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CONVERGENCE OF BELIEFS IN BAYESIAN NETWORK GAMES

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Convergence of Beliefs in Bayesian Network Games

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Abstract

In many contexts, players interact only with a subset of the whole population, i.e., players interact on a network. This paper a setting in which players are located on a network and play a fixed game with their neighbors. Players have incomplete information on the network structure. They have a common prior over the network, and in addition, they know the number of connections they have. That is, their type is their degree. We study the sensitivity of game-theoretic predictions to the specification of players' beliefs. We show that two priors are close in a strategic sense if and only if they assign similar probabilities to all local events, i.e., to all events involving the types of a player and his neighbors. This means that in order to fully explore the range of possible strategic outcomes, it suffices to vary the type distribution and the correlation among player types. On the other hand, it is not enough to vary only the type distribution, which has been the focus of much of the literature so far.

JEL classification: C72, D83, L14, Z13

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*Address: Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: willemien.kets@gmail.com. Tel: +31-13-4662478. Fax: +31-13-4663280. An early version of this paper has circulated under the title “Strategic convergence of beliefs in network games”. I am indebted to Mia Deijfen, Andrea Galeotti, Sanjeev Goyal, Atsushi Kajii, Stephen Morris, Adriaan Soetevent, Dolf Talman, Eric van Damme, Remco van der Hofstad, Vincent Vanmetelbosch, Mark Voorneveld, and in particular Wouter Kager for stimulating discussions and helpful comments and suggestions. Part of this work was completed during a visit to the Santa Fe Institute. Financial support from the James S. McDonnell Foundation Grant for Robustness in Social Processes is gratefully acknowledged.
1 Introduction

In many contexts, an agent’s well-being primarily depends on his own behavior and on the behavior of those with whom he has a direct relationship, rather than on the behavior of the population at large. Indeed, Goolsbee and Klénow (2002) and Tucker (2006) find that an individual’s decision to adopt a particular communication technology is primarily influenced by the adoption decisions of those with whom he interacts directly, rather than by the overall adoption level in the population. Also, an agent’s connections provide access to various resources such as information, knowledge and capital. For instance, a key success factor for a firm in a high-tech sector such as the biotechnology industry is its position in a network of R&D partnerships (Powell et al., 1996).\footnote{Other empirical studies that highlight the role of networks in economic settings include Coleman et al. (1966) and Conley and Udry (2005) on the diffusion of new technologies in medicine and agriculture, respectively, Granovetter (1974) on job search, and Fafchamps and Lund (2003) on informal insurance networks in developing countries.} Hence, in a variety of settings, the networks formed by agents’ relations are important in determining economic outcomes. These networks are generally large and complex, and evolve rapidly over time (e.g. Powell et al., 2005). This suggests that agents often do not know the exact structure of the network they belong to.\footnote{Krackhardt and Hanson (1993) report that informal networks are mostly unobservable to senior executives. Also, Powell et al. (1996, p.120) observe that in R&D collaborations in biotechnology, “beneath most formal ties […] lies a sea of informal relations”.} At the same time, it is unclear what beliefs agents have about their networks.\footnote{Evidence suggests that agents use simple heuristics (Janicik and Larrick, 2005), and that their perception of the network is biased (e.g. Kumbasar et al., 1994), even in an environment with strong incentives (Johnson and Orbach, 2002).} Hence, in settings where agents interact strategically with their neighbors on a network under incomplete information on the network structure, it is important to assess how game-theoretic predictions depend on the assumptions on players’ beliefs. This is the topic of the current paper.

More specifically, we study a setting in which players are located on a network and play a fixed game with their neighbors. Payoffs only depend on a player’s own action and characteristics and on the actions and characteristics of his neighbors. Players have a (common) prior over the network, and, in addition, they have some local information. Each player is informed of the number of neighbors he has in the network, i.e., a player’s type is his degree. This defines a Bayesian network game. Since the interest in such games is usually on the effect of network characteristics on the behavior of players, we focus on symmetric equilibria, as in much of the literature (cf. Galeotti et al., 2006; Jackson and ...
Yariv, 2007; Sundararajan, 2005). We define a function that for any two priors gives their strategic distance. Loosely speaking, the strategic distance between two priors is small if for any game in which players hold one of these priors, for any symmetric Bayesian-Nash equilibrium in that game, there is a symmetric approximate equilibrium in the associated game with the other prior such that ex ante expected payoffs are close under both equilibria (cf. Kajii and Morris, 1998). If that is the case, players can obtain approximately the same ex ante expected payoffs under both priors, and we say that the two priors are close in a strategic sense. We study the necessary and sufficient conditions for two priors to be close in a strategic sense. We thus consider a type of lower hemicontinuity of the correspondence of (interim) approximate equilibria in Bayesian network games (see Engl, 1995, for a discussion of different continuity concepts).

Our main result (Theorem 4.2) shows that two priors are close in a strategic sense if and only if they assign similar probabilities to local events, i.e., events that involve the types of a player and his neighbors. This result has two important implications. Firstly, it indicates that in order to fully explore the possible strategic outcomes in Bayesian network games, it is sufficient to vary the type distribution and the correlation among player types. Hence, on the one hand, varying the type distribution, as has been the focus of much of the literature so far (see below for a discussion of this literature), is not enough. On the other hand, the result limits the set of priors that one needs to consider, as we show that priors need only be varied along two dimensions.

Secondly, Theorem 4.2 implies that we can interpret a Bayesian network game as a set of overlapping “local” games, so that we do not need to concern ourselves with the nonlocal features of priors. This can best be understood by means of a concrete example. For instance, consider two priors, and suppose that one of the priors assigns positive probability only to networks that are isomorphic to the network in Figure 1(a), with each of the networks in this isomorphism class having equal probability, while the other prior assigns positive probability only to networks isomorphic to the network in Figure 1(b), and each of the networks in this isomorphism class has equal probability. Clearly, these priors are identical in terms of the probabilities assigned to local events, i.e., in terms of the events involving the types of a player and his direct neighbors, but very different in terms of the probabilities they assign to global events, i.e., to different networks. We show that, despite the differences on the global level, the two priors are identical in terms of their strategic implications.

The motivation for the question we study comes from empirical work. The last few years,

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4Roughly speaking, two networks are isomorphic if they have the same vertex set, and the vertices are connected in the same way. See Section 2 for a precise definition.
there has been a surge in empirical work on networks, owing to the availability of data on large-scale networks such as the World Wide Web (see Jackson, 2007, for an overview). This work has shown that networks that are relevant for economic applications are characterized by a number of properties. Some of these properties relate to the local environment of a player. For instance, an important property of networks is the distribution of the number of direct contacts that people have. Other properties are defined on a larger scale. The clustering coefficient of a network, for instance, quantifies the extent to which friends of your friends are also your friends. Another example is the degree correlation, i.e., the correlation in the number of contacts people have. An important question for game-theoretic applications is then how these different properties affect strategic interactions on networks.

So far, most of the literature has focused on the effect of varying the degree distribution, i.e., the distribution of player types, on game-theoretic outcomes, assuming that players’ types are independent (e.g. Galeotti and Vega-Redondo, 2005; Jackson and Yariv, 2007; López-Pintado, 2006; Sundararajan, 2005). An important question is whether game-theoretic predictions obtained under the assumption that players’ types are independent continue to hold if we relax this assumption. This paper shows that this is not the case. We show that for two priors to give rise to similar outcomes (from a player’s ex ante perspective), it is not sufficient that they are close in terms of the type distribution they induce, they also need to be close in terms of the correlation among player types. Hence, while varying the type distribution may be a good starting point, the current paper shows that one needs to go beyond the class of random network models with a given degree distribution to fully explore the range of strategic outcomes. At the same time, our result restricts the set of priors that one needs to consider, as we show that priors need only be varied along two dimensions, the distribution of types and the correlation among player types.
To illustrate these points, we present a simple example in Section 4.2. We study a game in which players can choose whether to invest or not. Not investing gives a payoff of zero, independent of others’ actions, while investing is only profitable if all neighbors invest. Hence, this is a game of strategic complements. We compare two priors which are identical in terms of the type distribution they induce, but which differ in terms of the correlation among types. We show that there exists a symmetric strategy profile that is a Bayesian-Nash equilibrium under one prior that is not an (approximate) equilibrium under the other prior, and vice versa, and ex ante expected payoffs under equilibria under the two priors differ. These priors are thus different in terms of strategic predictions.

The current paper is related to two distinct literatures. Firstly, this paper contributes to the literature on Bayesian network games (e.g. Galeotti et al., 2006; Galeotti and Vega-Redondo, 2005; Jackson and Yariv, 2007; Sundararajan, 2005). This literature studies the effect of network structure on game-theoretic outcomes. In particular, Galeotti et al. (2006) study the effect of varying the type distribution and the correlation among players’ types in a particular way in games with strategic complements and substitutes. They show that predictions change when the type distribution and the correlation among players’ types are varied. This illustrates that it is important to go beyond the assumption of independent types made in the earlier literature. The current paper complements the work of Galeotti et al. (2006) in two ways. First, we show that varying the type distribution and the type correlation, as Galeotti et al. (2006) do, is indeed sufficient to capture all possible strategic behavior in any class of Bayesian network games. Second, while Galeotti et al. (2006) focus on gradual changes in equilibrium behavior as priors are continuously varied in terms of the distribution of types and the type correlation, our results emphasize that it is possible to obtain qualitatively different outcomes if priors differ in these two dimensions (see e.g. the example in Section 4.2).

The second literature to which this paper is related is the literature on (payoff) continuity in games. Continuity issues in general Bayesian games have been studied by a number of authors (Kajii and Morris, 1998; Milgrom and Weber, 1985; Monderer and Samet, 1996). The question we study is similar to the question studied by Kajii and Morris (1998). While Kajii and Morris (1998) study payoff continuity in general Bayesian games, we restrict attention to the class of Bayesian network games. Moreover, we focus on symmetric equilibria. By exploiting the symmetry of the game, we are able to weaken the conditions of Kajii and Morris (1998). That this can be done is not obvious. While payoffs only depend directly on the actions of neighbors in our setting, actions and beliefs of those further away in the network may have a considerable effect on the payoffs to a player, through the effect on the neighbors
of those players and the neighbors of the neighbors of those players, and so on. There is thus a tension between the local nature of the payoffs and the interdependencies intrinsic to the network setting. Our results show that Bayesian network games can nevertheless be treated as a collection of overlapping local games. The value of this result is that it implies that priors need only be varied in terms of the type distribution and the correlation among player types that they induce to explore the possible strategic outcomes.

