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COHERENCY OF THE INDIRECT TRANSLOG DEMAND SYSTEM WITH BINDING NONNEGATIVITY CONSTRAINTS

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We study the incoherency problem of the indirect translog demand system in which nonnegativity constraints on all goods are explicitly taken into account. Two examples illustrate the nature of the problem: The set of constraints which are binding is not always uniquely defined, and thus the model may not have a reduced form. We derive sufficient conditions for coherency which are related to regularity properties of the demand system; these conditions appear to be 'almost necessary' for regularity.

1. Introduction

During the past few years, several authors have studied the problem of estimating systems of demand equations if a significant proportion of the observations in the sample contains zero expenditures on one or more goods. Wales and Woodland (1983) formulate a model based on the Kuhn–Tucker approach, which is consistent with utility-maximizing behaviour and allows for random preferences. In case an explicit specification of the direct utility function is available, this approach seems very natural and intuitively appealing.

The individual is assumed to solve the problem

$$\max_{x} G(x;\vartheta,u) \quad \text{s.t.} \quad v^T x \leq 1 \quad \text{and} \quad x \geq 0. \tag{1}$$

Here x is a K-dimensional vector of quantities, v is a K-dimensional vector of normalized prices ($v_i = p_i/M$, $i = 1,\dots,K$, where p_i is the price of good i and M is income), G is the (direct) utility function, depending on a vector ϑ of parameters (to be estimated) and a vector u of random variables (to allow for preference variation between individuals).

Ransom (1987) comments on the relationship between the Wales and Woodland model and the 'simultaneous equation' Tobit model of Amemiya

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(1974) for the special case of a quadratic utility function. He also discusses, for this special case, the issue of internal consistency, i.e., the question whether there is a one-to-one correspondence between all possible realizations of the random variables (except, perhaps, for those in a set of probability zero) and all possible quantity vectors x that could be observed (alternatively stated: do the probabilities sum to one?). In the literature, this property is also called coherency of the model [see, e.g., Gourieroux et al. (1980)]. Ransom finds that internal consistency is guaranteed if the (quadratic) direct utility function is strictly concave on the set of feasible quantities $\{x; x \geq 0\}$. This result is no surprise, since, if the utility function is strictly concave, (1) is an example of maximizing a strictly concave function over a (nonempty) compact convex set and standard Kuhn–Tucker theory assures us that this problem has one and only one solution.

Lee and Pitt (1986) propose to use duality theory and shadow prices in the Wales and Woodland model to be able to deal with more flexible demand systems for which no explicit specification of the direct utility function can be given. In particular, they pay attention to the indirect translog system [see Christensen, Jorgenson, and Lau (1975)]. Lee and Pitt do not address the issues of well-behaving of the indirect or direct utility function or internal consistency. In an earlier paper [Van Soest and Kooreman (1986)], we showed with a simple example in the three-goods case, that internal consistency can be severely violated for certain values of parameters of the indirect translog specification (which do not guarantee concavity of the direct utility function in a large enough region). In this paper we show that, provided that the parameters satisfy certain conditions, internal consistency is guaranteed and, moreover, that the direct utility function behaves well in the feasible region of the quantity-space, $\{x; v^T x \leq 1 \text{ and } x \geq 0\}$.

2. The framework

We start from the indirect utility function

$$H(p, M) = \sum_{i=1}^{K} \alpha_i \log(p_i/M) + \tfrac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \beta_{ij} \log(p_i/M) \log(p_j/M),$$

where $p = (p_1, \ldots, p_K)^T$ is a vector of prices, M denotes income, $\alpha = (\alpha_1, \ldots, \alpha_K)^T$ is a parameter vector, which is normalized such that $\sum_{i=1}^{K} \alpha_i = -1$ [α may include a random component, as in Lee and Pitt (1986)], and $B = (\beta_{ij})_{i,j=1}^{K}$ is a matrix of parameters. Without loss of generality, we assume that B is symmetric.

