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A LOWER BOUND FOR THE LAPLACIAN EIGENVALUES OF A GRAPH-PROOF OF A CONJECTURE BY GUO

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A lower bound for the Laplacian eigenvalues of a graph—proof of a conjecture by Guo

A. E. Brouwer & W. H. Haemers

Abstract

We show that if \( \mu_j \) is the \( j \)-th largest Laplacian eigenvalue, and \( d_j \) is the \( j \)-th largest degree (\( 1 \leq j \leq n \)) of a connected graph \( \Gamma \) on \( n \) vertices, then \( \mu_j \geq d_j - j + 2 \) (\( 1 \leq j \leq n - 1 \)). This settles a conjecture due to Guo.

Keywords: Graphs, Laplacian eigenvalues. JEL-code C0.

1 Introduction

Let \( \Gamma \) be a finite simple (undirected, without loops) graph on \( n \) vertices. Let \( X = \mathcal{V} \) be the vertex set of \( \Gamma \). Write \( x \sim y \) to denote that the vertices \( x \) and \( y \) are adjacent. Let \( d_x \) be the degree (number of neighbors) of \( x \).

The adjacency matrix \( A \) of \( \Gamma \) is the 0-1 matrix indexed by \( X \) with \( A_{xy} = 1 \) when \( x \sim y \) and \( A_{xy} = 0 \) otherwise. The Laplacian matrix of \( \Gamma \) is \( L = D - A \), where \( D \) is the diagonal matrix given by \( D_{xx} = d_x \), so that \( L \) has zero row and column sums.

The eigenvalues of \( A \) are called eigenvalues of \( \Gamma \). The eigenvalues of \( L \) are called Laplacian eigenvalues of \( \Gamma \). Since \( A \) and \( L \) are symmetric, these eigenvalues are real. Since \( L \) is positive semidefinite (indeed, for any vector \( u \) indexed by \( X \) one has \( u^T L u = \sum (u_x - u_y)^2 \) where the sum is over all edges \( xy \)), it follows that the Laplacian eigenvalues are nonnegative. Since \( L \) has zero row sums, 0 is a Laplacian eigenvalue. In fact the multiplicity of 0 as eigenvalue of \( L \) equals the number of connected components of \( \Gamma \).

Let \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0 \) be the Laplacian eigenvalues. Let \( d_1 \geq d_2 \geq \ldots \geq d_n \) be the degrees, ordered nonincreasingly. We will prove that \( \mu_i \geq d_i - i + 2 \) with basically one exception.

2 Exception

Suppose \( \mu_m = 0 < d_m - m + 2 \). Then \( d_m \geq m - 1 \), and we find a connected component with at least \( m \) vertices, hence with at least \( m - 1 \) nonzero Laplacian eigenvalues. It follows that this component has size precisely \( m \), and hence \( d_1 = \ldots = d_m = m - 1 \), and the component is \( K_m \). Now \( \Gamma = K_m + (n - m)K_1 \) is the disjoint union of a complete graph on \( m \) vertices and \( n - m \) isolated points. We'll see that this is the only exception.
3 Interlacing

Suppose $M$ and $N$ are real symmetric matrices of order $n$ and $m$ with eigenvalues $\lambda_1(M) \geq \ldots \geq \lambda_n(M)$ and $\lambda_1(N) \geq \ldots \geq \lambda_m(N)$, respectively. If $M$ is a principal submatrix of $N$, then it is well known that the eigenvalues of $M$ interlace those of $N$, that is,

$$\lambda_i(N) \geq \lambda_i(M) \geq \lambda_{n-m+i}(N) \text{ for } i = 1, \ldots, m.$$ 

Less well-known, (see for example [3]) is that the interlacing inequalities also hold if $M$ is the quotient matrix of $N$ with respect to some partition $X_1, \ldots, X_m$ of $\{1, \ldots, n\}$. This means that $(M_{i,j})$ equals the average row sum of the block of $N$ with rows indexed by $X_i$ and columns indexed by $X_j$.

4 The lower bound

Theorem 1 Let $\Gamma$ be a finite simple graph on $n$ vertices, with vertex degrees $d_1 \geq d_2 \geq \ldots \geq d_n$, and Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. If $\Gamma$ is not $K_m + (n-m)K_1$, then $\mu_m \geq d_m - m + 2$.

The case $m = 1$ of this theorem ($\mu_1 \geq d_1 + 1$ if there is an edge) is due to Grone & Merris [1]. The case $m = 2$ ($\mu_2 \geq d_2$ if the number of edges is not 1) is due to Li & Pan [4]. The case $m = 3$ is due to Guo [2], and he also conjectured the general result.

Let us separate out part of the proof as a lemma.

Lemma 2 Let $S$ be a set of vertices in the graph $\Gamma$ such that each vertex in $S$ has at least $e$ neighbors outside $S$. Let $m = |S|$, $m > 0$. Then $\mu_m \geq e$. If $S$ contains a vertex adjacent to all other vertices of $S$, and $e > 0$, then $\mu_m \geq e + 1$.