Payoff continuity has also been studied in other classes of games. Kets (2007a) studies payoff continuity in network games of incomplete information. The difference with the class of Bayesian network games in that there is additionally uncertainty about the network size. This makes that the player set is not common knowledge in these games; hence, they are not Bayesian games. Also, it means that, when the network can be of any (finite) size, the type set is countably infinite. By contrast, in our games, the player set and thus the network size is fixed (and common knowledge). The type set (the set of possible degrees) is then finite. In the class of network games of incomplete information, with a countably infinite type set, beliefs may be sensitive to small probability events (cf. Kajii and Morris, 1998). That is, the condition we derive for strategic closeness in Bayesian network games is not sufficient for network games of incomplete information. The reason is that small probability events may considerably affect outcomes through players’ conditional beliefs: even if an event has small prior probability, it may influence a player’s actions when he thinks (given his type) that it is likely that his neighbor think it is likely that their neighbors think it is likely... that the small probability event is true. In the current setting, with a finite type set, this is ruled out. This means that in Bayesian network games, it suffices to consider players’ prior beliefs over local events, i.e., to vary the type distribution and the correlation among neighbors’ types.

The remainder of this paper is organized as follows. Preliminaries are discussed in Section 2. Bayesian network games are defined in Section 3. The main result is presented in Section 4. Section 5 concludes. Proofs that are not included in the main text can be found in Appendix A.

2 Preliminaries

In our framework, players are located on a network. A network $g$ is a pair consisting of a finite, nonempty set $V(g)$ of vertices and a finite set $E(g)$ of edges, with an edge being an unordered pair of two distinct vertices. Let $g$ be a network. If $\{i,j\} \in E(g)$, where $i,j \in V(g), i \neq j$, then $i$ and $j$ are neighbors in $g$; alternatively, we say that $i$ and $j$ are
adjacent in $g$. For notational simplicity, an edge $\{i, j\} \in E(g)$ is sometimes denoted by $ij$.

Two networks $g, g'$ are isomorphic if $V(g) = V(g') =: V$ and there is a permutation $\pi$ of $V$ such that $\{i, j\} \in E(g)$ for $i, j \in V, i \neq j$, if and only if $\{\pi(i), \pi(j)\} \in E(g')$. This defines an equivalence relation; hence, the set of all networks with a given vertex set can be partitioned into a finite number of isomorphism classes, i.e., sets of isomorphic networks. In the current setting, we associate a player with each vertex, so that edges represent the relations between players. In the following, we therefore refer to players rather than to vertices.

Players’ beliefs are modeled by means of network belief systems. A network belief system is a probability space (i.e., a triple consisting of a sample space, a $\sigma$-algebra, and a probability measure on the $\sigma$-algebra) in which the sample space is a set of networks. Formally, denote the set of positive integers by $\mathbb{N}$, and let $\mathbb{N}_0$ denote the set of nonnegative integers. Let $n \in \mathbb{N}$ and $V^{(n)} := \{1, \ldots, n\}$. Let $\mathcal{G}^{(n)}$ be the set of all networks with player set $V^{(n)}$, and let $\mathcal{F}^{(n)}$ be the set of all subsets of $\mathcal{G}^{(n)}$. Let $\mathcal{M}^{(n)}$ be the set of all probability measures on the $\sigma$-algebra $\mathcal{F}^{(n)}$, and let $\mu \in \mathcal{M}^{(n)}$. Then, $(\mathcal{G}^{(n)}, \mathcal{F}^{(n)}, \mu)$ is a network belief system.

We are interested in the local environment of players. Let $i \in V^{(n)}$ and $g \in \mathcal{G}^{(n)}$. The neighborhood $N_i(g)$ of $i$ in $g$ is the set of neighbors of $i$ in $g$. The degree $D_i(g)$ of player $i$ in $g$ is the number of neighbors of $i$ in $g$, i.e., $D_i(g)$ is the cardinality of the set $N_i(g)$. We also consider the number of neighbors the neighbors of a given vertex have. Loosely speaking, the neighbor degree profile of a vertex in a given network is a list of the degrees of the neighbors of the vertex, in a non-increasing order. Note that the maximum degree a player can have is $n - 1$ if the total number of players is $n$. For $t \in \{1, \ldots, n - 1\}$, let

$$\Omega^K_t := \{(k_1, \ldots, k_t) \in \{1, \ldots, n - 1\}^t \mid k_1 \geq k_2 \geq \ldots \geq k_{t-1} \geq k_t\}.$$ 

For $t = 0$, let $\Omega^K_0 := \{0\}$, and define

$$\Omega^K := \bigcup_{t \in \{0,1,\ldots,n-1\}} \Omega^K_t.$$ 

Let $\mathcal{F}_K$ be the $\sigma$-field generated by the set of singletons of $\Omega^K$. For $g \in \mathcal{G}^{(n)}$ and $i \in V^{(n)}$ such that $D_i(g) = 0$, we set $K_i(g) := 0$. Otherwise, define

$$N_1 := N_i(g),$$ 

$$j(1) := \max\{j \in N_1 \mid D_j(g) \geq D_m(g) \text{ for all } m \in N_1\},$$ 

$$K_{i,1}(g) := D_{j(1)}(g),$$
and for $\ell = 2, \ldots, D_i(g)$:

$$N_\ell := N_{\ell-1} \setminus \{j(\ell - 1)\},$$

$$j(\ell) := \max\{j \in N_\ell \mid D_j(g) \geq D_m(g) \text{ for all } m \in N_\ell\},$$

$$K_{i,\ell}(g) := D_{j(\ell)}(g).$$

Then, $K_i(g) := (K_{i,1}(g), \ldots, K_{i,D_i(g)}(g))$ is the neighbor degree profile of $i$ in $g$.

**Example 2.1** Suppose we draw network $g$ in Figure 2 from the set $G^{(4)}$. The neighborhood of vertex 1 in $g$ is $N_1(g) = \{2, 3, 4\}$, and its degree in $g$ is $D_1(g) = 3$. The neighbor degree profile of vertex 1 in $g$ is $K_1(g) = (D_4(g), D_3(g), D_2(g)) = (2, 2, 1)$. ◀

We are interested in the case in which players are ex ante identical in terms of their network position. Throughout this paper, we therefore make the following assumption on network belief systems:

**Assumption 2.2 (Exchangeability)** Let $(G^{(n)}, \mathcal{F}^{(n)}, \mu)$ be a network belief system. The neighbor degree profiles $K_1, K_2, \ldots, K_n$ are exchangeable. That is, for any $k \in \{1, \ldots, n\}$, $i_1, \ldots, i_k \in V^{(n)}$, the random vector $(K_{i_1}, K_{i_2}, \ldots, K_{i_k})$ has the same distribution as the random vector $(K_{\pi(i_1)}, K_{\pi(i_2)}, \ldots, K_{\pi(i_k)})$ for any permutation $\pi : \{i_1, \ldots, i_k\} \to \{i_1, \ldots, i_k\}$. In particular, for all $i, j \in V^{(n)}$, for all $\theta \in \{0, \ldots, n-1\}$,

$$\mu(\{g \in G^{(n)} \mid D_i(g) = \theta\}) = \mu(\{g \in G^{(n)} \mid D_j(g) = \theta\}),$$

i.e., the probability that a player has a certain degree is the same for each player. ◀

Network belief systems $(G^{(n)}, \mathcal{F}^{(n)}, \mu)$ with this property exist. For instance, let $\mu$ be the uniform distribution on the finite set $G^{(n)}$.

**Definition 2.3** Let $(G^{(n)}, \mathcal{F}^{(n)}, \mu)$ be a network belief system such that Assumption 2.2 is satisfied. Then, the degree distribution of $(G^{(n)}, \mathcal{F}^{(n)}, \mu)$ is given by $(p^{(n)}(t))_{t \in \mathbb{N}_0}$, where

$$\forall t \in \mathbb{N}_0 : \quad p^{(n)}(t) := \mu(\{g \in G^{(n)} \mid D_1(g) = t\}).$$

That is, the degree distribution of a network belief system gives for each $t$ the probability that a player selected uniformly at random from the network has degree $t$. 8
Finally, for notational convenience, we assume:

**Assumption 2.4 (No isolated vertices)** The network belief system \((\mathcal{G}^{(n)}, \mathcal{F}^{(n)}, \mu)\) is such that with probability 1, each player has at least one neighbor. That is,

\[
\mu(\{g \in \mathcal{G}^{(n)} \mid D_i(g) > 0 \text{ for all } i \in V(g)\}) = 1.
\]

\(\Box\)

### 3 Bayesian network games

A Bayesian network game is a Bayesian game where the states of nature are networks drawn according to a network belief system and in which each player is informed of the number of neighbors he has. Formally, let \(n \in \mathbb{N}\). A *Bayesian network game* is a Bayesian game

\[
\langle N, \mathcal{G}^{(n)}, (A_i)_{i \in N}, (T_i)_{i \in N}, (\tau_i)_{i \in N}, \mu, (u_i)_{i \in N} \rangle,
\]

where \(N = \{1, \ldots, n\}\) is the set of players and \((\mathcal{G}^{(n)}, \mathcal{F}^{(n)}, \mu)\) is a network belief system on vertex set \(V^{(n)} = N\) such that Assumptions 2.2 and 2.4 are satisfied. The probability measure \(\mu\) is players’ (common) prior. Each player \(i \in N\) has a nonempty, finite set \(A_i\) of pure strategies or actions. If the state of nature/network is \(g \in \mathcal{G}^{(n)}\), player \(i\)'s private information is his degree. Hence, the set of types or signals of player \(i\) is \(T_i = \{0, \ldots, n - 1\} =: T\) and his signal function \(\tau_i : \mathcal{G}^{(n)} \rightarrow T\) assigns to each network \(g \in \mathcal{G}^{(n)}\) the degree \(\tau_i(g) := D_i(g)\) of player \(i\). Finally, each player \(i \in N\) has a von Neumann-Morgenstern utility function \(u_i : \times_{i \in N} A_i \times \mathcal{G}^{(n)} \rightarrow \mathbb{R}\).

Henceforth, we speak of type and neighbor type profile rather than of degree and neighbor degree profile. Also, we will refer to the type distribution of a network belief system that satisfies Assumption 2.2 to denote its degree distribution (Definition 2.3).