Introducing a vector $v = p/M$ of normalized prices, the indirect utility function can be written as

$$H(v) = \sum_{i=1}^{K} \alpha_i \log v_i + \tfrac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \beta_{ij} \log v_i \log v_j .$$

Note that

$$\frac{\partial H}{\partial M} = \left\{ 1 - \sum_{i=1}^{K} \sum_{j=1}^{K} \beta_{ij} \log v_i \right\} \Big/ M .$$

Since the indirect utility function must be increasing as a function of M, a necessary condition for the use of this specification in the neighbourhood of a given vector v is

$$D^*(v) = 1 - \sum_{i=1}^{K} \sum_{j=1}^{K} \beta_{ij} \log v_j > 0. \tag{2}$$

Application of Roy's identity yields the notional (uncompensated) demand equations

$$s_i^*(v) = z_i^*(v)/D^*(v),$$

where

$$z_i^*(v) = -\alpha_i - \sum_{j=1}^{K} \beta_{ij} \log v_j , \qquad i = 1, \dots, K.$$

Here $s^*(v) = (s_1^*(v), \dots, s_K^*(v))^T$ is the vector of optimal budget shares, some of which may be negative. We write $z^*(v) = (z_1^*(v), \dots, z_K^*(v))^T$.

3. Rationing

In the following, we consider a fixed vector v and we derive the optimal shares s_i that satisfy $s_i \geq 0$ ($i = 1, \dots, K$). With $s = (s_1, \dots, s_K)^T$ this can be written as $s \geq 0$.

Using the Kuhn–Tucker conditions, as in Lee and Pitt (1986) (and thus implicitly assuming, that the direct utility function 'behaves well in some large enough region'), the maximization problem can be written as:

Find a vector $\pi = (\pi_1, \ldots, \pi_K)^T$ of normalized shadow prices and a vector $s = (s_1, \ldots, s_k)^T$ of (optimal) shares such that

$$s_i = \left\{ z_i^* + \sum_{j=1}^{K} \beta_{ij} \left(\log v_j - \log \pi_j \right) \right\}$$

$$\Bigg/ \left\{ D^* + \sum_{j=1}^{K} \sum_{k=1}^{K} \beta_{jk} \left(\log v_j - \log \pi_j \right) \right\}, \qquad i = 1, \ldots, K,$$

and

$$s \geq 0, \qquad \pi \leq v, \qquad s^T(v - \pi) = 0.$$

With $y_j = \log v_j - \log \pi_j$, $j = 1, \ldots, K$, $y = (y_1, \ldots, y_K)^T$, and $e = (1, \ldots, 1)^T$, this can be written as:

Find a vector y satisfying

$$y \geq 0,$$

$$(z^* + By)/(D^* + y^T Be) \geq 0, \tag{3}$$

$$y^T(z^* + By) = 0.$$

[The corresponding shares are then given by $s = \{z^* + By\}/\{D^* + y^T Be\}$.]

The problem of internal consistency is illustrated by the following two examples with three commodities.

Example 1. Let

$$B = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = (1, 1, 1)^T.$$

Fig. 1 depicts the number of solutions of (3), for each realization of the vector $\alpha = (\alpha_1, \alpha_2, -1 - \alpha_1 - \alpha_2)^T$ of random variables. Each solution is characterized by some regime, i.e., a subset of $\{1, 2, 3\}$ indicating which constraints are binding.

