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de Zeeuw, A.J.

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Note on 'Nash and Stackelberg solutions in a differential game model of capitalism'

Aart de Zeeuw*

Tilburg University, 5000 LE Tilburg, The Netherlands
Free University, 1007 MC Amsterdam, The Netherlands

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Pohjola (1983) derives open-loop Stackelberg solutions for the Lancaster (1973) model of capitalism and compares the outcomes with the open-loop Nash outcome. Due to a shortcoming in the analysis only one open-loop Stackelberg solution with the workers as leader was found. This note shows that there are in fact infinitely many solutions. Furthermore, these solutions can be derived with standard optimal control techniques.

1. Introduction

Lancaster (1973) described capitalism as a differential game between workers and capitalists in which the workers determine their share of consumption in total output whereas the capitalists divide the remainder over investment and their own consumption. The purpose was to show the dynamic inefficiency of capitalism by comparing the noncooperative Nash outcome with the social optimum. Hoel (1978) extended this analysis by considering not only the social optimum but the whole set of Pareto efficient solutions.

Pohjola (1983) derived the open-loop Stackelberg solutions for the Lancaster model of capitalism under both workers' and capitalists' leadership. By comparing these outcomes with the open-loop Nash outcome it was shown that capitalism is in a stalemate, because both classes would prefer to act as the follower in the Stackelberg game. Başar, Haurie, and Ricci (1985) later

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analysed the feedback outcomes of the Lancaster game to show the impact of state information and the loss of commitment.

Following Wishart and Olsder (1979), Pohjola (1983) used generalised functions to handle the technical difficulties, but an error was made in the analysis. As a consequence of this only one open-loop Stackelberg solution was found in each of the two cases. In this note it is shown that there are in fact infinitely many solutions for the most probable parameter values. It is also shown that these solutions can be derived with standard optimal control techniques. The values of the objective functions are the same for all solutions, so that the remainder of the analysis in Pohjola (1983) still stands. In order to keep the note short only the game with the workers as leader is reconsidered.

Section 2 summarizes the Lancaster model of capitalism and derives by means of standard optimal control techniques the open-loop Stackelberg solutions under workers' leadership. Section 3 shows why Pohjola (1983) only found one solution. Section 4 is a short conclusion.

2. Open-loop Stackelberg solutions of the Lancaster game

The workers control their consumption rate u_1 and maximise their total consumption over a planning period

$$\int_0^T u_1(t) aK(t)\,\mathrm{d}t, \tag{1}$$

where K is the capital stock and a denotes the output–capital ratio. It is assumed that $c \le u_1(t) \le b$, $t \in [0,T]$, with $0 < c < b$ and $0.5 < b < 1$. The capitalists control the investment rate u_2 w.r.t. the remaining output ($0 \le u_2(t) \le 1$, $t \in [0,T]$) and maximise their total consumption over the planning period

$$\int_0^T [1-u_1(t)][1-u_2(t)] aK(t)\,\mathrm{d}t. \tag{2}$$

The capital accumulation can be written as

$$\dot{K}(t) = [1-u_1(t)] u_2(t) aK(t), \qquad K(0) = K_0. \tag{3}$$

The differential game (1)–(3) is called the Lancaster model of capitalism.

Suppose that the workers are the leader in the Stackelberg game and the capitalists are the follower. The Hamiltonian function for the rational reaction of the capitalists is given by

$$H_2(K,u_2,y_2,t) := [1-u_1(t)]\{1-u_2+y_2u_2\}aK. \tag{4}$$

Pontryagin's maximum principle yields the necessary conditions (3),

$$\hat{u}_2(t)=\begin{cases}1 & \text{if} \quad y_2(t)>1,\\ 0 & \text{if} \quad y_2(t)<1,\end{cases} \tag{5}$$

and

$$\dot{y}_2(t) = -[1-u_1(t)]\{1-\hat{u}_2(t)+y_2(t)\hat{u}_2(t)\}a,$$

$$y_2(T)=0. \tag{6}$$

According to Arrow's sufficiency theorem [see, e.g., Seierstad and Sydsaeter (1987, p. 107)], these conditions are also sufficient. The costate y_2 is continuous, and monotonically decreasing because $\dot{y}_2(t)<0$, $t\in(0,T)$. It follows that there are two possibilities:

(1) $y_2(0)\le 1$

In this case $u_2(t)=0$, $t\in(0,T]$, so that there is no investment and no capital accumulation. This can occur when the workers claim a too large consumption rate for themselves or when there is too little time to take advantage of the investment. The adjoint system (6) yields that the integral of u_1 over the time interval $[0,T]$ must be bigger than or equal to $T-1/a$. For T sufficiently large $[T>1/a(1-b)]$, this case can be ruled out, because $u_1(t)\le b$, $t\in[0,T]$.