We assume that there exists a finite set \(A\) such that \(A_i = A\) for all \(i \in N\). Furthermore, we assume that there exists a profile of local payoff functions \(v = (v_t)_{t \in T}\) that for each \(t \in T\) gives the payoff to a player of type \(t\). More specifically, for \(t = 0\), \(v_t\) is a real function on \(A\), and for each \((a_i, a_{-i}) \in A^n, g \in \mathcal{G}^{(n)}\) and \(i \in V^{(n)}\) such that \(\tau_i(g) = 0\), \(u_i(a_i, a_{-i}, g) = v_0(a_i)\), i.e., the payoffs to an isolated player only depend on his own type and action. For \(t > 0\), \(v_t\) is a real function on \(A \times A^t \times T^t\) that is symmetric in \(A^t\) and \(T^t\), i.e., for all permutations \(\pi_1, \pi_2\) on \(\{1, \ldots, t\}\), for all \(b \in A, (a_1, \ldots, a_t) \in A^t, (\theta_1, \ldots, \theta_t) \in T^t\),

\[
v_t(b, (a_1, \ldots, a_t), (\theta_1, \ldots, \theta_t)) = v_t(b, (a_{\pi_1(1)}, \ldots, a_{\pi_1(t)}), (\theta_{\pi_2(1)}, \ldots, \theta_{\pi_2(t)})).
\]
Then, for each \( i \in V^{(n)} \), \( g \in G^{(n)} \) and \( a = (a_1, \ldots, a_n) \in A^n \),

\[
    u_i(a, g) = v_{\tau_i(g)}(a_i, (a_j)_{j \in N_i(g)}, (\tau_j(g))_{j \in N_i(g)}).
\]

That is, a player’s payoff only depends on his own action and type, and the actions and types of his neighbors, and does so in an anonymous way. The bound \( B \) of a profile of local payoff functions \( v \) is defined as:

\[
    B := \max_{t \in T \setminus \emptyset; \theta \in T^{i,a,a'} \in A^{i+1}} \left\{ |v_{t}(a, \theta) - v_{t}(a', \theta)|, |v_{t}(a, \theta)| \right\}.
\]

This maximum exists, as the signal set \( T \) and the action set \( A \) are finite.

Throughout this paper, we fix the player set \( N \) and the action set \( A \). A Bayesian network game is then fully characterized by the common prior \( \mu \) and its profile \( v \) of local payoff functions. Henceforth, a Bayesian network game \( \langle N, G^{(n)}, (A_i)_{i \in N}, (T_i)_{i \in N}, (\tau_i)_{i \in N}, \mu, (u_i)_{i \in N} \rangle \) is therefore denoted by the pair \((\mu, v)\).

For \( i \in N \), a (mixed) strategy for player \( i \) is a function \( \sigma_i : T \to \Delta(A) \). Denote the set of all strategies by \( \Sigma \). The probability that action \( a_i \in A \) is played under strategy \( \sigma_i \) by player \( i \in N \) given that he has type \( t_i \in T \) is denoted by \( \sigma_i(a_i \mid t_i) \). A strategy profile is a function \( \sigma = (\sigma_i)_{i \in N} \in \Sigma^n \), with \( \sigma_i \) a strategy of player \( i \) for each \( i \in N \). For strategy profile \( \sigma = (\sigma_j)_{j \in N} \) and \( i \in N \), we write \( \sigma_{-i} \) to denote the strategy profile \( \sigma = (\sigma_j)_{j \in N \setminus \{i\}} \) of the opponents of \( i \). We say that a strategy profile \( \sigma \) is symmetric if \( \sigma_i = \sigma_j \) for all \( i, j \in N \).

We can now define expected payoffs. First, we introduce some notation that will be useful in the following. For \( t \in T \), \( F \in \mathcal{F}_K \), and \( \theta \in \Omega_K \), define

\[
    \mu(t) := \mu(\{g' \in G^{(n)} \mid D_1(g') = t\}),
\]

\[
    \mu(F) := \mu(\{g' \in G^{(n)} \mid K_1(g') \in F\}),
\]

\[
    \mu(\theta) := \mu(\{g' \in G^{(n)} \mid K_1(g') = \theta\}).
\]

By Assumption 2.2, \( \mu(t) \) is the prior probability that any fixed player has type \( t \), and \( \mu(F) \) is the prior probability that the neighbor type profile of any fixed player lies in the set \( F \). Finally, \( \mu(\theta) \) is the prior probability that a fixed player has neighbor type profile \( \theta \).

We also introduce some short-hand notation for various conditional probabilities. Let \( t \in T \) be such that \( \mu(t) > 0 \). For \( g \in G^{(n)} \), \( F \in \mathcal{F}_K \), and \( \theta \in \Omega_K \), let

\[
    \mu(\{g \mid t\}) := \mu(\{g \mid g' \in G^{(n)} \mid D_1(g') = t\}),
\]

\[
    \mu(F \mid t) := \mu(\{g' \in G^{(n)} \mid K_1(g') \in F\} \mid \{g' \in G^{(n)} \mid D_1(g') = t\}),
\]

\[
    \mu(\theta \mid t) := \mu(\{g' \in G^{(n)} \mid K_1(g') = \theta\} \mid \{g' \in G^{(n)} \mid D_1(g') = t\}).
\]
In words, $\mu(g \mid t)$ is the conditional probability that the network is $g$ given that player 1 has degree $t$. Similarly, $\mu(F \mid t)$ is the conditional probability that the neighbor type profile of player 1 lies in the set $F$ given that he has type $t$. Finally, $\mu(\theta \mid t)$ is the conditional probability that the neighbor type profile of player 1 is equal to $\theta$, given that he has type $t$.

Then, the interim expected payoff to player $i \in N$ of action $a_i \in A$ under common prior $\mu \in M^{(n)}$ when he receives signal $t_i \in T$ with $\mu(t_i) > 0$ and when the other players play according to the strategy profile $(\sigma_j)_{j \in N \setminus \{i\}} \in \Sigma^{n-1}$ is given by

$$\varphi_i(a_i, \sigma_{-i}; t_i, \mu) := \sum_{g \in G^{(n)}} \mu(g \mid t_i) u_i(a_i, (\sigma_j(\tau_j(g)))_{j \in N \setminus \{i\}}, g)$$

$$= \sum_{g \in G^{(n)}} \mu(g \mid t_i) v_t(a_i, \sigma_{N_i(g)}, \tau_{N_i(g)})$$

where we have defined $\sigma_{N_i(g)} := (\sigma_j(\tau_j(g)))_{j \in N_i(g)}$ and $\tau_{N_i(g)} := (\tau_j(g))_{j \in N_i(g)}$. Similarly, the ex ante expected payoff to a player $i \in N$ of the strategy profile $\sigma$ when players’ prior is $\mu$ is

$$\Phi_i(\sigma; \mu) := \sum_{g \in G^{(n)}} \mu(\{g\}) u_i(\sigma, g)$$

$$= \sum_{t_i \in T: \mu(t_i) > 0} \sum_{a_i \in A} \sigma_i(a_i \mid t_i) \varphi_i(a_i, \sigma_{-i}; t_i, \mu).$$

**Definition 3.1** Let $\varepsilon \geq 0$. A strategy profile $\sigma \in \Sigma^n$ is an (interim) $\varepsilon$-equilibrium of a Bayesian network game $(\mu, v)$ if for each player $i \in N$, for each $t_i \in T$ with $\mu(t_i) > 0$, each $a_i \in A$ with $\sigma_i(a_i \mid t_i) > 0$,

$$\varphi_i(a_i, \sigma_{-i}; t, \mu) \geq \varphi_i(b_i, \sigma_{-i}; t, \mu) - \varepsilon$$

for all $b_i \in A$. That is, in an $\varepsilon$-equilibrium, a player can gain at most $\varepsilon$ from unilateral deviation. An $\varepsilon$-equilibrium is symmetric if it is a symmetric strategy profile.

A Bayesian-Nash equilibrium is a 0-equilibrium. By standard arguments, a Bayesian-Nash equilibrium exists for Bayesian network games. We also have the following result:

**Proposition 3.2** Let $(\mu, v)$ be a Bayesian network game. Then there exists a symmetric Bayesian-Nash equilibrium of $(\mu, v)$.

For a proof, see Appendix A. For $\varepsilon \geq 0$, denote the set of symmetric $\varepsilon$-equilibria of the Bayesian network game $(\mu, v)$ by $N^\varepsilon(\mu, v)$. In particular, the set $N^0(\mu, v)$ denotes the set of symmetric Bayesian-Nash equilibria of $(\mu, v)$. 11
When players play according to a symmetric strategy profile, we can simplify the expressions for players’ expected payoffs. A symmetric strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma^n$ can be denoted by $\hat{\sigma} := (\hat{\sigma}_t^i)_{t \in T}$, with $\hat{\sigma}_t^i(a) = \sigma_i(a \mid t)$ for any $i \in N$ the probability that a player of type $t \in T$ takes action $a \in A$. Let $\sigma$ be a symmetric strategy profile, and let $\hat{\sigma} = (\hat{\sigma}_t^i)_{t \in T}$, with, for all $t \in T$, $\hat{\sigma}_t^i(\cdot) = \sigma_i(\cdot \mid t)$ for any $i \in N$. For $t \in T$ and type profile $\theta = (\theta_1, \ldots, \theta_t) \in \Omega_K$, we write $\hat{\sigma}_\theta$ to denote $(\hat{\sigma}_{\theta_1}^i, \ldots, \hat{\sigma}_{\theta_t}^i)$. Then, for $t \in T$ such that $\mu(t) > 0$, and $a \in A$, we define

$$\hat{\varphi}_t(a, \hat{\sigma}; \mu) := \sum_{\theta \in \Omega_K^t} \mu(\theta \mid t) v_t(a, \hat{\sigma}_\theta, \theta)$$

$$= \varphi_i(a, \sigma_{-i}; t, \mu) \text{ for any } i \in N$$

to be the interim expected payoff to an arbitrary player of type $t$ of action $a$ when players play according to the symmetric strategy profile $\sigma$ and the common prior is $\mu$. Similarly, for a symmetric strategy profile $\sigma \in \Sigma^n$, we define

$$\hat{\Phi}(\hat{\sigma}; \mu) := \sum_{\mu(t) > 0} \mu(t) \sum_{a \in A} \hat{\sigma}_t^i(a) \hat{\varphi}_t(a, \hat{\sigma}; \mu)$$

$$= \Phi_i(\sigma; \mu) \text{ for any } i \in N$$

to be the ex ante expected payoff to an arbitrary player when players play according to the symmetric strategy profile $\sigma$ and the prior is $\mu$.

