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Abstract

In this paper, the classical theory of two-person cooperative games is extended to two-person cooperative games with interval uncertainty. The core, balancedness, superadditivity and related topics are studied. Solutions called ψ^α-values are introduced and characterizations are given.

Keywords: Cooperative game theory, Interval uncertainty, Core, Value, Balancedness

Classification: JEL code C71

1 Introduction

Classical cooperative game theory deals with coalitions who coordinate their actions and pool their winnings. One of the problems is how to divide the rewards or costs among the members of the formed coalition. Generally, the situations here are considered from a deterministic point of view. For further information about classical cooperative game theory the reader is referred to the books by Branzei et al. (2005) and Tijs (2003). However, in most economical situations potential rewards or costs are not known precisely, but often it is possible to estimate intervals to which they belong. In Yager and Kreinovich (2000) an algorithm for fair division under interval uncertainty is

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presented using the work on interval analysis by Moore (1979). Cooperative games arising from bankruptcy situations with interval uncertainty, called (cooperative) interval games, were introduced and analyzed by Branzei et al. (2003) and Branzei et al. (2004). In a classical bankruptcy situation a certain amount of money (estate) has to be divided among some people (claimants) who have individual claims on the estate, and the total claim is weakly larger than the estate (cf. Aumann and Maschler (1985), Curiel et al. (1987), O'Neill (1982)). When the estate and/or the claims may belong to intervals of real numbers we have bankruptcy situations under interval uncertainty. In Carpente et al. (2005) a method is proposed to associate a coalitional interval game to each strategic game. Throughout the above literature we can find motivations, from different points of view, for the study of interval games. Here, *a cooperative interval game* is defined as an ordered pair $< N, w >$ where N is the set of players, and w is the characteristic function which assigns to each coalition S a closed interval $w(S)$ in $\mathcal{R}$. We introduce the notion of the core set of a cooperative interval game and various notions of balancedness. Then we focus on two-person (cooperative) interval games and extend to these games well-known results for classical two-person cooperative games. Moreover, we define and analyze specific solution concepts on the class of two-person interval games, such as the mini-core set and the ψ^α-values. The mini-core set is determined by considering the upper bound of the worths of the one-player coalitions in the two-person case. If a mini-core allocation is proposed, then no one-player coalition has any incentive to split off from the grand coalition for each selection of the interval games.

The paper is organized as follows. In Section 2 we recall basic definitions and results on balancedness for classical cooperative games. In Section 3 we introduce some definitions for n-person cooperative games under interval uncertainty and focus on balancedness. Section 4 deals with two-person interval games and their solutions: balancedness, the mini-core set and its relation with the core set, the ψ^α-values and their axiomatic characterizations. We conclude in Section 5 with some remarks on further research.

2 Preliminaries on classical games in coalitional form

We give in the following some definitions and a theorem concerning classical games in coalitional form. For an extensive description of classical games in coalitional form see Tijs (2003) and Branzei et al. (2005).

A *cooperative n-person game in coalitional form* is an ordered pair $< N, v >$, where $N = \{1, 2, ..., n\}$ *(the set of players)* and $v : 2^N \to \mathcal{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$. This function v is called the characteristic function of the game, $v(S)$ is called the *worth (or value)* of coalition S. Often we identify a game $< N, v >$ with its characteristic function v.

The set G^N of coalitional games with player set N forms with the usual operators of addition and scalar multiplication of functions a $(2^{|N|} - 1)$-dimensional linear space; a basis of this space is supplied by the *unanimity games* u_T (or $< N, u_T >$), $T \in 2^N \setminus \{\emptyset\}$, which are defined by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S \\ 0 & \text{otherwise.} \end{cases}$$

One can easily check that for each $v \in G^N$ we have $v = \sum_{T \in 2^N \setminus \emptyset} c_T u_T$ with $c_T = \sum_{S: S \subset T} (-1)^{|T| - |S|} v(S)$.

A *payoff vector* $x \in \mathcal{R}^n$ is called an imputation for the game $< N, v >$ if

(i) x is *individually rational*, i.e., $x_i \geq v(\{i\})$ for all $i \in N$,

(ii) x is *efficient (Pareto optimal)*, i.e., $\sum_{i=1}^n x_i = v(N)$.

