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Rodriguez, J.C.

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OPTION PRICING AND MOMENTUM

By Juan Carlos Rodríguez

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Option pricing and momentum

Juan Carlos Rodríguez*
Tilburg University and CentER

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Abstract

If managers are reluctant to fully adjust dividends to changes in earnings, stock returns and changes in the dividend yield will tend to be negatively correlated. When this is the case, stock returns will exhibit positive autocorrelation, or momentum. This paper studies the pricing of options in such a situation, within a new model in which the dividend yield is an affine function of past stock returns. The model accommodates momentum in stock returns under complete markets, and, moreover, it renders preference-free formulas for European options. A momentum-inducing dividend yield implies that calls will be overpriced (underpriced) relative to puts after stock price increases (declines), a prediction in line with the findings of recent empirical research in finance, and that the Black-Scholes formula with constant dividend yield underprices out-of-the money options.

Keywords: Options, Momentum, Stochastic convenience yield

JEL Classification: G13

*Address for correspondence: Dept. of Finance, Tilburg University. P.O. Box 90153, 5000 LE Tilburg, The Netherlands. Tel: +31 13 466-3262. Fax: +31 13 466-287. Email: j.c.rodriguez@uvt.nl.

1 Introduction

If a stock pays a stochastic dividend, the correlation between changes in the dividend yield and the returns on the stock will at least partially determine the autocorrelation of stock returns. A positive correlation between returns and the dividend yield will induce return mean reversion; a negative correlation between returns and the dividend yield will induce return continuation, or momentum. In reality, stock returns and changes in the dividend yield are negatively correlated, which can be explained by the reluctance of managers to fully adjust dividends to changes in earnings (see Lintner (1956)). As a consequence of this managerial behavior, the returns on stocks paying a stochastic dividend may exhibit a tendency to be positively autocorrelated.

Since the seminal work of Merton (1973), option valuation models of dividend paying assets have usually assumed a constant dividend yield. This assumption can be justified in the case on an individual stock, because options are relatively short-lived contracts and dividends are unlikely to change much during the life of an option. It is less appropriate for an index, because there will normally be a random flow of dividends paid on the index underlying stocks even over a short period of time. Moreover, Harvey and Whaley (1992) report that the assumption of a constant dividend yield, in the case of an index, can lead to large pricing errors, and Broadie, Detemple, Ghysels and Torres (2000) document that a stochastic dividend yield may partially affect the early exercise decision of American options.

Other consequences of a stochastic dividend yield on option pricing are also worth exploring. If the dividend yield induces momentum, the variance of stock returns will be larger than the variance corresponding to a random walk, and so the Black-Scholes model will underprice options at all maturities. Moreover, the past performance of the stock, through its influence on the dividend yield, may affect the expected return on the stock under the risk-neutral measure. If this happens, calls will look overpriced (underpriced), and puts underpriced (overpriced), relative to Black Scholes, after a row of positive (negative) returns.

These issues have more than a mere theoretical interest. In a recent paper, Amin, Coval and Seyhun (2004) find evidence that past returns on the S&P100 index influence OEX option prices. They document that violations of put-call parity condition (for American options) depend on stock market momentum. In particular strongly positive (negative) past market returns lead to these violations by increasing call (put) prices. They also report that put-call volatility spreads depend on past market returns, with calls relatively overpriced after large stock market increases, and puts relatively overpriced after large stock market declines.