## 4 Strategic convergence

### 4.1 Strategic distance

Our objective is to define a “measure” of similarity of priors such that if two priors are similar according to this measure, then, for each Bayesian network game in which beliefs are given by one of the priors, for each symmetric Bayesian-Nash equilibrium of the game, there exists a symmetric approximate equilibrium in the game with the same profile of local payoff functions but with beliefs given by the other prior such that ex ante payoffs are close under the two equilibria. If that is the case, then, for all possible payoff functions, players can obtain approximately the same payoffs (ex ante) under both priors. In that case, the two priors are similar from players’ (ex ante) perspective. At the same time, we do not want to make the conditions on priors to be similar any stricter than necessary. We thus look for the weakest condition that guarantees that the above holds.
Formally, let \( \mu, \mu' \in \mathcal{M}^{(n)} \), and let \( v = (v_t)_{t \in T} \) be a profile of local payoff functions. For each \( \varepsilon \geq 0 \), define

\[
\chi(\mu, \mu'; v, \varepsilon) := \sup_{\sigma \in \mathcal{N}^{(\mu,v)}} \inf_{\sigma' \in \mathcal{N}^{(\mu',v)}} |\hat{\Phi}(\hat{\sigma}; \mu) - \hat{\Phi}(\hat{\sigma'}; \mu')|,
\]

where \( \hat{\Phi} \) is the ex ante expected payoff function given profile \( v \) of local payoff functions, and \( \sigma \) and \( \sigma' \) are the symmetric strategy profiles corresponding to \( \hat{\sigma} \) and \( \hat{\sigma'} \), respectively. Hence, \( \chi(\mu, \mu'; v, \varepsilon) \) is a measure of the difference in outcomes under \( \mu' \) and \( \mu \) in terms of ex ante expected payoffs when players play according to a symmetric strategy. More specifically, for a given \( \varepsilon \geq 0 \), for each symmetric Bayesian-Nash equilibrium under \( \mu \), we first fix a symmetric \( \varepsilon \)-equilibrium under \( \mu' \) which minimizes the (absolute) difference in ex ante expected payoffs under both equilibria, and we then take a symmetric Bayesian-Nash equilibrium under \( \mu \) which maximizes this difference. This formalizes the idea that for each symmetric Bayesian-Nash equilibrium of a Bayesian network game with one of the priors, there exists some symmetric approximate equilibrium of the Bayesian network game with the other prior, such that ex ante expected payoffs are similar under both equilibria. However, the function \( \chi(\mu, \mu'; v, \varepsilon) \) is not symmetric in \( \mu \) and \( \mu' \), as we would want. To obtain a symmetric function of \( \mu \) and \( \mu' \), let

\[
\chi^*(\mu, \mu'; v, \varepsilon) := \max \{ \chi(\mu, \mu'; v, \varepsilon), \chi(\mu', \mu; v, \varepsilon) \}.
\]

We refer to \( \chi^*(\mu, \mu'; v, \varepsilon) \) as the strategic distance between \( \mu \) and \( \mu' \) for the profile \( v \) given \( \varepsilon \). The supremum of \( \chi^*(\mu, \mu'; v, \varepsilon) \) over \( v \) is called the strategic distance between \( \mu \) and \( \mu' \) given \( \varepsilon \).

Note that when \( \varepsilon \) increases, the set of symmetric \( \varepsilon \)-equilibria weakly increases, as more and more symmetric strategies will satisfy the equilibrium criterion, and the (absolute) difference in ex ante expected payoffs will decrease weakly. Hence, the interesting case is when \( \varepsilon \) comes arbitrarily close to 0. This leads us to the following definition (cf. Kajii and Morris, 1998):

**Definition 4.1** Take any \( \mu \in \mathcal{M}^{(n)} \), and consider a sequence \( (\mu^k)_{k \in \mathbb{N}} \) in \( \mathcal{M}^{(n)} \). The sequence \( (\mu^k)_{k \in \mathbb{N}} \) converges strategically to \( \mu \) if for each profile \( v \) of local payoff functions and for each \( \varepsilon > 0 \) we have that

\[
\lim_{k \to \infty} \chi^*(\mu, \mu^k; v, \varepsilon) = 0.
\]

In the next section, we give an example which illustrates the factors that are important for strategic convergence.
4.2 Example: Local investment

For reasons that will become clear shortly, let $\mathcal{N}$ be the set of integers that can be written as

$$N_\nu = (2^1 + 1)n_1 + (2^2 + 1)n_2 + \cdots + (2^\nu + 1)n_\nu$$

for some $\nu \in \mathbb{N}$, with $n_\nu = 1$ and for each $\ell \in \{1, \ldots, \nu - 1\}$, $n_\ell = 2^\ell n_{\ell+1}$.

Let $\nu \in \mathbb{N}$ and consider the following game. There is a set of $n = N_\nu$ players. Each player has two actions, $S$ and $R$. Action $S$ is the safe action. It always gives a payoff of 0, independent of a player’s type or the actions and types of his neighbors. The payoffs to the risky action $R$ depend on the actions of a player’s neighbors in the network. More precisely, the payoffs to a player of type $t \in T, t > 0$, of action $R$ when the action and type profiles of his neighbors are $a^{(t)} = (a_1^{(t)}, \ldots, a_t^{(t)}) \in A^t$ and $\theta^{(t)} = (\theta_1^{(t)}, \ldots, \theta_t^{(t)}) \in T^t$, respectively, are:

$$v_t(R, a^{(t)}, \theta^{(t)}) = \begin{cases} 3c & \text{if } a_\ell^{(t)} = R \text{ for all } \ell \in \{1, \ldots, t\}, \\ -c & \text{otherwise}, \end{cases}$$

where $c > 0$ is some constant. An interpretation of this game is that players need to decide whether to invest (play $R$) or not (play $S$). Investment is risky. Only if all his neighbors invest, a player gets a positive payoff from investing, otherwise he looses. Clearly, this is a game of (strict) strategic complements, since the incentives for a player to invest increase strictly when the number of neighbors who invest increases.

We consider two priors on $(G^{(n)}, F^{(n)})$, the independent types prior and the core-periphery prior. The core-periphery prior $\mu_{cp}$ assigns probability one to the isomorphism class of networks that, for $\ell \in \{1, \ldots, \nu\}$, consist of $n_\ell$ components with $2^\ell + 1$ players, of which one player—the core player—is connected to all other players, and the other $2^\ell$ players—the peripheral players—are connected to the core player and to $2^\ell - 1$ peripheral players. Hence, the type (degree) of the core player in a component with $2^\ell + 1$ players is $2^\ell$, and the type of the peripheral players in such a component is $2^{\ell-1}$. We assume that each of the networks in the isomorphism class has equal probability. See Figure 3 for components that occur with positive probability when $\nu$ is at least 3. Note that we can only construct such networks when the number of players is an element of $\mathcal{N}$.

If we define $n_0 = n_\nu+1 = 0$, it can easily be verified that the type distribution under $\mu_{cp}$ is given by $(\xi^{(n)}(t))_{t \in \mathbb{N}_0}$, where

$$\xi^{(n)}(t) := \begin{cases} \frac{1}{N_\nu} (n_{\log_2(t)} + 2tn_{\log_2(t)+1}) & \text{if } t \in \{1, 2, 4, \ldots, 2^\nu\}, \\ 0 & \text{otherwise}. \end{cases}$$
Figure 3: Components that occur with positive probability under the core-periphery prior when \( \nu \) is at least 3. The core players are indicated with white dots, the peripheral players by black dots.

It can be readily checked that \( \sum_{t \in \mathbb{N}_0} \xi^{(n)}(t) = 1 \). In addition, it can be shown that for each \( t \in \mathbb{N}_0 \), \( \xi^{(n)}(t) \) converges to some \( \xi(t) \) when \( \nu \to \infty \), with

\[
\sum_{t \in \mathbb{N}_0} \xi(t) = 1, \quad \sum_{t \in \mathbb{N}_0} t \xi(t) < \infty.
\]

That is, the function \( \xi: \mathbb{N}_0 \to [0,1] \) is a distribution (probability mass function) and has a finite first moment. We refer to \( (\xi(t))_{t \in \mathbb{N}_0} \) as the limiting (type) distribution. Furthermore, it is not hard to verify that for \( \ell = 2, 3, \ldots, \nu - 1 \) the conditional probability that a player of type \( t = 2^\ell \) is a core player is

\[
\frac{1 \cdot n_\ell}{1 \cdot n_\ell + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{1}{3},
\]

independent of \( \ell \), where we have used that \( n_\ell = 2^\ell n_{\ell+1} \). For future reference, note that a player with type \( 2^\ell \) who is a core player interacts with players of type \( t' = 2^{\ell-1} \). Similarly, the conditional probability that a player with type \( t = 2^\ell \) is a peripheral player is

\[
\frac{2^{\ell+1} \cdot n_{\ell+1}}{1 \cdot n_\ell + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{2}{3}.
\]

In that case, he interacts with players of type \( t' = 2^{\ell+1} \).

We now define the independent types prior. We follow the literature (e.g. Galeotti et al., 2006; Jackson and Yariv, 2007; Sundararajan, 2005) by assuming that under the independent types prior, players believe that the type distribution is given by some fixed distribution and that players’ types are independent.\(^5\)

\(^5\)Note that these assumptions require some bounded rationality on the part of players, for two reasons. Firstly, there exists no prior on the finite set \( \mathcal{G}^{(n)} \) that gives rise to independent types. Moreover, the independent types prior, players assign positive probability to a player having a type (degree) that exceeds
Hence, we assume that for each \( t \in \mathbb{N}_0 \), players’ prior belief that the type of an arbitrary player is \( t \) is \( \mu_{\text{ind}}(t) := \xi(t) \). We now derive the conditional probability that a fixed player has a given neighbor type profile, given his type. First note that if the probability that a player selected uniformly at random from the network has type (degree) \( t \) is \( \xi(t) \), then the probability that a neighbor of a player selected uniformly at random from the network has degree \( t \) is proportional to \( t \xi(t) \), as a player of degree \( t \) has \( t \) times more neighbors than a player of degree 1. Hence, for each \( t \in \mathbb{N}_0 \), the probability that the neighbor of a player selected uniformly at random from the network has type (degree) \( t \) is

\[
\sum_{s \in \mathbb{N}_0} \frac{t \xi(t)}{s \xi(s)}.
\]

Then, for each neighbor type profile \( \theta = (\theta_1, \ldots, \theta_t) \in \Omega^t_n \), for all \( k \in \mathbb{N}_0 \), let \( c_k(\theta) \) be the number of elements in \( \theta \) that are equal to \( k \). Define

\[
M(\theta) := \frac{t!}{\prod_{k \in \mathbb{N}_0} c_k(\theta)!},
\]

where we recall that \( 0! = 1 \). That is, \( M(\theta) \) is the multinomial coefficient corresponding to \( \theta \). Then, for each \( t \in \mathbb{N}_0 \) such that \( \mu(t) > 0 \), for each \( \theta = (\theta_1, \ldots, \theta_t) \in \Omega^t_n \), the conditional belief that a player’s neighbor type profile is \( \theta \) given that he has type \( t \) is

\[
\mu_{\text{ind}}(\theta \mid t) = M(\theta) \prod_{s \in \mathbb{N}_0, \xi(s) > 0} \left( \frac{s \xi(s)}{\sum_{s \in \mathbb{N}_0} r \xi(r)} \right)^{c_s(\theta)},
\]

where we have used that \( x^0 = 1 \) for \( x > 0 \). In words, the conditional distribution of neighbors’ type, given that the “central” player has type \( t \) is given by the multinomial distribution. That is, the types of the player’s neighbors are drawn in \( t \) independent trials, with the probability that a neighbor has type \( s \) given by \( \eta(s) \). It is important to note that \( \mu_{\text{ind}}(\theta \mid t) \) does not depend on \( t \).