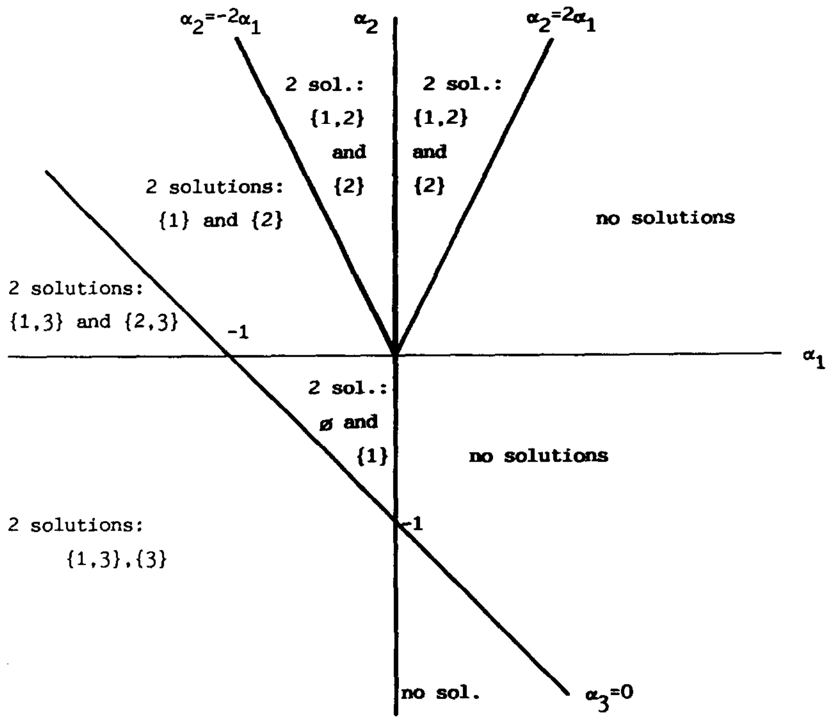

Fig. 1. *Example 1*: an incoherent demand system. The number of solutions and the corresponding regimes for each $(\alpha_1, \alpha_2, -1-\alpha_1-\alpha_2)^T$.

For example, regime {2} yields the system

$$y_1 = 0,$$

$$y_3 = 0,$$

$$(-\alpha_2 + 4y_2)/(1+6y_2) = 0,$$

$$y_2 > 0,$$

$$(\alpha_1 + 2y_2)/(1+6y_2) > 0,$$

$$-\alpha_3/(1+6y_2) > 0.$$

And this system yields a solution iff $\alpha_3 < 0$, $\alpha_2 > 0$, and $\alpha_2 > 2\alpha_1$. Fig. 1 indicates for each vector α those regimes that yield a solution. For α with $\alpha_1 > 0$ and $\alpha_2 < 2\alpha_1$, no solution is found, and for other α's (except for some set of probability zero) there are two solutions. For none of the α's there is exactly one solution, so coherency is severely violated.

Example 2. Let

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = (1,1,1)^T.$$

The regions in α-space for which solutions for the different regimes exist are given in fig. 2. In this case, for each α, system (3) yields exactly one solution and the model is internally consistent.

The following proposition gives sufficient conditions for internal consistency of the model. It is easily proved using the notation introduced to describe system (3).