The set of imputations of $< N, v >$ is denoted by $I(v)$. Note that $I(v) = \emptyset$ if and only if $v(N) < \sum_{i \in N} v(\{i\})$.

The core of a game (cf. Gillies (1953)) is a central set-valued solution concept in game theory.

The *core* of a game $< N, v >$ is the set

$$C(v) = \left\{ x \in I(v) | \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

If $x \in C(v)$, then no coalition $S \neq N$ has any incentive to split off if x is the proposed reward allocation in N, because the total amount $\sum_{i \in S} x_i$ allocated

to S is not smaller than the amount $v(S)$ which the players can obtain by forming the subcoalition.
For a two-person game $< N, v >$, $I(v) = C(v)$.
A map $\lambda : 2^N \setminus \{\emptyset\} \to \mathcal{R}_+$ is called a *balanced map* if $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N$. Here e^S is the *characteristic vector* for coaliton S with

$$e_i^S = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \in N \setminus S. \end{cases}$$

A collection B of coalitions is called a *balanced collection* if there is a balanced map λ such that

$$B = \left\{ S \in 2^N \setminus \{\emptyset\} \, | \lambda(S) > 0 \right\}.$$

An n-person game $< N, v >$ is called a *balanced game* if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathcal{R}_+$ we have $\sum_S \lambda(S) v(S) \leq v(N)$.
The importance of this notion becomes clear in the following theorem proved by Bondareva (1963) and Shapley (1967). This theorem characterizes games with a non-empty core.

THEOREM 2.1. *Let $< N, v >$ be an n-person game. Then the following two assertions are equivalent:*

(i) $C(v) \neq \emptyset$,

(ii) $< N, v >$ *is a balanced game.*

Let $\pi(N)$ be the set of all permutations $\sigma : N \to N$.

The set $P^\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$ consists of all predecessors of i with respect to the permutation σ.
Let $v \in G^N$ and $\sigma \in \pi(N)$. The *marginal vector* $m^\sigma(v) \in \mathcal{R}^n$ with respect to σ and v has as i-th coordinate $m_i^\sigma(v) = v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))$ for each $i \in N$.
The Shapley value (cf. Shapley (1967)) is one of the most interesting one-point solution concepts in classical cooperative game theory. The Shapley value associates to each n-person game one (payoff) vector in $\mathcal{R}^n$.
The Shapley value $\Phi(v)$ of a game $v \in G^N$ is the average of the marginal vectors of the game, i.e.

$$\Phi(v) = \frac{1}{n!} \sum_{\sigma \in \pi(N)} m^\sigma(v).$$

Marginal vectors of a two-person game $< N, v >$ are

$$m^{(12)}(v) = (v(\{1\}), v(\{1,2\}) - v(\{1\})),$$

and

$$m^{(21)}(v) = (v(\{1,2\}) - v(\{1\}), v(\{2\})).$$

For a two person game $< N, v >$ we have

$$\Phi_i(v) = v(\{i\}) + \frac{v(\{1,2\}) - v(\{1\}) - v(\{2\})}{2}, \ i = \{1,2\}.$$

Note that for a two-person game $< N, v >$, the Shapley value is the standard solution which is in the middle of the core and the marginal vectors are the extreme points of the core whose average gives the Shapley value.
A game $< N, v >$ is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$. In a superadditive game it is advantageous for the players to cooperate.
A two-person cooperative game $< N, v >$ is superadditive if and only if $v(\{1\}) + v(\{2\}) \leq v(\{1,2\})$ holds. Note that a two-person cooperative game $< N, v >$ is superadditive if and only if the game is balanced.