---

\( n - 1 \), which is clearly impossible. However, these assumptions can be justified in the following way. Kets (2007b) discusses a network belief system that, given a distribution \( \eta \) with support in \( \mathbb{N}_0 \), gives rise to a prior on \( \mathcal{G}^{(n)} \) that induces a degree distribution (type distribution) that is close to \( (\eta(t))_{t \in \mathbb{N}_0} \) such that degrees (types) are almost independent when \( n \) is large. Furthermore, as shown in the next section, priors that are close in terms of the type distribution and the correlation among player types that they induce are similar in terms of game-theoretic predictions. This means that the results we would obtain under a type distribution close to \( (\xi(t))_{t \in \mathbb{N}_0} \) and under almost independent types will be very similar in game-theoretic terms to the results we derive here for type distribution \( (\xi(t))_{t \in \mathbb{N}_0} \) and independent types.

\( \text{Note that } \mu_{\text{ind}}(t), t \in \mathbb{N}_0, \text{ and } \mu_{\text{ind}}(\theta \mid t) \text{ for } t \in \mathbb{N}_0, \theta \in \Omega^t_n, \text{ are not derived from some prior on a set of networks, as in the rest of the paper.} \)
Hence, the core-periphery prior and the independent types prior are very similar in terms of the type distribution they induce. Under the independent types prior, the type distribution is exactly \((\xi(t))_{t \in \mathbb{N}_0}\), while under the core-periphery prior it is close to \((\xi(t))_{t \in \mathbb{N}_0}\) (assuming that the number of players is large). However, the two priors are very different in the type correlation they induce. Under the independent types prior, types are independent. By contrast, under the core-periphery prior, players of type 2 only interact with players of type 1, 2 and 4, players of type 4 only interact with players of type 2, 4 and 8, and so on.

An important question is whether the two priors are similar from a game-theoretic perspective. It is easy to see that under both priors, there is a symmetric Bayesian-Nash equilibrium in which all players invest, regardless of their type, and a symmetric Bayesian-Nash equilibrium in which no player invests for any type he ends up having. There are also so-called threshold equilibria in which players invest if and only if their type is above or below some threshold. We show that there is a threshold equilibrium under the independent types prior such that there is no corresponding symmetric approximate equilibrium under the core-periphery prior and vice versa. Hence, the set of equilibria changes substantively when we change the correlation among player types.

We start by showing that there is a threshold equilibrium under the independent types prior such that there is no corresponding approximate equilibrium under the core-periphery prior. First, for \(t, \bar{t} \in \mathbb{N}_0\), define

\[
\begin{align*}
f(t; \bar{t}) := & 3c \cdot \left( \frac{\sum_{t' \leq \bar{t}} t' \xi(t')} {\sum_{s \in \mathbb{N}_0} s \xi(s)} \right)^t - c \cdot \left( 1 - \frac{\sum_{t' \leq \bar{t}} t' \xi(t')} {\sum_{s \in \mathbb{N}_0} s \xi(s)} \right)^t.
\end{align*}
\]

When the number of players is sufficiently large, there is a unique \(\bar{t} \in \{1, 2, 3, \ldots, 2^\nu - 1\}\) such that

\[
f(t; \bar{t}) \geq 0 \iff t \leq \bar{t}.
\]

In that case, the expected payoffs under the independent types prior to a player of type \(t\) who chooses action \(R\) when other players follow the strategy of investing if and only if their type does not exceed the threshold \(\bar{t}\) are given by \(f(t; \bar{t})\). Then, it is easy to see that there is a symmetric Bayesian-Nash equilibrium under the independent types prior in which players invest if and only if their type is at most \(\bar{t}\).²

---

²Such a threshold exists. For each \(\bar{t} \in \mathbb{N}_0\), \(f(t; \bar{t})\) is declining in \(t\), and for each \(t \in \mathbb{N}_0\), \(f(t; \bar{t}_1) > f(t; \bar{t}_2)\) whenever \(\bar{t}_1 > \bar{t}_2\). Hence, there exists a unique \(\bar{t} \in \mathbb{N}_0\) such that \(f(t; \bar{t}) \geq 0\) if and only if \(t \leq \bar{t}\); by choosing the number of players large enough, we have \(\bar{t} \in \{1, 2, 3, \ldots, 2^\nu - 1\}\).

²This result is not in contradiction with Proposition 2 of Galeotti et al. (2006), which shows that under independent types and strict strategic complements, every symmetric Bayesian-Nash equilibrium is monotone.
By contrast, there does not exist a corresponding \( \varepsilon \)-equilibrium under the core-periphery prior for \( \varepsilon \) sufficiently small. To see this, suppose by contradiction that there would exist a threshold \( \bar{t} \) such that players would invest if and only if their type is at most \( \bar{t} \), and consider the lowest type \( s_{\min} := \min\{s \in \{1, 2, 4, \ldots, 2^{\nu}\} \mid s > \bar{t}\} \) that does not invest under this proposed equilibrium. The conditional probability that a player of type \( t = 2^\ell \), where \( \ell \in \{1, 2, \ldots, \nu - 1\} \), is a core player rather than a peripheral player, i.e., that all his neighbors invest under the proposed equilibrium, is
\[
\frac{1 \cdot n_\ell}{1 \cdot n_\ell + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{1}{3},
\]
independent of \( t \). Consequently, the interim expected payoffs to a player with type \( s_{\min} \) of \( R \) under the proposed equilibrium are
\[
\frac{3c}{3} - \frac{2c}{3} = \frac{c}{3} > 0.
\]
Hence, for \( \varepsilon < c/3 \), it is an \( \varepsilon \)-best response to choose \( R \) for players with type \( s_{\min} \). But then, by the same argument, players with the next lowest type that does not invest under the proposed strategy will also find it profitable to invest (in terms of \( \varepsilon \)-best responses), and so on. Hence, there exists no \( \varepsilon \)-equilibrium under the core-periphery prior corresponding to the threshold equilibrium under the independent types prior if \( \varepsilon \) is sufficiently small.

We now show that there is a threshold equilibrium under the core-periphery prior which is not an (approximate) equilibrium under the independent types prior. Let \( \hat{t} \in \{1, 2, 3, \ldots, 2^{\nu-1}\} \), and consider the symmetric strategy profile in which players invest if and only if their type is at least \( \hat{t} \). As the interim expected payoffs of \( R \) to players of type \( t \) are declining in \( t \) for any such threshold strategy under the independent types prior, this strategy cannot be an \( \varepsilon \)-equilibrium under this prior for \( \varepsilon \) sufficiently small. However, such a strategy is a Bayesian-Nash equilibrium for any \( \hat{t} \in \{1, 2, 3, \ldots, 2^{\nu-1}\} \) under the core-periphery prior. Fix \( \hat{t} \), and suppose players play \( R \) if and only if their type is at least \( \hat{t} \). Consider a player of type \( t = 2^\ell \geq \hat{t} \). With conditional probability
\[
\frac{2^{\ell+1} \cdot n_{\ell+1}}{1 \cdot n_\ell + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{2}{3},
\]
all his neighbors play \( R \), so that he earns a payoff of \( 3c \); with conditional probability
\[
\frac{1 \cdot n_\ell}{1 \cdot n_\ell + 2^{\ell+1} \cdot n_{\ell+1}} = \frac{1}{3},
\]
increasing in type (in the current setting, if low types invest, then high types invest, but not vice versa), as they assume that payoffs satisfy some additional property that is not satisfied by the current example.
some neighbors play $S$, giving him a payoff of $-c$. His interim expected payoffs are thus $(6c/3) - (2c/3) > 0$, so that he cannot gain by deviating. Now consider a player of type $t < \hat{t}$. With probability 1, at least some of his neighbors play $S$, so his best response is to play $S$ as well. Hence, under the core-periphery prior, there exists a threshold equilibrium in which players invest if and only if their type exceeds some threshold.

These examples show that strategy profiles that are Bayesian-Nash equilibria under one prior, may not be (approximate) equilibria under a prior which only differs from the first prior in the type correlation it induces. Note that if the number of players is sufficiently large, there are multiple threshold strategies that induce a symmetric Bayesian-Nash equilibrium under the core-periphery prior. By contrast, there is a unique threshold equilibrium strategy under the independent types prior. Hence, by choosing the constant $c$ appropriately, we can find a threshold equilibrium under the core-periphery prior such that there is no symmetric approximate equilibrium under the independent types priors that is close to this threshold equilibrium in terms of ex ante expected payoffs. Hence, even though the priors are very close in terms of the type distribution they induce (for a large number of players), the strategic distance between them (given $c$ and $\varepsilon$) can be large.

Similar examples can be constructed for other games, e.g. games with strategic substitutes (cf. Galeotti et al., 2006). In the next section, we show that the type distribution and the type correlation indeed determine the strategic distance between priors.

### 4.3 Main result

The example in the previous section suggests that differences in correlation among player types are an important determinant of the strategic distance between two priors. It is intuitive that also the type distribution induced by priors plays an important role. As we show in Lemma 4.3 below, closeness of priors in terms of the type distribution and the correlation among player types is equivalent to closeness in terms of the prior probabilities assigned to local events, i.e., events involving the type of a player and his neighbors. Hence, for $\mu, \mu' \in \mathcal{M}^{(n)}$, define

$$d^*(\mu, \mu') := \max_{F \in \mathcal{F}_K} |\mu(F) - \mu'(F)|.$$ 

That is, $d^*(\mu, \mu')$ measures the difference in probabilities assigned by $\mu$ and $\mu'$ to local events, or, equivalently (by Lemma 4.3), the difference in the type distribution and the type correlations induced by $\mu$ and $\mu'$.

Theorem 4.2 establishes that convergence of priors in terms of prior probabilities assigned to local events is in fact necessary and sufficient for strategic convergence.
**Theorem 4.2** Let \( \mu \in \mathcal{M}^{(n)} \) and let \( (\mu^k)_{k \in \mathbb{N}} \) be a sequence in \( \mathcal{M}^{(n)} \). Then, \( (\mu^k)_{k \in \mathbb{N}} \) converges strategically to \( \mu \) if and only if
\[
\lim_{k \to \infty} d^s(\mu, \mu^k) = 0.
\]
The proof of Theorem 4.2 follows from Proposition 4.5 and Lemma 4.6. Proposition 4.5 shows that if two priors \( \mu, \mu' \) are close in terms of the prior probabilities assigned to local events, then for any Bayesian network game, for any symmetric Bayesian-Nash equilibrium of the game in which players hold the prior \( \mu \), there exists a symmetric approximate equilibrium in the game with prior \( \mu' \) such that ex ante payoffs are similar. Proposition 4.5 uses Lemma 4.3 and Lemma 4.4.