Proof Consider the principal submatrix $L_S$ of $L$ with rows and columns indexed by $S$. Let $L(S)$ be the Laplacian of the subgraph induced on $S$. Then $L_S = L(S) + D$ where $D$ is the diagonal matrix such that $D_{ss}$ is the number of neighbors of $s$ outside $S$. Since $L(S)$ is positive semidefinite and $D \geq eI$, all eigenvalues of $L_S$ are not smaller than $e$, and by interlacing $\mu_m \geq e$.

Now suppose that $S = \{s_0\} \cup T$, where $s_0$ is adjacent to all vertices of $T$. Throw away all edges entirely outside $S$, and possibly also some edges leaving $S$, so that each vertex of $S$ has precisely $e$ neighbours outside $S$. Also throw away all vertices outside $S$ that now are isolated. Since these operations do not increase $\mu_m$, it suffices to prove the claim for the resulting situation.

Consider the quotient matrix $Q$ of $L$ for the partition of the vertex set $X$ into the $m+1$ parts $\{s\}$ for $s \in S$ and $X \setminus S$. We find, with $r = |X \setminus S|$,

$$Q = \begin{pmatrix}
  e + m - 1 & -1 & \ldots & -1 & -e \\
  -1 & -e & \ddots & \ddots & \ddots \\
  \vdots & \ddots & L_T & \ddots & \ddots \\
  -1 & \ddots & -e & \ddots & \ddots \\
  -e/r & -e/r & \ldots & -e/r & em/r
\end{pmatrix}.$$
Consider the quotient matrix $R$ of $Q$ for the partition of the vertex set $X$ into the 3 parts $\{s_0\}$, $T$, $X \setminus S$. Then

$$R = \begin{pmatrix} e + m - 1 & 1 - m & -e \\ -1 & e + 1 & -e \\ -e/r & -e(m - 1)/r & em/r \end{pmatrix}. $$

The eigenvalues of $R$ are $0$, $e + m$, and $e + me/r$, and these three numbers are also the eigenvalues of $Q$ for (right) eigenvectors that are constant on (left) eigenvectors perpendicular to these, so of the form $(0, u^\top, 0)$ with $\sum u = 0$. Now $LTu = \theta u$, but $LT = L(T) + (e + 1)I$ and $L(T)$ is positive semidefinite, so $\theta \geq e + 1$.

Since $me/r \geq 1$ (each vertex in $S$ has $e$ neighbors outside $S$ and $|S| = m$, so at most $me$ vertices in $X \setminus S$ have a neighbor in $S$), it follows that all eigenvalues of $Q$ except for the smallest are not less than $e + 1$.

By interlacing, $\mu_m \geq e + 1$.

**Proof** (of the theorem). Since $\mu_m \geq 0$ we are done if $d_m \leq m - 2$. So, suppose that $d_m \geq m - 1$.

Let $\Gamma$ have vertex set $X$, and let $x_i$ have degree $d_i$ ($1 \leq i \leq n$). Put $S = \{x_1, \ldots, x_m\}$. Put $e = d_m - m + 1$, then we have to show $\mu_m \geq e + 1$.

Each point of $S$ has at least $e$ neighbours outside. If each point of $S$ has at least $e + 1$ neighbours outside, then we are done by the lemma. And if not, then a point in $S$ with only $e$ neighbours outside is adjacent to all other vertices in $S$, and we are done by the lemma, unless $e = 0$.

Suppose first that $\Gamma$ is $K_m$ with a pending edge attached, possibly together with some isolated vertices. Then $\Gamma$ has Laplacian spectrum $m + 1$, $m^{m-2}$, $1$, $0^{n-m}$, with exponents denoting multiplicities, and equality holds. And if $\Gamma$ is $K_m + K_2 + (n - m - 2)K_1$, it has spectrum $m^{m-1}$, $2$, $0^{n-m}$, and the inequality holds.

Let $T$ be the subset of $S$ whose vertices have at most $m - 2$ neighbours in $S$. The case $T = \emptyset$ has been treated above. For all $s \in T$ with fewer than $m - 2$ neighbours in $S$, delete edges between $s$ and $X \setminus S$ such that the row of $L_S$, indexed by $s$ gets row sum $1$. Since $d_m = m - 1$ we can always do so. Also delete possible isolated vertices. By interlacing, $\mu_i$ has not been increased, so it suffices to show that for the remaining graph $\mu_m \geq 1$. Again consider the partition of $X$ into $m + 1$ parts consisting of $\{s\}$ for each $s \in S$, and $X \setminus S$, and let $Q$ be the corresponding quotient matrix of $L$. By interlacing it suffices to show that the second smallest eigenvalue of $Q$ is at least $1$. Put $r = |X \setminus S|$ and $t = |T|$, then $0 < r \leq t$, and

$$Q = \begin{pmatrix} mI & J & 0 \\ -J & L_T + (m - t + 1)I & -1 \\ 0^\top & -1/r & t/r \end{pmatrix} $$