3 Cooperative games under interval uncertainty

In the following we will develop a theory of cooperation under interval uncertainty, inspired by the classical cooperative game theory (cf. Branzei et al. (2005) and Tijs (2003)).
A *cooperative n-person* interval game in coalitional form is an ordered pair $< N, w >$ where $N := \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathcal{R})$ is the characteristic function which assigns to each coalition $S \in 2^N$ a closed interval $w(S) \in I(\mathcal{R})$ where $I(\mathcal{R})$ is the set of all closed intervals in $\mathcal{R}$ such that $w(\emptyset) = [0, 0]$.
For each $S \in 2^N$, the *worth set* (or *worth interval*) of the coalition S in the interval game, $w(S)$, is a closed interval which will be denoted by $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the lower bound and $\overline{w}(S)$ is the upper bound of $w(S)$.
Note that if all the worth intervals are *degenerate intervals*, i.e., $\underline{w}(S) = \overline{w}(S)$, then the interval game $< N, w >$ corresponds to the classical cooperative game $< N, v >$ where $v(S) = \underline{w}(S)$.

Let $< N, w >$ be an interval game; then $v : 2^N \to \mathcal{R}$ is called a *selection* of w if $v(S) \in w(S)$ for each $S \in 2^N$. We denote the set of selections of w by $Sel(w)$.
The imputation set of an interval game $< N, w >$ is defined by

$$I(w) = \cup \{I(v) | v \in Sel(w)\}.$$

The *core set* of an interval game $< N, w >$ is defined by

$$C(w) = \cup \{C(v) | v \in Sel(w)\}.$$

$C(w) \neq \emptyset$ if and only if there exists a $v \in Sel(w)$ with $C(v) \neq \emptyset$.
The family of all interval games with player set N is denoted by IG^N.
If all the worth intervals of an interval game $w \in IG^N$ are degenerate intervals, then $I(w) = I(\underline{w}) = I(\overline{w})$ and $C(w) = C(\underline{w}) = C(\overline{w})$.
Note that $v(S) \in w(S)$ is a real number, but $w(S) = [\underline{w}(S), \underline{w}(S)]$ is a degenerate interval which is a set consisting of one point.
An interval game $< N, w >$ is *strongly balanced* if for each balanced map λ it holds that $\sum \lambda(S)\overline{w}(S) \leq \underline{w}(N)$. The family of all strongly balanced interval games with player set N is denoted by BIG^N.

PROPOSITION 3.1. *Let $< N, w >$ be an interval game. Then, the following three statements are equivalent:*

(i) *For each $v \in Sel(w)$ the game $< N, v >$ is balanced.*

(ii) *For each $v \in Sel(w)$, $C(v) \neq \emptyset$.*

(iii) *The interval game $< N, w >$ is strongly balanced.*

Proof. $(i) \Leftrightarrow (ii)$ follows from Theorem 2.1.
$(i) \Leftrightarrow (iii)$ follows using the inequalities $\underline{w}(N) \leq v(N) \leq \overline{w}(N)$ and $\sum \lambda(S)\underline{w}(S) \leq \sum \lambda(S)v(S) \leq \sum \lambda(S)\overline{w}(S)$ for each balanced map λ. □

It follows from Proposition 3.1 that for a strongly balanced game $< N, w >$, $C(w) \neq \emptyset$ since for all $v \in Sel(w)$, $C(v) \neq \emptyset$.
We call an interval game $< N, w >$ *strongly unbalanced,* if there exists a balanced map λ such that $\sum \lambda(S)\underline{w}(S) > \overline{w}(N)$. Then, $C(v) = \emptyset$ for all $v \in Sel(w)$, which implies that $C(w) = \emptyset$.
If all the worth intervals of an interval game $< N, w >$ are degenerate intervals then strongly balancedness corresponds to balancedness and strongly unbalancedness corresponds to unbalancedness in classical cooperative game $< N, v >$.

4 On two-person cooperative games under interval uncertainty

4.1 Balancedness and related topics

We simply use $w(1), w(2)$ and $w(1,2)$ instead of $w(\{1\}), w(\{2\})$ and $w(\{1,2\})$. Let $< N, w >$ be a two-person interval game. Then, we define:

(i) the *pre-imputation set*

$$I^*(w) := \left\{x \in \mathcal{R}^2 | x_1 + x_2 \in w(1,2)\right\},$$

(ii) the *imputation set*

$$I(w) := \left\{x \in \mathcal{R}^2 | x_1 \geq \underline{w}(1), x_2 \geq \underline{w}(2), x_1 + x_2 \in w(1,2)\right\},$$