(2) $y_2(0)>1$

In this case there is a point in time $\hat{t}_2$ with $y_2(\hat{t}_2)=1$, where the capitalists switch from full investment $u_2(t)=1$, $t\in[0,\hat{t}_2)$, to no investment $u_2(t)=0$, $t\in(\hat{t}_2,T]$.

The rational reaction of the capitalists leads to the following constraints for the maximisation problem of the workers:

(i) Before $\hat{t}_2$ there is capital accumulation according to

$$\dot{K}(t)=[1-u_1(t)]aK(t), \qquad K(0)=K_0, \tag{7}$$

and after $\hat{t}_2$ the capital stock is fixed: $K(t)=K(\hat{t}_2)$, $t\in[\hat{t}_2,T]$.

(ii) After $\hat{t}_2$ the consumption rate u_1 has to satisfy

$$\dot{y}_2(t) = -[1 - u_1(t)]a,$$

$$y_2(\hat{t}_2) = 1, \qquad y_2(T) = 0,$$

which yields

$$\int_{\hat{t}_2}^{T} u_1(t)\,\mathrm{d}t = T - \hat{t}_2 - 1/a. \tag{8}$$

The objective functional of the workers becomes

$$\int_0^{\hat{t}_2} u_1(t) aK(t)\,\mathrm{d}t + \{a(T - \hat{t}_2) - 1\}K(\hat{t}_2). \tag{9}$$

The workers have to choose $u_1(t)$, $t \in [0, \hat{t}_2]$, and $\hat{t}_2 \in (0, T)$ in order to maximise (9) subject to (7), and have to satisfy the constraint (8). The maximisation problem is a simple optimal control problem with a scrap value and a variable final time. The Hamiltonian function for this maximisation problem is given by

$$H_1(K, u_1, y_1, t) := \{u_1 + y_1[1 - u_1]\}aK. \tag{10}$$

Necessary and sufficient conditions [see, e.g., Seierstad and Sydsaeter (1987, pp. 397–399)] for the optimum are (7),

$$\hat{u}_1(t) = \begin{cases} b & \text{if} \quad y_1(t) < 1, \\ c & \text{if} \quad y_1(t) > 1, \end{cases} \tag{11}$$

$$\dot{y}_1(t) = -\{\hat{u}_1(t) + y_1(t)[1 - \hat{u}_1(t)]\}a,$$

$$y_1(\hat{t}_2) = a(T - \hat{t}_2) - 1, \tag{12}$$

and

$$\{a(T - \hat{t}_2) - 2\}[1 - \hat{u}_1(\hat{t}_2)]a\hat{K}(\hat{t}_2) = 0. \tag{13}$$

From (13) it follows that

$$\hat{t}_2 = T - 2/a, \tag{14}$$

so that $y_1(\hat{t}_2) = 1$. Because $\dot{y}_1(t) < 0$, $t \in (0, \hat{t}_2)$, y_1 is monotonically decreasing. This implies that $\hat{u}_1(t) = c$, $t \in [0, \hat{t}_2)$.

If $c \leq 0.5$, then the constraint (8) can be met, so that there is a multiplicity of open-loop Stackelberg solutions with the workers as leader:

$$\hat{u}_1(t) = c, \qquad \hat{u}_2(t) = 1, \qquad t \in [0, T - 2/a),$$

$$\int_{T-2/a}^{T} \hat{u}_1(t) = 1/a, \qquad \hat{u}_2(t) = 0, \qquad t \in (T - 2/a, T]. \tag{15}$$