**Lemma 4.3** Let \( \mu \in \mathcal{M}^{(n)} \), and let \( (\mu^k)_{k \in \mathbb{N}} \) be a sequence in \( \mathcal{M}^{(n)} \). Let \( T' \) be the set of types \( t \in T \) such that \( \mu(t) > 0 \) and \( \mu^k(t) > 0 \) for all \( k \in \mathbb{N} \). Suppose that \( T' \) is nonempty, and that there exists \( c > 0 \) such that \( \mu^k(t) \geq c \) for all \( t \in T' \) uniformly over \( k \in \mathbb{N} \). Then,
\[
\lim_{k \to \infty} \max_{F \in \mathcal{F}_K} |\mu(F) - \mu^k(F)| = 0 \iff \begin{cases} 
\lim_{k \to \infty} \max_{t \in T} |\mu(t) - \mu^k(t)| = 0, \\
\lim_{k \to \infty} \max_{t \in T', F \in \mathcal{F}_K} |\mu(F | t) - \mu^k(F | t)| = 0.
\end{cases}
\]

**Proof.** See Appendix A. \( \square \)

**Lemma 4.4** Let \( \mu, \mu' \in \mathcal{M}^{(n)} \), and let \( \gamma > 0 \). Let \( v \) be a profile of local payoff functions with bound \( B \). There exists \( \delta > 0 \) such that if \( \sigma \) is a symmetric Bayesian-Nash equilibrium of \( (\mu, v) \) and \( d^s(\mu, \mu') \leq \delta \), then there exists a symmetric \( 3\gamma B \)-equilibrium \( \sigma' \) of the game \( (\mu', v) \) with \( \sigma'(. | t) = \sigma(. | t) \) for all \( t \in T \) such that \( \mu(t) > 0 \) and \( \mu'(t) > 0 \).

**Proof.** Define
\[
S_{\mu, \mu'} := \{ t \in T \mid \mu(t) > 0 \text{ and } \mu'(t) > 0 \}
\]
to be the set of types that occur with positive probability under both \( \mu \) and \( \mu' \). Recall that \( \hat{\sigma} = (\hat{\sigma}_t)_{t \in T} \) is defined by:
\[
\forall t \in T, a \in A : \quad \hat{\sigma}_t(a) = \sigma_t(a | t) \quad \text{for any } i \in N.
\]
Set \( \hat{\sigma}'_t := \hat{\sigma}_t \) for all types \( t \in S_{\mu, \mu'} \). For \( t \notin S_{\mu, \mu'} \), take \( \hat{\sigma}'_t \) such that \( (\hat{\sigma}'_t)_{t \notin S_{\mu, \mu'}} \) induces a symmetric Bayesian-Nash equilibrium of the reduced game where each player \( i \in N \) with a type \( t \in S_{\mu, \mu'} \) is required to play \( \hat{\sigma}'_t = \hat{\sigma}_t \). Such an equilibrium exists by Proposition 3.2. By construction, \( \sigma' \) is a best response for players with types \( t \notin S_{\mu, \mu'} \). We need to show that \( \sigma' \) is a \( 3\gamma B \)-best response for a type \( t \in S_{\mu, \mu'} \). First, let
\[
S'_{\mu'} := \{ t \in T \mid \mu'(t) > 0 \}
\]
...
be the set of types that have positive probability under \( \mu' \). Also, let \( H \in \mathcal{F}_K \) be the event that a player interacts with at least one player with a type that has positive probability under \( \mu' \) but not under \( \mu \), i.e.,

\[
H = \bigcup_{t \in T} \{(\theta_1, \ldots, \theta_t) \in \Omega_K^t \mid \exists \ell \in \{1, \ldots, t\} : \theta_\ell \in S_{\mu'} \setminus S_{\mu,\mu'} \},
\]

and let \( H^c \) be the complement (relative to \( \Omega_K \)) of \( H \). By definition,

\[
\mu(H \mid t) = 0 \text{ for all } t \in S_{\mu,\mu'}.
\] (4.1)

By Lemma 4.3, there is a \( \delta > 0 \) such that if \( d^*(\mu, \mu') \leq \delta \),

\[
\max_{t \in S_{\mu,\mu'}; F \in \mathcal{F}_K} |\mu(F \mid t) - \mu'(F \mid t)| \leq \gamma.
\] (4.2)

Combining (4.1) and (4.2) gives

\[
\forall t \in S_{\mu,\mu'} : \quad \mu'(H \mid t) \leq \gamma.
\] (4.3)

Let \( t \in S_{\mu,\mu'} \), and let \( a, b \in A \) with \( \hat{\sigma}_t(a) > 0 \). Then,

\[
|\hat{\varphi}_t(a, \hat{\sigma}'; \mu') - \hat{\varphi}_t(b, \hat{\sigma}'; \mu')| \leq \sum_{t \in H} \mu'(\theta \mid t) |v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)| + \sum_{\theta \in H^c} \mu'(\theta \mid t) |v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)|.
\] (4.4)

By (4.3), recalling that the bound on \( v \) is \( B \), the first sum in (4.4) is at most \( \gamma B \). To evaluate the second sum, first note that the neighbors of a player with neighbor type profile \( \theta \in H^c \) play according to \( \hat{\sigma} \). As \( a \) lies in the support of the symmetric Bayesian-Nash equilibrium \( \sigma \) of \( (\mu, v) \),

\[
\sum_{\theta \in \Omega_K} \mu(\theta \mid t) v_t(a, \hat{\sigma}_\theta, \theta) \geq \sum_{\theta \in \Omega_K} \mu(\theta \mid t) v_t(b, \hat{\sigma}_\theta, \theta).
\] (4.5)

Using that \( \mu(\theta \mid t) = 0 \) for all \( \theta \in H \), we can rewrite (4.5) to find:

\[
\sum_{\theta \in H^c} \mu(\theta \mid t) [v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)] \geq - \sum_{\theta \in H} \mu(\theta \mid t) [v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)] = 0.
\] (4.6)

Define \( G_t := \{\theta \in H^c \mid \mu(\theta \mid t) - \mu'(\theta \mid t) > 0\} \) and let \( G_t^c \) be the complement of \( G_t \) relative to \( H^c \). For notational simplicity, define

\[
V_{\mu,\mu'}(a, b; \hat{\sigma}) := \left| \sum_{\theta \in H^c} (\mu(\theta \mid t) - \mu'(\theta \mid t)) (v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)) \right|.
\]
Using (4.2), it follows that
\[
V_{\mu, \mu'}(a, b; \hat{\sigma}) \leq \sum_{\theta \in G} (\mu(\theta | t) - \mu'(\theta | t)) |v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)| + \sum_{\theta \in G^c} (\mu'(\theta | t) - \mu(\theta | t)) |v_t(a, \hat{\sigma}_\theta, \theta) - v_t(b, \hat{\sigma}_\theta, \theta)| \\
\leq 2\gamma B. \tag{4.7}
\]
Combining (4.6) and (4.7) gives
\[
|\hat{\phi}_t(a, \hat{\sigma}; \mu') - \hat{\phi}_t(b, \hat{\sigma}; \mu')| \leq 3\gamma B. \quad \Box
\]

**Proposition 4.5** Let \( \mu, \mu' \in \mathcal{M}^{(n)} \), and fix \( \gamma > 0 \). Let \( v \) be a profile of local payoff functions with bound \( B \), and let \( \delta > 0 \) as in Lemma 4.4. Let \( \eta \in (0, \delta] \), and suppose that \( d^*(\mu, \mu') \leq \eta \).

Then, if \( \sigma \) is a symmetric Bayesian-Nash equilibrium of the game \( (\mu, v) \), there exists a symmetric \( 3\gamma B \)-equilibrium \( \sigma' \) of the game \( (\mu', v) \) such that
\[
|\hat{\Phi}(\hat{\sigma}; \mu') - \hat{\Phi}(\hat{\sigma'}; \mu')| \leq 4\eta B,
\]
where \( \hat{\sigma} = (\hat{\sigma}_t)_{t \in T} \) and \( \hat{\sigma}' = (\hat{\sigma}'_t)_{t \in T} \) are defined by \( \hat{\sigma}_t = \sigma_t(\cdot | t) \) and \( \hat{\sigma}'_t = \sigma'_t(\cdot | t) \) for any \( i \in N \) for all \( t \in T \).

**Proof.** Let \( \sigma \) be a symmetric Bayesian-Nash equilibrium of \( (\mu, v) \). By Lemma 4.4, there exists a symmetric \( 3\gamma B \)-equilibrium \( \sigma' \) of the game \( (\mu', v) \) such that \( \mathcal{G} \) for \( t \in T \) such that \( \mu(t) > 0 \) and \( \mu'(t) > 0 \). Define
\[
G := \{ \theta \in \Omega_K \mid \mu(\theta) - \mu'(\theta) > 0 \},
\]
and let \( G^c \) be the complement of \( G \) relative to \( \Omega_K \). Define the function \( \zeta : \Omega_K \to T \) by \( \zeta(\theta) = t \) whenever \( \theta \in \Omega_K^t \). That is, the function \( \zeta \) gives the type of a player for each possible neighbor type profile he may have. Then,
\[
|\hat{\Phi}(\hat{\sigma}; \mu') - \hat{\Phi}(\hat{\sigma}; \mu)| \leq \sum_{\theta \in G} (\mu(\theta) - \mu'(\theta)) \sum_{a \in A} \hat{\sigma}_\zeta(\theta)(a) |v_{\zeta}(a, \hat{\sigma}_\theta, \theta)| + \sum_{\theta \in G^c} (\mu'(\theta) - \mu(\theta)) \sum_{a \in A} \hat{\sigma}_\zeta(\theta)(a) |v_{\zeta}(a, \hat{\sigma}_\theta, \theta)| \\
\leq 2\eta B. \tag{4.8}
\]

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Also, define
\[ F_{\mu'} := \{ \theta \in \Omega_K \mid \mu'((\theta)) > 0 \}, \]
\[ F_{\mu,\mu'} := \{ \theta \in \Omega_K \mid \mu((\theta)) > 0 \text{ and } \mu'((\theta)) > 0 \}. \]

Then, as \( \mu(F_{\mu'} \setminus F_{\mu,\mu'}) = 0 \) by definition,
\[ \mu'(F_{\mu'} \setminus F_{\mu,\mu'}) \leq \eta. \]

Recalling that \( \hat{\sigma}'_t = \hat{\sigma}_t \) for \( t \) such that \( \mu(t) > 0 \) and \( \mu'(t) > 0 \), this yields
\[
|\hat{\Phi}(\hat{\sigma}'; \mu') - \hat{\Phi}(\hat{\sigma}; \mu')| \\
\leq \sum_{\theta \in F_{\mu,\mu'}} \mu'(\theta) \left| \sum_{a \in A} \hat{\sigma}'(\theta)(a) v(\theta)(a, \hat{\sigma}'(\theta), \theta) - \sum_{a \in A} \hat{\sigma}'(\theta)(a) v(\theta)(a, \hat{\sigma}(\theta), \theta) \right| + \\
\sum_{\theta \in F_{\mu'} \setminus F_{\mu,\mu'}} \mu'(\theta) \left| \sum_{a \in A} \hat{\sigma}'(\theta)(a) v(\theta)(a, \hat{\sigma}'(\theta), \theta) - \sum_{a \in A} \hat{\sigma}'(\theta)(a) v(\theta)(a, \hat{\sigma}(\theta), \theta) \right| \\
= \sum_{\theta \in F_{\mu'} \setminus F_{\mu,\mu'}} \mu'(\theta) \left| \sum_{a \in A} \hat{\sigma}'(\theta)(a) v(\theta)(a, \hat{\sigma}'(\theta), \theta) - \sum_{a \in A} \hat{\sigma}'(\theta)(a) v(\theta)(a, \hat{\sigma}(\theta), \theta) \right| \\
\leq 2\eta B. \tag{4.9}
\]

Combining (4.8) and (4.9) gives
\[ |\hat{\Phi}(\hat{\sigma}; \mu) - \hat{\Phi}(\hat{\sigma}'; \mu')| \leq 4\eta B. \quad \square \]

Proposition 4.5 establishes the sufficiency of our condition for strategic convergence. Lemma 4.6 below shows that the condition that \( d^* \) should be small is also necessary for strategic convergence.