(iii) the *mini-core set*

$$MC(w) := \left\{x \in \mathcal{R}^2 | x_1 \geq \overline{w}(1), x_2 \geq \overline{w}(2), x_1 + x_2 \in w(1,2)\right\},$$

(iv) the *core set*

$$C(w) := \left\{x \in \mathcal{R}^2 | x_1 \geq \underline{w}(1), x_2 \geq \underline{w}(2), x_1 + x_2 \in w(1,2)\right\}.$$

Notice that for two-person interval games the imputation set and the core set are equal. Moreover, if an interval game is strongly balanced then its mini-core set is nonempty and it is a subset of the core set of the game.
The next example is intended to give insight into the core set and mini-core set of a two person (strongly balanced) game $< N, w >$.

Example 4.1. *Let* $N = \{1, 2\}$, $w \in IG^{\{1,2\}}$ *such that*

$$w(\emptyset) = [0, 0], w(1) = [1, 3], w(2) = [2, 5], w(1, 2) = [10, 12].$$

In Figure 1, the mini-core set and the core set are depicted.

This is a strongly balanced game since $\overline{w}(1)+\overline{w}(2) = 3+5 \leq \underline{w}(1, 2) = 10$.

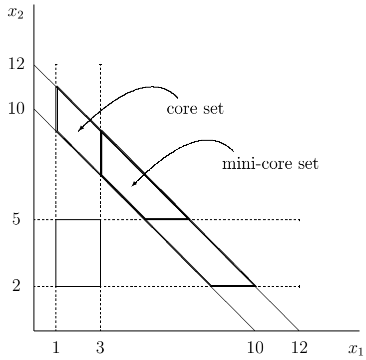

Figure 1: The mini-core set and the core set of a strongly balanced game

Now, we describe the core set and the mini-core set of a two-person interval game in terms of its selections.
Let

$$s_1 \in w(1) = [\underline{w}(1), \overline{w}(1)], s_2 \in w(2) = [\underline{w}(2), \overline{w}(2)], t \in w(1,2) = [\underline{w}(1,2), \overline{w}(1,2)]$$

and denote by $w^{s_1,s_2,t}$ the *selection* of w corresponding to s_1, s_2 and t. Then,

$$C(w) = \cup \left\{ C(w^{s_1,s_2,t}) | (s_1, s_2, t) \in w(1) \times w(2) \times w(1,2) \right\}.$$

Furthermore,

$$MC(w) = \cup \left\{ C(w^{s_1,s_2,t}) | s_1 \in [\overline{w}(1), \overline{w}(1)], s_2 \in [\overline{w}(2), \overline{w}(2)], t \in w(1,2) \right\}.$$

So,

$$MC(w) \subset \cup \left\{ C(w^{s_1,s_2,t}) | s_1 \in w(1), s_2 \in w(2), t \in w(1,2) \right\}.$$

The mini-core set $MC(w)$ is interesting because for each s_1, s_2 and t all points in $MC(w)$ with $x_1 + x_2 = t$ are also in $C(w^{s_1,s_2,t})$. Note that all points in the mini-core set of w are individually rational points for each selection $w^{s_1,s_2,t}$, and each selection $w^{s_1,s_2,t}$ can be written as a linear combination of unanimity games in the following way

$$w^{s_1,s_2,t} = s_1 u_{\{1\}} + s_2 u_{\{2\}} + (t - s_1 - s_2) u_{\{1,2\}}.$$

Let A and B be two intervals. We say that A is left to B, denoted by $A \preceq B$, if for each $a \in A$ and for each $b \in B$, $a \leq b$.

A two-person interval game $< N, w >$ is called *superadditive*, if

$$w(1) + w(2) \preceq w(1,2)$$

where $w(1) + w(2) = \{s_1 + s_2 | s_1 \in w(1), s_2 \in w(2)\}$ and $t \in w(1,2)$.
If $w \in IG^{\{1,2\}}$ is a superadditive game, then for each s_1, s_2 and t we have $s_1 + s_2 \leq t$. So, each selection $w^{s_1,s_2,t}$ of w is balanced. We conclude that if $\overline{w}(1) + \overline{w}(2) \leq \underline{w}(1,2)$ is satisfied, then each selection $w^{s_1,s_2,t}$ of w is superadditive. Hence, a two-person interval game $< N, w >$ is superadditive if and only if $< N, w >$ is strongly balanced.

