and we are done by the lemma, unless $e = 0$.

Suppose first that $\Gamma$ is $K_m$, with a pendant edge attached, possibly together with some isolated vertices. Then $\Gamma$ has Laplacian spectrum $m + 1$, $m^{m-2}$, 1, $0^{n-m}$, with exponents denoting multiplicities, and equality holds. And if $\Gamma$ is $K_m + K_2 + (n - m - 2)K_1$, it has spectrum $m^{m-1}$, 2, $0^{n-m}$, and the inequality holds.

Let $T$ be the subset of $S$ whose vertices have at most $m - 2$ neighbours in $S$. The case $T = \emptyset$ has been treated above. For all $s \in T$ with fewer than $m - 2$ neighbours in $S$, delete edges between $s$ and $X \setminus S$ such that the row of $L_S$, indexed by $s$ gets row sum 1. Since $d_m = m - 1$ we can always do so. Also delete possible isolated vertices. By interlacing, $\mu_i$ has not been increased, so it suffices to show that for the remaining graph $\mu_m \geq 1$. Again consider the partition of $X$ into $m + 1$ parts consisting of $\{s\}$ for each $s \in S$, and $X \setminus S$, and let $Q$ be the corresponding quotient matrix of $L$. By interlacing it suffices to show that the second smallest eigenvalue of $Q$ is at least 1. Put $r = |X \setminus S|$ and $t = |T|$, then $0 < r \leq t$, and
\[
Q = \begin{pmatrix}
mI - J & -J \\
-J & L_T + (m - t + 1)I & -1 \\
0^\top & -1/ri & t/r
\end{pmatrix}
\]
($J$ is the all-ones matrix, and $0$ and $1$ denote the all-zeros and the all-ones vector, respectively). Now $Q$ has a $3 \times 3$ quotient matrix
\[
R = \begin{pmatrix}
t & -t & 0 \\
t - m & m - t + 1 & -1 \\
0 & -t/r & t/r
\end{pmatrix}
\]
The three eigenvalues of $R$ are $0 \leq x \leq y$ (say). We easily have that
\[
(1 - x)(1 - y) = \det(I - R) = t - 1 + (m - 1)(t/r - 1) \geq 0,
\]
and $\mu_m \geq e + 1$. \hfill $\Box$
which implies that $x \geq 1$ (since $x \leq y \leq 1$ contradicts $x + y = \text{trace } R > m + 1$). These three values are also eigenvalues of $Q$ with (right) eigenvectors constant over the partition. The remaining eigenvalues have (left) eigenvectors that are orthogonal to the characteristic vectors of the partition, and these eigenvalues remain unchanged if a multiple of $J$ is added to a block of the partition of $Q$. So they are also eigenvalues of

$$Q' = \begin{pmatrix} mI & O \\ O & L_T + (m-t+1)I \\ 0^\top & 0^\top \\ 0 & 1 \end{pmatrix},$$

which are clearly at least 1. So we can conclude that $\mu_m \geq 1$. □

## 5 Equality

There are many cases of equality (that is, $\mu_m = d_m - m + 2$), and we do not have a complete description.

For $m = 1$ we have equality, i.e., $\mu_1 = d_1 + 1$, if and only if $\Gamma$ has a vertex adjacent to all other vertices.

For $m = n$ we have equality, i.e., $0 = \mu_n = d_n - m + 2$, if and only if the complement of $\Gamma$ has maximum degree 1.

The path $P_3 = K_{1,2}$ has Laplacian eigenvalues 3, 1, 0 and degrees 2, 1, 1 with equality for $m = 0, 1, 2$, and is the only graph with equality for all $m$.

The complete graph $K_m$ with a pending edges attached at the same vertex has spectrum $a + m, m^{m-2}, 1^{m-1}$, 0, with exponents denoting multiplicities. Here $d_m = m - 1$, with equality for $m$ (and also for $m = 1$).

The complete graph $K_m$ with a pending edges attached at each vertex has spectrum $a + b, a^{b-1}, b^{a-1}, 0$. For $(a = 1$ or $a \geq b$ and $b \geq 2$ we have $d_2 = a = \mu_2$. This means that all graphs $K_{1,b}$ and all graphs between $K_{2,a}$ and $K_{a,a}$ have equality for $m = 2$.

The following describes the edge-minimal cases of equality.

**Proposition 3** Let $\Gamma$ be a graph satisfying $\mu_m = d_m - m + 2$ for some $m$, and such that for each edge $e$ the graph $\Gamma \setminus e$ has a different $m$-th largest degree or a different $m$-th largest eigenvalue. Then one of the following holds.

(i) $\Gamma$ is a complete graph $K_m$ with a single pending edge.

(ii) $m = 2$ and $\Gamma$ is a complete bipartite graph $K_{2,d}$.

(iii) $\Gamma$ is a complete graph $K_m$ with a pending edges attached at each vertex. Here $d_m = m + a - 1$.

**Proof** This is a direct consequence of the proof of the main result. □

Many further examples arise in the following way. Any eigenvector $u$ of $L = L(\Gamma)$ remains eigenvector with the same eigenvalue if one adds an edge between two vertices $x$ and $y$ for which $u_x = u_y$. If $\Gamma$ had equality,
and adding the edge does not change $d_m$ or the index of the eigenvalue $\mu_m$, then the graph $\Gamma'$ obtained by adding the edge has equality again.

The eigenvector for the eigenvalue $a + 1$ for $K_m$ with a pending edges attached at each vertex, is given by: 1 on the vertices of degree 1, and $-a$ on the vertices in the $K_m$. So, equality will persist when arbitrary edges between the outside vertices are added to this graph, as long as the eigenvalue keeps its index and $d_m$ does not change.

The eigenspace of $K_{a,b}$ for the eigenvalue $a$ is given by: values summing to 0 on the $b$-side, and 0 on the $a$-side. Again we can add edges.

For example, the graphs $K_{2,d}$ with $d \geq 2$ have $d_2 = d = \mu_2$ with equality for $m = 2$. Adding an edge on the 3-side of $K_{2,3}$ gives a graph with spectrum 5, 4, 3, 2, 0, and the eigenvalue 3 is no longer 2nd largest. Adding an edge on the 4-side of $K_{2,4}$ gives a graph with spectrum 6, 4, 4, 2, 2, 0, and adding two disjoint edges gives 6, 4, 4, 4, 2, 0, and in both cases we still have equality for $m = 2$.

References