($J$ is the all-ones matrix, and $0$ and $1$ denote the all-zeros and the all-ones vector, respectively). Now $Q$ has a $3 \times 3$ quotient matrix

$$R = \begin{pmatrix} t & -t & 0 \\ t - m & m - t + 1 & -1 \\ 0 & -t/r & t/r \end{pmatrix} $$

The three eigenvalues of $R$ are $0 \leq x \leq y$ (say). We easily have that

$$(1 - x)(1 - y) = \det(I - R) = t - 1 + (m - 1)(t/r - 1) \geq 0, $$

3
which implies that $x \geq 1$ (since $x \leq y \leq 1$ contradicts $x + y = \text{trace } R > m + 1$). These three values are also eigenvalues of $Q$ with (right) eigenvectors constant over the partition. The remaining eigenvalues have (left) eigenvectors that are orthogonal to the characteristic vectors of the partition, and these eigenvalues remain unchanged if a multiple of $J$ is added to a block of the partition of $Q$. So they are also eigenvalues of

$$Q' = \begin{pmatrix} mI & O & 0 \\ O & L_{\mathcal{T}} + (m - t + 1)I & 0 \\ 0^\top & 0^\top & 1 \end{pmatrix},$$

which are clearly at least 1. So we can conclude that $\mu_m \geq 1$.

\section{Equality}

There are many cases of equality (that is, $\mu_m = d_m - m + 2$), and we do not have a complete description.

For $m = 1$ we have equality, i.e., $\mu_1 = d_1 + 1$, if and only if $\Gamma$ has a vertex adjacent to all other vertices.

For $m = n$ we have equality, i.e., $0 = \mu_n = d_n - m + 2$, if and only if the complement of $\Gamma$ has maximum degree 1.

The path $P_3 = K_{1,2}$ has Laplacian eigenvalues 3, 1, 0 and degrees 2, 1, 1 with equality for $m = 0, 1, 2$, and is the only graph with equality for all $m$.

The complete graph $K_m$ with a pending edges attached at the same vertex has spectrum $a + m, m^{m-2}, 1^m, 0$, with exponents denoting multiplicities. Here $d_m = m - 1$, with equality for $m$ (and also for $m = 1$).

The complete graph $K_m$ with a pending edges attached at each vertex has spectrum $\frac{1}{2}(m + a + 1 \pm \sqrt{(m + a + 1)^2 - 4m})^{m-1}, a + 1, 1^{m(a-1)}, 0$, with $\mu_m = a + 1 = d_m - m + 2$.

The complete bipartite graph $K_{a,b}$ has spectrum $a + b, a^{b-1}, b^{a-1}, 0$. For $(a = 1$ or $a \geq b$) and $b \geq 2$ we have $d_2 = a = \mu_2$. This means that all graphs $K_{1,b}$, and all graphs between $K_{2,a}$ and $K_{a,a}$ have equality for $m = 2$.

The following describes the edge-minimal cases of equality.

\textbf{Proposition 3} Let $\Gamma$ be a graph satisfying $\mu_m = d_m - m + 2$ for some $m$, and such that for each edge $e$ the graph $\Gamma \setminus e$ has a different $m$-th largest degree or a different $m$-th largest eigenvalue. Then one of the following holds.

(i) $\Gamma$ is a complete graph $K_m$ with a single pending edge.

(ii) $m = 2$ and $\Gamma$ is a complete bipartite graph $K_{2,d}$.

(iii) $\Gamma$ is a complete graph $K_m$ with a pending edges attached at each vertex. Here $d_m = m + a - 1$.

\textbf{Proof} This is a direct consequence of the proof of the main result. \hfill \Box

Many further examples arise in the following way. Any eigenvector $u$ of $L = L(\Gamma)$ remains eigenvector with the same eigenvalue if one adds an edge between two vertices $x$ and $y$ for which $u_x = u_y$. If $\Gamma$ had equality,
and adding the edge does not change $d_m$ or the index of the eigenvalue $\mu_m$, then the graph $\Gamma'$ obtained by adding the edge has equality again.

The eigenvector for the eigenvalue $a + 1$ for $K_m$ with a pending edges attached at each vertex, is given by: 1 on the vertices of degree 1, and $-a$ on the vertices in the $K_m$. So, equality will persist when arbitrary edges between the outside vertices are added to this graph, as long as the eigenvalue keeps its index and $d_m$ does not change.

The eigenspace of $K_{a,b}$ for the eigenvalue $a$ is given by: values summing to 0 on the $b$-side, and 0 on the $a$-side. Again we can add edges.

For example, the graphs $K_{2,d}$ with $d \geq 2$ have $d_2 = d = \mu_2$ with equality for $m = 2$. Adding an edge on the 3-side of $K_{2,3}$ gives a graph with spectrum 5, 4, 3, 2, 0, and the eigenvalue 3 is no longer 2nd largest. Adding an edge on the 4-side of $K_{2,4}$ gives a graph with spectrum 6, 4, 4, 2, 2, 0, and adding two disjoint edges gives 6, 4, 4, 4, 2, 0, and in both cases we still have equality for $m = 2$.

References