4.2 ψ^{α}-values and their axiomatization

In this subsection optimism vectors will play a role.
Let $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$, which we call the *optimism vector*, and $w \in IG^{\{1,2\}}$. We define:

$$s_1^{\alpha_1}(w) := \alpha_1 \overline{w}(1) + (1 - \alpha_1)\underline{w}(1), s_2^{\alpha_2}(w) := \alpha_2 \overline{w}(2) + (1 - \alpha_2)\underline{w}(2).$$

We are interested in maps $\kappa : [a,b] \to \mathcal{R}^2$ where $[a,b]$ is a closed interval in $\mathcal{R}$ with properties:

(i) for each $a \leq x_1 \leq x_2 \leq b$, $\kappa_1(x_1) \leq \kappa_1(x_2)$, $\kappa_2(x_1) \leq \kappa_2(x_2)$;

(ii) for each $x \in [a,b]$, $\kappa_1(x) + \kappa_2(x) = x$.

In the following, we call such maps *monotonic curves*, and we denote by $\mathcal{K}(\mathcal{R}^2)$ the set of all monotonic curves in $\mathcal{R}^2$.
A map $F : IG^{\{1,2\}} \to \mathcal{K}(\mathcal{R}^2)$ assigning to each interval game w a unique curve $F(w) : [\underline{w}(1,2), \overline{w}(1,2)] \to \mathcal{R}^2$ for $t \in [\underline{w}(1,2), \overline{w}(1,2)]$, $i \in \{1,2\}$, in $\mathcal{K}(\mathcal{R}^2)$ is called a solution.
We say that $F : IG^{\{1,2\}} \to \mathcal{K}(\mathcal{R}^2)$ has the property of

(i) *efficiency (EFF)*, if for all $w \in IG^{\{1,2\}}$, $t \in [\underline{w}(1,2), \overline{w}(1,2)]$; $\sum_{i \in N} F(w)(t)_i = t$.

(ii) *α-symmetry (α-SYM)*, if for all $w \in IG^{\{1,2\}}$, $t \in [\underline{w}(1,2), \overline{w}(1,2)]$, with $s_1^{\alpha_1}(w) = s_2^{\alpha_2}(w)$, we have $F(w)(t)_1 = F(w)(t)_2$.

(iii) *covariance with respect to translations (COV)*, if for all

$$w \in IG^{\{1,2\}}, t \in [\underline{w}(1,2), \overline{w}(1,2)] \text{ and } a = (a_1, a_2) \in \mathcal{R}^2,$$

we have $F(w + \hat{a})(a_1 + a_2 + t) = F(w)(t) + a$.

Here, $\hat{a} \in IG^{\{1,2\}}$ is defined by

$$\hat{a}(\{1\}) = [a_1, a_1], \hat{a}(\{2\}) = [a_2, a_2], \hat{a}(\{1,2\}) = [a_1 + a_2, a_1 + a_2],$$

and $w + \hat{a} \in IG^{\{1,2\}}$ is defined by

$$(w + \hat{a})(s) = w(s) + \hat{a}(s) \text{ for } s \in \{\{1\}, \{2\}, \{1,2\}\}.$$

For each $w \in IG^{\{1,2\}}$ and $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ we define the map
$\psi^{\alpha} : IG^{\{1,2\}} \to \mathcal{K}(\mathcal{R}^2)$ such that

$$\psi^{\alpha}(w)(t) := (s_1^{\alpha_1}(w) + \beta, s_2^{\alpha_2}(w) + \beta),$$

where $\beta = \frac{1}{2}(t - s_1^{\alpha_1}(w) - s_2^{\alpha_2}(w))$.
The next example illustrates the solution ψ^{α} with $\alpha = (0,0)$ and its relations with the mini-core set.