In this paper I show that these empirical findings can be accounted for by an option pricing model with a stochastic dividend yield. To study this issue, I introduce a model in which the dividend yield is an affine function of past index performance. The model accommodates momentum in stock returns under complete markets, and it has the standard Black-Scholes (1973) model as a special case. Moreover, the model exhibits dependence of expected returns on past stock performance under the risk-neutral measure, and renders preference-free formulas for European options. In line with Amin et al. (2004) empirical findings, the model predicts that calls will be overpriced (underpriced) relative to puts after stock market increases (declines), and that put-call volatility spreads will be affected likewise by stock price performance. The price effects are more pronounced for out-of-the-money options. In particular, the model prices out- of-the-money calls and puts uniformly above the standard Black-Scholes formula, with call (put) price differences increasing (decreasing) with stock performance. These differences are economically significant. For example, suppose that the annual interest rate is 5%, the current stock price is 40, and the strike price is 45. With a 20% annual returns volatility and a 4% constant dividend yield, the Black-Scholes price of a 6-month call, under the random walk assumption, is 0.7073. Under a 4% mean stochastic dividend yield inducing a first autocorrelation of monthly returns as low as 1.7%, the price of the same call is 0.7303 (3.26% difference) after a 5% decline in the stock price, and 0.7618 (7.71% difference) after a 5% increase in the stock price. These results suggest that a momentum generating dividend yield may partially explain known biases of the Black-Scholes model.

The key assumption of the model introduced in this paper is that the dividend yield is an affine function of a weighted sum of past stock returns. This assumption makes changes in the dividend yield perfectly negatively correlated with spot returns, and it is aimed to capture the empirical fact that when stock prices move, dividends follow but at a slower pace. As an example, in the postwar period the index S&P500, on which many options are written, exhibits as correlation between index returns and changes in the dividend yield of -90.5%, so a perfect negative correlation, although strong, seems to be a reasonable assumption[1]. More importantly, this assumption leads to closed form solutions for option prices, which, in the case of the model studied in this paper, are preference-free and therefore potentially implementable.

[1] The correlation is calculated on monthly returns from 1-1946 to 12-2005. Data is taken from the Shiller database (http://www.econ.yale.edu/~shiller/data.htm).

The literature on option pricing with stochastic dividend yield is scarce. Geske (1978) was the first to derive an option pricing formula when the underlying stock has a stochastic dividend yield, and to point out that a major channel through which stochastic dividends may affect the option price is their impact on the variance of stock returns. However, there are important differences between Geske's model and mine. First, due to market incompleteness (he assumes that stock returns and dividend yield are imperfectly correlated) Geske (1978) must rely on an equilibrium argument (Rubinstein (1976)) to obtain the option formula, which depends on the CAPM market price of risk. Second, Geske (1978) does not consider the possible impact of past stock performance on option prices. Finally, although he suggests that a stochastic dividend yield may partly account for Black-Scholes biases, Geske (1978) does not pursue systematically the issue in his paper.

Lioui (2006) discusses derivative valuation of a stock with a stochastic dividend yield under complete markets. However, one of the main points in Lioui (2006) is that, even under market completeness, the stochastic dividend yield complicates the implementation of option formulas, because it is necessary to compute a risk premium. In this paper I find a parameterization of the dividend yield leading to option prices for which no risk premium must be estimated, which suggests that Lioui's (2006) results may not be general. Moreover, Liuoi (2006) discusses neither momentum, nor the possible impact of past stock performance on options pricing, as this paper does.

Another related paper is Lo and Wang (1995), who study option pricing when stock returns are predictable. Lo and Wang (1995) argue that predictability will have an effect on option pricing through the estimation of the variance of stock returns. If returns are predictable (and then autocorrelated), the estimate of the instantaneous variance can be seriously mispecified if it is computed under the wrong assumption that the stock price follows a random walk. In particular, the variance will be underestimated when stock returns are negatively autocorrelated, and overestimated when stock returns are positively autocorrelated. Note that this implies that Black-Scholes underprices options when returns exhibit mean reversion, and overprices options when returns exhibit momentum. Lo and Wang (1995) assume a nondividend paying stock. In contrast, in this paper predictability effects are induced by a stochastic dividend yield, and because of this the pricing consequences differ strongly from the pricing consequences derived in Lo and Wang (1995).

The structure of the paper is as follows. The model is presented in section 2 (some mathematical derivations are included in an Appendix at the end of the paper). The price distribution under the martingale measure is obtained in section 3. Option prices are derived in section 4. Section 5 presents numerical results. Finally, section 6 concludes.