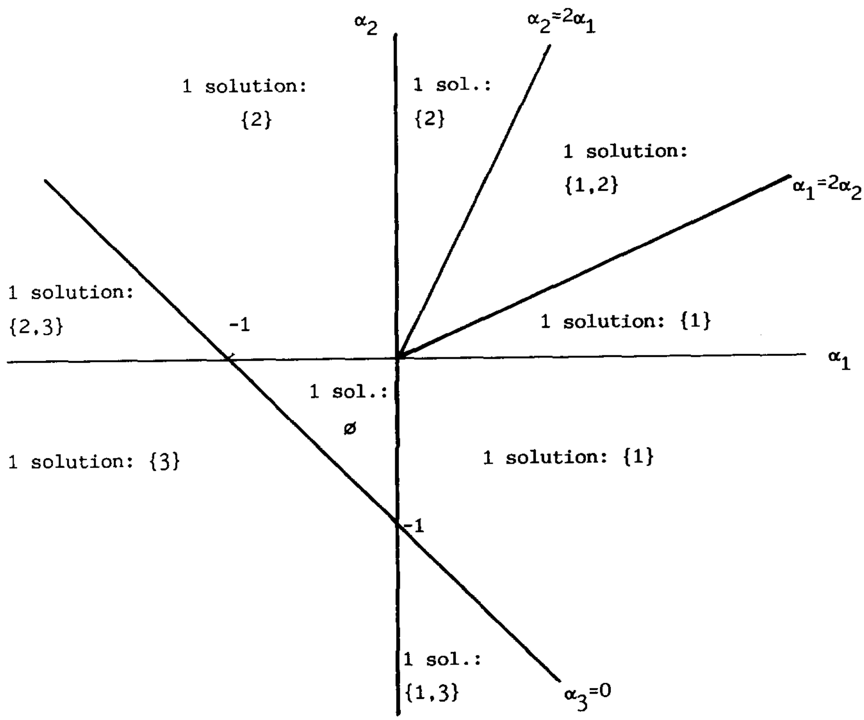

Fig. 2. *Example 2*: a coherent demand system. The number of solutions and the corresponding regimes for each $(\alpha_1, \alpha_2, -1-\alpha_1-\alpha_2)^T$.

Proposition 1. Assume

(A1) *B is positive definite,*
(A2) $D^* > 0$,
(A3) $Be \geq 0$,

Then the problem given by (3) has one and only one solution for all K-dimensional vectors α.

Proof. $D^* > 0$ and $Be \geq 0$ imply that $D^* + y^T Be > 0$ for all $y \geq 0$. This means, that (3) can be rewritten as:

Find a vector y satisfying

$$\begin{aligned} &y \geq 0, \\ &z^* + By \geq 0, \\ &y^T(z^* + By) = 0. \end{aligned} \tag{4}$$

This problem is known in the literature as the linear complementarity problem [cf. Amemiya (1974) or Eaves (1971)]. It has one unique solution if all principal minors of B are positive. Since B is symmetric, this means that B must be positive definite. ■

Assumptions (A2) and (A3) are necessary to guarantee

$$D^*(\pi) = D^* + y^T Be > 0,$$

for all vectors of shadow prices π corresponding to all feasible vectors of shares $s \geq 0$. As stated in (2), $D^*(\pi) > 0$ is a necessary condition for good behaviour of the indirect utility function, and therefore assumptions (A2) and (A3) seem quite reasonable.

Assumption (A1) arises in a similar way, if we consider concavity of the corresponding cost function.

4. Concavity in the feasible region of the shares space

The cost function is only well-behaving in the neighbourhood of a vector π of shadow prices corresponding to a given vector s of shares, if the matrix of second-order partial derivatives of the cost function with respect to prices is negative semidefinite. This condition is equivalent to the requirement that the matrix of Allen–Uzawa elasticities of substitution be negative semidefinite, which, in case of the translog function, means that the matrix

$$C(s) = -\Delta(s) + ss^T - \left(D^* + y^T Be\right)^{-1} \times \left\{ B - s(Be)^T - Bes^T + e^T Bess^T \right\}$$

must be negative semidefinite [see, e.g., Barnett et al. (1985)]. Here $\Delta(s)$ denotes the diagonal matrix with s_i $(i = 1, \ldots, K)$ on the diagonal and y corresponds to s, i.e., $y_j = \log v_j - \log \pi_j$, where $\pi = (\pi_1, \ldots, \pi_K)^T$ is the vector of shadow prices corresponding to the vector s of shares.

Proposition 2. Assume

(A1) *B is positive definite,*
(A2) *$D^* > 0$,*
(A3) *$Be \geq 0$.*

Then concavity holds for the optimal vector of shares, i.e., the vector $s \geq 0$ corresponding to the solution of (3).

Proof. The proof is an immediate consequence of the following two lemmas.

Lemma 1. If $s \geq 0$ and $s_1 + \ldots + s_k = 1$, then $\Delta(s) - ss^T$ is positive semidefinite.

Lemma 2. If B is positive definite, then $B - s(Be)^T - Bes^T + e^T Bess^T$ is positive semidefinite for all s.