If $c > 0.5$, then the constraint (8) cannot be met. To put it differently, the workers have to choose $\hat{t}_2$ in the time interval $[T - 1/a(1-b),\ T - 1/a(1-c)]$ in order to be able to meet the constraint (8). The optimal $\hat{t}_2$, given by (14), now lies on the right of this. Because the left-hand side of (13) is positive on this time interval, the maximisation problem of the workers has the corner solution

$$\hat{t}_2 = T - 1/a(1-c), \tag{16}$$

with $y_1(\hat{t}_2) > 1$ and again $\hat{u}_1(t) = c$, $t \in [0, \hat{t}_2)$. There is now only one open-loop Stackelberg solution with the workers as leader:

$$\hat{u}_1(t) = c, \qquad \hat{u}_2(t) = 1, \qquad t \in [0, T - 1/a(1-c)),$$

$$\hat{u}_1(t) = c, \qquad \hat{u}_2(t) = 0, \qquad t \in (T - 1/a(1-c), T]. \tag{17}$$

The conclusion is that in the case $c \leq 0.5$ there are infinitely many open-loop Stackelberg solutions with the workers as leader, given by (15). An example is the bang–bang control in Pohjola (1983) where the workers continue at the point in time $T - 2/a$ with the low consumption rate c and switch to the high consumption rate b at the point in time $T - (1-2c)/a(b-c)$. Another example is the situation where the workers claim an average consumption rate $\hat{u}_1(t) = 0.5$ on the whole time interval $(T - 2/a, T]$. It does not matter how the workers spread their total consumption over that time interval. As long as they announce consumption rates with the same level of modesty they induce the capitalists to invest longer than in the open-loop Nash outcome, which is the typical result in these analyses.

3. The derivation with generalised functions

Following Wishart and Olsder (1979), Pohjola (1983) performs the analysis in the much more complex space of generalised functions, in which the function $\hat{u}_2$, given by (5), has a time derivative equal to the delta function $-\delta(t-\hat{t}_2)$. One error is made in the analysis. In contrast to what is presented in table 4 [Pohjola (1983, p. 183)], the costate z [Pohjola (1983, eq. (24))] is constant and equal to $-y_1(\hat{t}_2)\hat{K}(\hat{t}_2)$ on the whole time interval $(\hat{t}_2, T]$ [see, e.g., Gel'fand and Shilov (1964)]. Note that the costate y_1 here is not the same as in section 2 of this note. It follows that the switching function B [Pohjola (1983, eq. (27))] is constant and equal to $[1-y_1(\hat{t}_2)]\hat{K}(\hat{t}_2)$ on the whole time interval $(\hat{t}_2, T]$. Therefore the switching function B cannot determine a switch in the optimal consumption rate $\hat{u}_1$ in this time interval.

One can proceed as follows. The adjoint systems for y_2 and y_1 [Pohjola (1983, eqs. (19), (23))] become on the time interval $[\hat{t}_2, T]$

$$\dot{y}_2(t) = -a[1-\hat{u}_1(t)],$$

$$y_2(\hat{t}_2) = 1, \qquad y_2(T) = 0,$$

and

$$\dot{y}_1(t) = -a\hat{u}_1(t),$$

$$y_1(T) = 0.$$

There are three possibilities:

(1) $B(t) > 0$, $t \in (\hat{t}_2, T]$ or $y_1(\hat{t}_2) < 1$, so that $\hat{u}_1(t) = b$, $t \in (\hat{t}_2, T]$. This leads to a contradiction with $b > 0.5$.

(2) $B(t) = 0$, $t \in (\hat{t}_2, T]$ or $y_1(\hat{t}_2) = 1$. This yields $\hat{t}_2 = T - 2/a$ and finally leads to the multiple open-loop Stackelberg solutions for $c \leq 0.5$, given by (15).

(3) $B(t) < 0$, $t \in (\hat{t}_2, T]$ or $y_1(\hat{t}_2) > 1$, so that $\hat{u}_1(t) = c$, $t \in (\hat{t}_2, T]$. This yields $\hat{t}_2 = T - 1/a(1-c)$ and finally leads to the single open-loop Stackelberg solution for $c > 0.5$, given by (17).

4. Conclusion

This note shows that there are infinitely many open-loop Stackelberg solutions for the Lancaster model of capitalism under workers' leadership. Furthermore, it is shown that it is not necessary to employ optimal control theory in the space of generalised functions, because the problem of the leader can be seen as a simple optimal control problem with a scrap value and a variable final time.

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