2 Stock price dynamics

Let's assume a frictionless financial market in which trading is continuous. The stock price S_t satisfies the following differential equation:

$$\frac{dS_t}{S_t} = (\mu - \delta_t)\, dt + \sigma dW_t, \tag{1}$$

where μ is the total instantaneous expected return, δ_t is the stochastic dividend yield, and σ is the instantaneous return volatility. The only source of risk in the economy is a standard Wiener process, W_t, defined on a filtered probability space $(\Omega, \mathcal{F}, \Pi)$.

Let's now define $s_t = \log(S_t)$. Then, from equation (1):

$$ds_t = \left(\mu - \frac{1}{2}\sigma^2 - \delta_t\right) dt + \sigma dW_t, \tag{2}$$

Now define m_t as a weighted sum of past stock log returns, as:

$$m_t = \int_0^t e^{-\omega(t-u)} ds_u, \tag{3}$$

where $\omega \geq 0$ determines the weight of past returns. Next, to capture the dependence of the dividend yield on the price performance of the stock, let's assume that the dividend yield is an affine function of m_t in the following way:

$$\delta_t = \delta - \phi m_t, \tag{4}$$

where δ is a constant and $\phi \geq 0$ is the loading of performance on the dividend yield. If $\phi = 0$, we have a constant dividend yield. On the other hand, $\phi > 0$ makes the dividend yield negatively related to past stock performance. This is aimed to capture that dividends move in the direction of returns, but at a slower pace, a stylized fact that can be explained by the reluctance of managers to fully adjust dividends to changes in earnings[2]. By making explicit the dependence of the dividend yield on stock returns,

[2]For an early paper showing a model of partial adjustment of dividends to changes in earnings, see Lintner (1956)

equation (4) implies that past stock performance will affect the prices of options, an empirical fact first documented in Amin et al. (2004).

After these definitions, equation (2) can be rewritten as:

$$ds_t = \left(\mu - \frac{1}{2}\sigma^2 - \delta\right) dt + \phi m_t dt + \sigma dW_t, \tag{5}$$

Note that equations (3) and (5) determine endogenously the dynamics of m_t. Differentiating both sides of equation (3) gives:

$$dm_t = ds_t - \omega m_t dt, \tag{6}$$

which, after inserting equation (5), becomes:

$$dm_t = -(\omega - \phi)(m_t - \theta) dt + \sigma dW_t, \tag{7}$$

where $\theta = \frac{\mu - \frac{1}{2}\sigma^2 - \delta}{\omega - \phi}$. That is, m_t follows an Ornstein-Uhlenbeck process with long run mean θ and mean reversion speed $\omega - \phi$. To guaranty that the m_t process is stationary, the restriction $\omega - \phi > 0$ must be imposed. Equations (5) and (7) describe the evolution of the stock price and the dividend yield. Note that, although (7) depends on the history of the stock, the system (S_t, m_t) is Markovian.

By construction, the dividend yield is instantaneously perfectly correlated to stock returns, and this correlation is negative. As pointed out above, approximation is reasonable for indices such as the S&P500, which has a correlation between changes in the dividend yield and index returns of about -90%. Stock returns will show continuation, or positive autocorrelation, when they are negatively correlated to changes in the dividend yield. The following lemmas show this formally. First, define τ-period returns as:

$$\rho_{t+\tau} = s_{t+\tau} - s_t.$$

Then, integrating equation (5) gives:

$$\begin{aligned}\rho_{t+\tau} = {} & \frac{\omega}{\omega - \phi}\left(\mu - \delta - \frac{1}{2}\sigma^2\right)\tau + \frac{\phi}{\omega - \phi}(m_t - \theta)\left(1 - e^{-(\omega-\phi)\tau}\right) + \\ & \sigma \int_t^{t+\tau} \left[1 + \frac{\phi}{\omega - \phi}\left(1 - e^{-(\omega-\phi)(t+\tau-u)}\right)\right] dW_u.\end{aligned} \tag{8}$$