**Lemma 4.6** Let \( \delta \in [0, 1] \) and let \( \mu, \mu' \in \mathcal{M}^{(n)} \). If
\[ d^*(\mu, \mu') > \delta, \]
then there exists a profile \( v \) of local payoff functions with bound \( B = 1 \) and a symmetric Bayesian-Nash equilibrium \( \sigma \) of the game \((\mu, v)\) such that for any symmetric \( \delta \)-equilibrium \( \sigma' \) of \((\mu', v)\), it holds that
\[ |\hat{\Phi}(\hat{\sigma}; \mu) - \hat{\Phi}(\hat{\sigma}'; \mu')| > \delta, \]
where \( \hat{\sigma} = (\hat{\sigma}_t)_{t \in T} \) and \( \hat{\sigma}' = (\hat{\sigma}'_t)_{t \in T} \) are defined by \( \hat{\sigma}_t = \sigma_i(\cdot \mid t) \) and \( \hat{\sigma}'_t = \sigma'_i(\cdot \mid t) \) for any \( i \in N \) for all \( t \in T \).
**Proof.** By assumption, there exists a set of neighbor type profiles $F \in \mathcal{F}_K$ such that $|\mu(F) - \mu'(F)| > \delta$. For each $t \in T, t > 0, a \in A, a^{(t)} \in A^t$ and $\theta \in \Omega_K$, let

$$v_t(a, a^{(t)}, \theta) = \begin{cases} 1 & \text{if } \theta \in F, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$|\hat{\Phi}(\hat{\tau}; \mu) - \hat{\Phi}(\hat{\tau}'; \mu')| > \delta$$

for any two symmetric strategy profiles $\sigma, \sigma' \in \Sigma^n$. □

We can now prove Theorem 4.2:

**Proof.** (If) Let $v$ be a profile of local payoff functions with bound $B$. Let $\gamma > 0$ be arbitrarily small, and let $\delta > 0$ be as in Lemma 4.4. Take any $\epsilon \in (0, \delta]$. Since $d^*(\mu, \mu^k) \to 0$ as $k \to \infty$, it holds that $d^*(\mu, \mu^k) \leq \epsilon$ for all $k$ sufficiently large. Hence, by Proposition 4.5,

$$\chi^*(\mu, \mu^k; v, 3B\gamma) \leq 4B\epsilon$$

for $k$ sufficiently large. That is, if $d^*(\mu, \mu^k) \to 0$ as $n \to \infty$, then, for any $v$ and any $c > 0$, we have $\chi^*(\mu, \mu^k; v, c) \to 0$ as $k \to \infty$.

(Only if) Let $\mu, \mu' \in \mathcal{M}^n$. For $\delta \in [0, 1)$, if $d^*(\mu, \mu') > \delta$, then, by Lemma 4.6, there exists a profile of local payoff functions $v$ with bound $B = 1$ and a symmetric Bayesian-Nash equilibrium $\sigma \in \Sigma^n$ of $(\mu, v)$ such that for any symmetric $\delta$-equilibrium $\sigma' \in \Sigma^n$ of $(\mu', v)$,

$$|\hat{\Phi}(\hat{\tau}; \mu) - \hat{\Phi}(\hat{\tau}'; \mu')| > \delta.$$

□

Theorem 4.2 shows that for two priors to be close in a strategic sense, it is necessary and sufficient for them to be close in terms of prior probabilities they assign to local events, i.e., events that involve the types of a player and his neighbors. This result has two important implications. Firstly, this result means that in order to explore the full range of strategic outcomes in Bayesian network games, it is sufficient to vary the type distribution and the correlation among player types. Hence, on the one hand, it suggests that varying the type distribution, as has been the focus of much of the literature so far, is often not enough. On the other hand, it limits the set of priors that one needs to consider. We show that priors need only be varied along two dimensions. A second important implication is that we can interpret a Bayesian network game as a set of overlapping “local games”, and that we do not need to concern ourselves with the nonlocal features of network belief systems. If we refer back to the networks in Figure 1(a) and (b), and consider two priors, one which gives probability 1 to the network in Figure 1, and the other giving probability 1 to the network in Figure 1, we see that Theorem 4.2 tells us that these priors are identical in terms of their
game-theoretic predictions, even though they are very different in terms of the networks they predict.

Hence, by exploiting the symmetry of the game and the local features of the payoff functions, it is possible to weaken the conditions of Kajii and Morris (1998) for this particular class of Bayesian games. Kajii and Morris (1998) show that for general Bayesian games with finite type sets, priors need to be close in terms of the prior probabilities they assign to all possible events. By contrast, we only require that priors are close in terms of the prior probabilities assigned to local events. While it is not surprising that we can weaken the general result of Kajii and Morris (1998) for a subclass of games, it yields the useful insight that we can treat Bayesian network games as a collection of overlapping local games, and that the important features of priors in terms of strategic outcomes are the type distribution and correlation among player types that they induce.

We end this section with a discussion of our framework and our assumptions. Firstly, in this paper, we have focused on symmetric equilibria, as this is the focus of much of the literature on Bayesian network games (e.g. Galeotti et al., 2006; Jackson and Yariv, 2007; Sundararajan, 2005). It is possible to derive similar results for general Bayesian-Nash equilibria, though it will not be possible to exploit the symmetry of the game as we have done here. If one would consider general equilibria, results similar to those of Kajii and Morris (1998) would be obtained.

Secondly, our definition of strategic closeness requires that ex ante expected payoffs be close in equilibria under two priors, i.e., we focus on payoff continuity. An alternative continuity notion would require that with high probability, a player and his neighbors follow the same strategies under the two priors (cf. Monderer and Samet, 1996). Indeed, from the proof of Proposition 4.5, it follows that if two priors are close in terms of the measure $d^*$, then for each symmetric Bayesian-Nash equilibrium under one of the priors, there exists an approximate equilibrium under the other prior which coincides with the first equilibrium for all types that have positive probability under both priors, i.e., there is also continuity in terms of strategies. However, in the current setting (unlike in the setting of e.g. Monderer and Samet, 1996), we also have to consider the difference in prior probabilities that players have a given type under the two priors in order to ensure that the two priors give rise to similar outcomes from a player’s ex ante perspective. Hence, the appropriate definition of strategic closeness in the current setting considers differences in ex ante expected payoffs.

Thirdly, while we study the general case in which payoffs depend on the actions and types of a player and his neighbors, one could also consider the special case in which a player’s payoffs depend only on his own action and type and on the actions of his neighbors, and not
on his neighbors’ types. Obviously, for this subclass of games, the condition we derived for strategic convergence is still sufficient, though it may not be necessary. Our conjecture is that the condition cannot be weakened substantially for this subclass of games.

Finally, in line with the literature, we have studied games in which a player’s payoffs only depend on the actions and types of his direct neighbors. Our result can easily be generalized to the case where a player’s payoff depends on the actions and types of those within \( k \) steps in the network, for some \( k \in \mathbb{N} \). Of course, when \( k \) increases, the condition for two priors to be close becomes more strict, ultimately recovering the condition of Kajii and Morris (1998) that priors need to be close in terms of the prior probabilities they assign to global events. Indeed, when the payoffs to a player depend on the actions and types of all others in the network (even on the actions and types of those with whom he is not directly connected), the game can be alternatively modeled as a standard Bayesian game, with some suitable restrictions on payoffs.

5 Conclusions

Networks are ubiquitous in economics, and they can have a large effect on economic outcomes. The current paper considers a setting in which players are located on a network and play a fixed game with their neighbors. Players have incomplete information on the network structure. They have a common prior on a given class of networks, and, in addition, they have some local information on the network structure. Given the complexity of many social and economic networks, it is important to study whether game-theoretical predictions are sensitive to assumptions on players’ beliefs.

In the current paper, we have studied the conditions that are necessary and sufficient for two (common) priors to be close in a strategic sense. More specifically, we have studied the conditions under which for any Bayesian network game in which players hold one of these priors, for any symmetric equilibrium in that game, there is a symmetric approximate equilibrium in the associated game with the other prior such that ex ante expected payoffs are close under the two equilibria. Our main result (Theorem 4.2) states a necessary and sufficient condition for two priors to be close in this sense is that they be close in terms of the prior probabilities they assign to local events, i.e., events involving the type of an arbitrary player and his neighbors. An equivalent condition is that two priors be close in terms of the type distribution and the correlation among player types they induce (Lemma 4.3). Hence, the essential features of a prior in Bayesian network games are the type distribution and the type correlation it induces.
This result suggests that one needs to go beyond priors with independent types, which has been the focus of much of the literature so far. We have illustrated this point in Section 4.2, where we show that priors with the same type distribution can give rise to very different equilibria in a simple game, depending on the correlation among player types. The current result also puts restrictions on the set of priors one needs to consider. We show that one only needs to vary the type distribution and the correlation among types.

There are several directions for further research. Firstly, the current result indicates that it is important to systematically assess the effect of varying the type distribution and the correlations among player types on game-theoretic outcomes. While several authors study the effect of varying the type distribution in specific games (e.g. Jackson and Yariv, 2007; Sundararajan, 2005), there is little work on the effect on game-theoretic outcomes of changing the correlation among players’ types. Galeotti et al. (2006) analyze the effect of some specific changes in the type distribution and the correlation among player types in certain classes of games. However, there is no systematic exploration of the effect of changing the type correlations. Such an analysis will not be easy. There are two prime difficulties. The first is that it is not clear how type correlation should be measured. Galeotti et al. (2006) define the concepts of positive and negative association, but these only seem to capture some dimensions of type correlation. The second is that it is hard to define suitable network belief systems in which the appropriate dimensions of type correlations can be varied continuously in the appropriate way.