EXAMPLE 4.2. *Consider a bankruptcy situation with two claimants with demands $d_1 = 70$ and $d_2 = 90$ and (uncertain) estate $E = [100, 120]$.*
Then, the characteristic function of the interval game is as follows:

$$w(\emptyset) = [0,0], w(1) = [(\underline{E} - d_2)_+, (\overline{E} - d_2)_+] = [10, 30]$$

$$w(2) = [(\underline{E} - d_1)_+, (\overline{E} - d_1)_+] = [30, 50], w(1,2) = [100, 120].$$

This is a strongly balanced game, since $\overline{w}(1) + \overline{w}(2) = 30 + 50 \leq \underline{w}(1,2) = 100$,

$$\psi^{(0,0)}(w)(t) = (10 + \beta, 30 + \beta) \text{ with } \beta = \frac{1}{2}(t - 40) \text{ and } t \in [100, 120].$$

Figure 2 illustrates that for all $t \in [100, 120]$, $\psi^{(0,0)}(w)(t) \in MC(w^{(0,0,t)})$; L in this figure denotes the set $\{\psi^{(0,0)}(w)(t) | t \in [100, 120]\}$.

Next, we give an axiomatic characterization of the ψ^{α}-value for $\alpha \in [0,1] \times [0,1]$.

PROPOSITION 4.1. *The ψ^{α}-value satisfies the properties EFF, α-SYM and COV.*

Proof. (i) For all $w \in IG^{\{1,2\}}$ and $t \in [\underline{w}(1,2), \overline{w}(1,2)]$, the solution ψ^{α} satisfies the efficiency (EFF) property since

$$\psi^{\alpha}(w)(t)_1 + \psi^{\alpha}(w)(t)_2 = s_1^{\alpha_1}(w) + s_2^{\alpha_2}(w) + 2\beta = t.$$

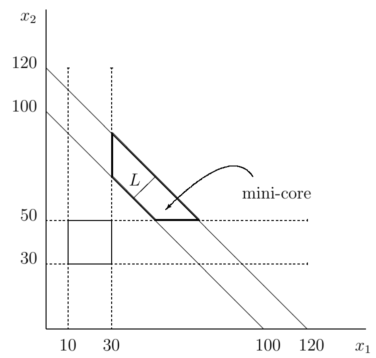

Figure 2: The mini-core set and the $\psi^{(0,0)}$-values of the game $< N, w >$

(ii) For all $w \in IG^{\{1,2\}}$ and $t \in [\underline{w}(1,2), \overline{w}(1,2)]$, the solution ψ^α satisfies the α-symmetry (α-SYM) property since $s_1^{\alpha_1}(w) = s_2^{\alpha_2}(w)$ implies $\psi^\alpha(w)(t)_1 = s_1^{\alpha_1}(w) + \beta = s_2^{\alpha_2}(w) + \beta = \psi^\alpha(w)(t)_2$.

(iii) Take $w \in IG^{\{1,2\}}$, $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ and $a \in \mathcal{R}^2$. The solution ψ^α satisfies the covariance with respect to translations (COV) property since

$$\psi^\alpha(w + \hat{a})(a_1 + a_2 + t) = (s_1^{\alpha_1}(w + \hat{a}) + \hat{\beta}, s_2^{\alpha_2}(w + \hat{a}) + \hat{\beta})$$

Then,

$$\psi^\alpha(w + \hat{a})(a_1 + a_2 + t) = (s_1^{\alpha_1} + \beta, s_2^{\alpha_2} + \beta) + (a_1, a_2) = \psi^\alpha(w)(t) + a.$$

Note that

$$\beta = \hat{\beta} = \frac{1}{2}(\hat{t} - s_1^{\alpha_1}(w + \hat{a}) - s_2^{\alpha_2}(w + \hat{a}))$$

where $\hat{t} = a_1 + a_2 + t$.

□

THEOREM 4.1. *The ψ^α-value is the unique solution satisfying EFF, α-SYM and COV properties.*

Proof. Suppose the solution $F : IG^{\{1,2\}} \to \mathcal{K}(\mathcal{R}^2)$ satisfies the three properties above. We show that $F = \psi^\alpha$.
Take $w \in IG^{\{1,2\}}$ and let $a = (s_1^{\alpha_1}(w), s_2^{\alpha_2}(w))$. Then, $s^\alpha(w - \hat{a}) = (0, 0)$.