Proof of Lemma 1.[1] Without loss of generality we may assume $s_i > 0$ $(i = 1, \ldots, K)$. We must prove

$$F(x) := x^T \left\{ \Delta(s) - ss^T \right\} x \geq 0 \quad \text{for all vectors } x.$$

Using standard Lagrange technique, it is easy to show that on the ellipsoid

$$\left\{ x = (x_1, \ldots, x_K)^T;\ \sum_{j=1}^{K} s_j x_j^2 = 1 \right\}$$

the minimum value attained by F equals zero (and the maximum is one). Since $F(\lambda x) = \lambda^2 F(x)$ for all real λ, this implies that $F(x) \geq 0$ for all vectors x. ■

[1]A more general version of this lemma is proved in Bekker (1986, p. 69).

Proof of Lemma 2. Let x be any K-dimensional nonzero vector. Then

$$x^T\left(B - s(Be)^T - Bes^T + e^TBess^T\right)x$$

$$= x^TBx - 2(x^Ts)(x^TBe) + (x^Ts)^2 e^TBe.$$

This is a quadratic function of x^Ts, with discriminant

$$D = 4\left\{(x^TBe)^2 - (x^TBx)(e^TBe)\right\}.$$

If B is positive definite, then $D \leq 0$ (Cauchy–Schwarz), so the function does not change sign and, because $x^TBx > 0$, it is always nonnegative. ■

5. Conclusion

Proposition 1 can intuitively be seen as a consequence of Proposition 2: Concavity of the cost function corresponds to concavity of the direct utility function on the feasible region $S = \{s;\, s \geq 0 \text{ and } s^Te \leq 1\}$. Since S is convex, the concave direct utility function attains a unique maximum on S.

The main problem left seems to be the question whether the assumptions (A1), (A2), and (A3) in Propositions 1 and 2 are necessary. If they are not, imposition of them might destroy the second-order flexibility of the indirect translog specification. It is easy to see, that concavity at all vertices $(0,\dots,0,1,0,\dots,0)^T$ of the feasible region S implies, together with assumptions (A2) and (A3), that all principal $(K-1)\times(K-1)$ submatrices of B must be positive definite. This means that, if observations in the neighborhood of all these vertices exist, the assumptions do not seem to strong. But what can be left out if, for instance, there is one commodity with a positive share for everyone, as often happens in practice?

Finally, note that assumptions (A1), (A2), and (A3) do not say anything about α. Hence if the assumptions are met the α_i's may vary randomly and no truncation of their distribution is necessary to assure that probabilities add up to one. Random variation of the α_i's implies that all feasible vectors of shares could be optimal, and thus makes the assumptions (A1), (A2), and (A3) 'almost necessary'.

References

Amemiya, T., 1974, Multivariate regression and simultaneous equation models when the dependent variables are truncated normal, Econometrica 42, 999–1012.

Barnett, W.A., Y.W. Lee, and M.D. Wolfe, 1985, The three-dimensional global properties of the minflex Laurent, generalized Leontief and translog flexible functional forms, Journal of Econometrics 30, 3–31.

Bekker, P., 1986, Essays on identification in linear models with latent variables (Tilburg University, Tilburg).
Christensen, L., D. Jorgenson, and L.J. Lau, 1975, Transcendental logarithmic utility functions, American Economic Review 53, 367–383.
Eaves, B.C., 1971, The linear complementarity problem, Management Science 17, 612–634.
Gourieroux, C., J. Laffont, and A. Montfort, 1980, Coherency conditions in simultaneous linear equation models with endogenous switching regimes, Econometrica 48, 675–695.
Lee, L.F. and M.M. Pitt, 1986, Microeconometric demand systems with binding non-negativity constraints: The dual approach, Econometrica 54, 1237–1243.
Ransom, M.R., 1987, A comment on consumer demand systems with binding non-negativity constraints, Journal of Econometrics 34, 354–359.
Van Soest, A. and P. Kooreman, 1986, Comment on 'Microeconometric demand systems with binding non-negativity constraints: The dual approach,' Research memorandum 226 (Tilburg University, Tilburg).
Wales, T.J. and A.D. Woodland, 1983, Estimation of consumer demand systems with binding non-negativity constraints, Journal of Econometrics 21, 263–285.