Another direction for future research is to study the sensitivity of game-theoretic predictions to the specification of players’ information. As in much of the literature, we have assumed that players only know the number of connections they have. However, in reality, players may have different “observational horizons” (Friedkin, 1983). It is an open question how this heterogeneity in information affects outcomes. Moreover, game-theoretic outcomes in network games will generally be different under incomplete and complete information (see Galeotti et al. (2006) for an example). It is not clear how predictions change if players’ information is varied from knowing only their direct environment to knowing the full network structure. The current paper illustrates that it is important to study such sensitivity questions.
Appendix A  Proofs

A.1 Proof of Proposition 3.2

Define the strategic game

\[ G := \langle N, (M_i)_{i \in N}, (\bar{\Phi}_i(\cdot; \mu))_{i \in N} \rangle, \]

where for each \( i \in N \), the set of pure strategies \( M_i \) is the set of maps \( m_i : T \to A \). Hence, we have \( M_i =: M \) for all \( i \in N \), and the set \( M \) is finite. For each \( i \in N \), the payoff function \( \bar{\Phi}_i(\cdot; \mu) \) is defined by:

\[ \forall m \in M^n : \bar{\Phi}_i(m; \mu) := \sum_{g \in G^{(n)}} \mu(g) v_{\tau_i(g)}(m_i(\tau_i(g)), (m_j(\tau_j(g)))_{j \in N_i(g)}, (\tau_j(g))_{j \in N_i(g)}). \]

Mixed strategies are obtained by randomizing over strategies in the set \( M \). Denote the set of mixed strategies in \( G \) by \( \Delta(M) \). Payoffs can be extended to mixed strategies in the standard way. That is, for each \( i \in N \), \( \beta \in (\Delta(M))^n \),

\[ \bar{\Phi}_i(\beta; \mu) := \sum_{g \in G^{(n)}} \mu(g) \sum_{m_i \in M} \beta_i(m_i) \sum_{m \in M^{\tau_i(g)}} \left( \prod_{j \in N_i(g)} \beta_j(m_j) \right) v_{\tau_i(g)}(m_i(\tau_i(g)), m_{N_i(g)}, \tau_{N_i(g)}), \]

where we have defined \( m_{N_i(g)} := (m_j(\tau_j(g)))_{j \in N_i(g)} \) and \( \tau_{N_i(g)} := (\tau_j(g))_{j \in N_i(g)} \).

The proof now follows from two steps:

**Step 1:** There exists \( \beta = (\beta_j)_{j \in N} \in (\Delta(M))^n \) with \( \beta_i = \beta_j \) for all \( i, j \in N \) such that for all \( i \in N \),

\[ \bar{\Phi}_i(\beta; \mu) \geq \bar{\Phi}_i(\beta'_i, \beta_{-i}; \mu) \]

for all \( \beta'_i \in \Delta(M) \).

**Proof of Step 1:** The set \( \Delta(M) \) is a nonempty, convex and compact subset of the Euclidean space \( \mathbb{R}^{\vert M \vert} \), and, by standard arguments, \( \bar{\Phi}_i(\cdot; \mu) \) is continuous in \( \beta = (\beta_i, \beta_{-i}) \) and quasi-concave in \( \beta_i \). Furthermore, the game \( G \) is symmetric by Assumption 2.2 and the symmetry of the payoff functions. Define the correspondence \( B \) on \( \Delta(M) \) by:

\[ \forall \beta \in \Delta(M) : B(\beta) := \arg \max_{\alpha \in \Delta(M)} \bar{\Phi}_i(\alpha, \beta, \ldots, \beta; \mu) \text{ for any } i \in N, \]

i.e., \( B(\beta) \) is the set of best responses (mixed strategies) of a player when other players play according to \( \beta \). By standard arguments, the correspondence \( B \) is nonempty, convex-valued, and upper-hemicontinuous (e.g. Fudenberg and Tirole, 1991, pp. 29–30). Hence, by
Kakutani’s fixed point theorem (e.g. Ok, 2007, p. 331), a fixed point exists for $B$, i.e., there exists $\beta \in \Delta(M)$ such that $\beta \in B(\beta)$.

**Step 2:** Let $\beta = (\beta_j)_{j \in N} \in (\Delta(M))^n$ with $\beta_i = \beta_j$ for all $i, j \in N$ be such that for all $i \in N$,

$$\tilde{\Phi}_i(\beta; \mu) \geq \tilde{\Phi}_i(\beta'_i, \beta_{-i}; \mu)$$

for all $\beta'_i \in \Delta(M)$, and define $\sigma = (\sigma_j)_{j \in N} \in \Sigma^n$ by:

$$\forall i \in N, a_i \in A, t_i \in T : \quad \sigma_i(a_i \mid t_i) = \sum_{m_i \in M: m_i(t_i) = a_i} \beta_i(m_i).$$

Then, for each $i \in N$,

$$\Phi_i(\sigma; \mu) \geq \Phi_i(\sigma'_i, \sigma_{-i}; \mu)$$

for all $\sigma'_i \in \Sigma$.

**Proof of Step 2:** From substituting the relevant expressions, we obtain

$$\forall i \in N : \Phi_i(\sigma; \mu) = \tilde{\Phi}_i(\beta; \mu).$$

Suppose by contradiction that there exists $i \in N$ such that for some $\sigma'_i \in \Sigma$,

$$\Phi_i(\sigma'_i, \sigma_{-i}; \mu) > \Phi_i(\sigma; \mu).$$

Define $\beta'_i \in \Delta(M)$ by:

$$\forall m_i \in M : \quad \beta'_i(m_i) := \prod_{t_i \in T} \sigma'_i(m_i(t_i) \mid t_i).$$

Then, again by substitution,

$$\tilde{\Phi}(\beta'_i, \beta_{-i}; \mu) = \Phi_i(\sigma'_i, \sigma_{-i}; \mu) > \Phi_i(\sigma; \mu) = \tilde{\Phi}(\beta; \mu),$$

which contradicts that no player in $G$ can gain by deviating unilaterally from $\beta$.

From Step 1 and 2 it follows that there exists $\sigma = (\sigma_j)_{j \in N} \in \Sigma^n$ such that for all $i \in N$,

$$\Phi_i(\sigma; \mu) \geq \Phi_i(\sigma'_i, \sigma_{-i}; \mu)$$

for all $\sigma'_i \in \Sigma$, and $\sigma_i = \sigma_j$ for all $i, j \in N$, i.e., there exists a symmetric Bayesian-Nash equilibrium for $(\mu, v)$. 

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A.2 Proof of Lemma 4.3

(If) Suppose that \( \lim_{k \to \infty} \max_{F \in \mathcal{F}_K} |\mu(F) - \mu^k(F)| = 0 \). Then, clearly, for all \( k \in \mathbb{N} \),

\[
\max_{t \in T} |\mu(t) - \mu^k(t)| \leq \max_{F \in \mathcal{F}_K} |\mu(F) - \mu^k(F)|,
\]

and hence

\[
\lim_{k \to \infty} \max_{t \in T} |\mu(t) - \mu^k(t)| = 0.
\]

Also, for all \( k \in \mathbb{N} \),

\[
\max_{F \in \mathcal{F}_K} \left| \mu(F \mid t) - \mu^k(F \mid t) \right| = \max_{F \in \mathcal{F}_K} \left| \frac{\mu(F \cup \Omega^t_K)}{\mu(\Omega^t_K)} - \frac{\mu^k(F \cup \Omega^t_K)}{\mu^k(\Omega^t_K)} \right|
\]

\[
= \max_{F \in \mathcal{F}_K} \left| \frac{\mu(F \cup \Omega^t_K) - \mu^k(F \cup \Omega^t_K)}{\mu(\Omega^t_K)} + \frac{\mu(F \cup \Omega^t_K) - \mu^k(F \cup \Omega^t_K)}{\mu^k(\Omega^t_K)} \right|
\]

\[
\leq \max_{F \in \mathcal{F}_K} \frac{1}{\mu^k(t)} |\mu(F \cup \Omega^t_K) - \mu^k(F \cup \Omega^t_K)| + \max_{F \in \mathcal{F}_K} \frac{\mu(F \mid t)}{\mu^k(t)} |\mu(t) - \mu^k(t)|
\]

\[
\leq \left( \frac{2}{3} \right) \cdot \max_{F \in \mathcal{F}_K} |\mu(F) - \mu^k(F)|
\]

where we have used the triangle inequality for the first inequality. Hence,

\[
\lim_{k \to \infty} \max_{F \in \mathcal{F}_K} \left| \mu(F \mid t) - \mu^k(F \mid t) \right| = 0.
\]

(Only if) Suppose that

\[
\lim_{k \to \infty} \max_{t \in T'} |\mu(t) - \mu^k(t)| = 0 \quad \text{(A.1)}
\]

and

\[
\lim_{k \to \infty} \max_{F \in \mathcal{F}_K} \left| \mu(F \mid t) - \mu^k(F \mid t) \right| = 0. \quad \text{(A.2)}
\]

Fix \( \varepsilon > 0 \). For each \( F \in \mathcal{F}_K \) and each \( k \in \mathbb{N} \), it holds that

\[
|\mu(F) - \mu^k(F)| = \left| \sum_{t \in T'} [\mu(F \mid t) - \mu^k(F \mid t)] \mu(t) + \sum_{t \in T'} \mu^k(F \mid t)[\mu(t) - \mu^k(t)] \right|
\]

\[
\leq \sum_{t \in T'} |\mu(F \mid t) - \mu^k(F \mid t)| \mu(t) + \sum_{t \in T'} \mu^k(F \mid t)|\mu(t) - \mu^k(t)|.
\]
By (A.1) and (A.2), there exists $Q \in \mathbb{N}$ such that for all $t \in T'$, $k > Q$ implies that

$$|\mu(F | t) - \mu^k(F | t)| < \varepsilon \cdot \left(\frac{c}{1 + c}\right)$$

and

$$|\mu(t) - \mu^k(t)| < \varepsilon \cdot \left(\frac{c}{1 + c}\right).$$

Hence, for $k > Q$,

$$|\mu(F) - \mu^k(F)| \leq \varepsilon \cdot \left(\frac{c}{1 + c}\right) \left[\sum_{t \in T'} \mu(t) + \sum_{t \in T'} \mu^k(F | t)\right]$$

$$\leq \varepsilon \cdot \left(\frac{c}{1 + c}\right) \cdot \left[1 + \frac{1}{c} \sum_{t \in T'} \mu^k(F | t)\mu^k(t)\right]$$

$$\leq \varepsilon \cdot \left(\frac{c}{1 + c}\right) \cdot \left[1 + \frac{1}{c}\right]$$

$$= \varepsilon,$$

and hence $\lim_{k \to \infty} |\mu(F) - \mu^k(F)| = 0$. As this holds for all $F \in \mathcal{F}_K$, we have

$$\lim_{k \to \infty} \max_{F \in \mathcal{F}_K} |\mu(F) - \mu^k(F)| = 0.$$

References