By α-SYM and EFF, for each $\tilde{t} = t - a_1 - a_2$ with $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ we have $F(w - \hat{a})(\tilde{t}) = (\frac{1}{2}\tilde{t}, \frac{1}{2}\tilde{t}) = \psi^\alpha(w - \hat{a})(\tilde{t})$. Hence, $F(w - \hat{a}) = \psi^\alpha(w - \hat{a})$. By COV of F and ψ^α we obtain $F(w)(t) = F(w - \hat{a})(\tilde{t}) + a = \psi^\alpha(w - \hat{a})(\tilde{t}) + a = \psi^\alpha(w)(t)$ for each $w \in IG^{\{1,2\}}$ and $t \in [\underline{w}(1,2), \overline{w}(1,2)]$. From Proposition 4.1 it follows that ψ^α satisfies EFF, α-SYM and COV.
So, ψ^α is the only solution with these three properties.

□

The marginal curves for a two-person game $< N, w >$ are defined by $m^{\sigma,\alpha}(w) : [\underline{w}(1,2), \overline{w}(1,2)] \to \mathcal{R}^2$, where

$$m^{(1,2),\alpha}(w)(t) = (s_1^{\alpha_1}(w), t - s_1^{\alpha_1}(w)), m^{(2,1),\alpha}(w)(t) = (t - s_2^{\alpha_2}(w), s_2^{\alpha_2}(w)).$$

Note that each point of the marginal curve $m^{(1,2),\alpha}(w) : [\underline{w}(1,2), \overline{w}(1,2)] \to \mathcal{R}^2$ corresponds to a marginal vector of a selection of w, since for all $\alpha \in [0,1] \times [0,1]$ and for all $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ we have $m^{(1,2),\alpha}(w)(t) = m^{(1,2)}(v)$, where $v : 2^{\{1,2\}} \to \mathcal{R}$ is the characteristic function of the game with

$$v(\emptyset) = 0, v(1) = s_1^{\alpha_1}(w)(t), v(2) = s_2^{\alpha_2}(w)(t) \text{ and } v(1,2) = t.$$

Similarly, $m^{(2,1),\alpha}(w)(t) = m^{(2,1)}(v)$ for all $\alpha \in [0,1] \times [0,1]$ and for all $t \in [\underline{w}(1,2), \overline{w}(1,2)]$.
In case $w(S)$ is for each $S \in 2^N$ a degenerate interval, we have $m^{\sigma,\alpha}(w)(t) = m^\sigma(v)$ for all $\alpha \in [0,1] \times [0,1]$ and for all $t \in [\underline{w}(1,2), \underline{w}(1,2)]$ with

$$v(\emptyset) = 0, v(1) = \underline{w}(1), v(2) = \underline{w}(2) \text{ and } v(1,2) = \underline{w}(1,2).$$

Let us consider the Shapley-like solutions (Shapley (1953)) of the form $\varphi^\alpha : IG^{\{1,2\}} \to \mathcal{K}(\mathcal{R}^2)$ defined by

$$\varphi^\alpha(w) = \frac{1}{2}(m^{(1,2),\alpha}(w) + m^{(2,1),\alpha}(w))$$

for each $w \in IG^{\{1,2\}}$ and for each $\alpha \in [0,1] \times [0,1]$.
Then, for each $t \in [\underline{w}(1,2), \underline{w}(1,2)]$ it holds that $\varphi^\alpha(w)(t) = \psi^\alpha(w)(t)$.
So, φ^α coincides with ψ^α.

5 Concluding remarks

It would be interesting to extend our results to n-person games under interval uncertainty and to study cooperation under interval uncertainty in different Operations Research Game situations (cf. Borm et al. (2001)) such as flow game situations where the capacities are intervals, minimum cost spanning tree situations where the costs of the edges are intervals, airport game situations where the costs of the pieces of runways are intervals, etc. Also environmental problems such as carbon dioxide emission reduction and fish quota (cf. Weber et al. (2007)) could be approached in an interval uncertainty manner, and then cooperative games with interval uncertainty might be useful.

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