From equation (8) it is possible to calculate the unconditional variance of $\rho_{t+\tau}$:

$$Var\left(\rho_{t+\tau}\right)=\frac{\sigma^2}{\left(\omega-\phi\right)^2}\left[\omega^2\tau-\frac{2\phi}{\omega-\phi}\left(\omega-\frac{\phi}{2}\right)\left(1-e^{-(\omega-\phi)\tau}\right)\right],$$

and the covariance between ρ_t and $\rho_{t+\tau}$ (see the Appendix for details on the derivations of these two equations):

$$Cov\left(\rho_t,\rho_{t+\tau}\right)=\frac{\sigma^2}{\left(\omega-\phi\right)^2}\frac{\phi}{\omega-\phi}\left(\omega-\frac{\phi}{2}\right)\left(1-e^{-(\omega-\phi)\tau}\right)^2.$$

Therefore, the autocorrelation of τ-period returns can be expressed as:

$$\alpha\left(\rho_t,\rho_{t+\tau}\right)=\frac{\frac{\phi}{\omega-\phi}\left(\omega-\frac{\phi}{2}\right)\left(1-e^{-(\omega-\phi)\tau}\right)^2}{\omega^2\tau-\frac{2\phi}{\omega-\phi}\left(\omega-\frac{\phi}{2}\right)\left(1-e^{-(\omega-\phi)\tau}\right)}. \tag{9}$$

The following two lemmas show that $\phi>0$, that is, negative correlation between changes in the convenience yield and stock returns is a sufficient condition for momentum. Lemma 1 demonstrates that if $\phi>0$ the unconditional variance of τ-period returns is larger than the variance corresponding to the random walk ($\phi=0$) for $\tau>0$. The second lemma shows that the sign of the first autocorrelation of stock returns is equal to the sign of ϕ.

Lemma 1: If $\phi>0$, $Var\left(\rho_{t+\tau}\right)\geq\sigma^2\tau$. The inequality is strict for $\tau>0$.

Proof: Write:

$$Var\left(\rho_{t+\tau}\right)=\sigma^2\tau\frac{\omega^2-2\phi\left(\omega-\frac{\phi}{2}\right)\frac{\left(1-e^{-(\omega-\phi)\tau}\right)}{(\omega-\phi)\tau}}{\left(\omega-\phi\right)^2}.$$

If $\tau=0$, $Var\left(\rho_{t+\tau}\right)=\sigma^2\tau=0$. So it is necessary to show that for $\tau>0$:

$$\frac{\omega^2-2\phi\left(\omega-\frac{\phi}{2}\right)\frac{\left(1-e^{-(\omega-\phi)\tau}\right)}{(\omega-\phi)\tau}}{\left(\omega-\phi\right)^2}>1.$$

Note that this follows from the fact that for $\tau>0$:

$$\frac{\left(1-e^{-(\omega-\phi)\tau}\right)}{\left(\omega-\phi\right)\tau}<1.$$

Then, for $\phi > 0$:

$$\omega^2 - 2\phi\left(\omega - \frac{\phi}{2}\right)\frac{\left(1 - e^{-(\omega-\phi)\tau}\right)}{(\omega - \phi)\,\tau} > \omega^2 - 2\phi\left(\omega - \frac{\phi}{2}\right) = (\omega - \phi)^2 .$$

Therefore:

$$Var\left(\rho_{t+\tau}\right) > \sigma^2\tau\frac{(\omega-\phi)^2}{(\omega-\phi)^2} = \sigma^2\tau,$$

and the lemma is proved.

■

Note that $Var\left(\rho_{t+\tau}\right) \approx \sigma^2\tau$ for $\tau \approx 0$, and that for large τ, $Var\left(\rho_{t+\tau}\right) \approx \sigma^2\tau\left(\frac{\omega}{\omega-\phi}\right)^2 > \sigma^2\tau$.

Lemma 2: $\phi > 0$ implies positive autocorrelation of returns.

Proof: Note that from $\omega - \phi > 0$ we have that:

$$sign\ Cov\left(\rho_t, \rho_{t+\tau}\right) = sign\ \phi.$$

Also, from Lemma 1, the denominator in (9) is positive. Therefore:

$$sign\ \alpha\left(\rho_t, \rho_{t+\tau}\right) = sign\ \phi,$$

and this completes the proof of the lemma.

■

Momentum also implies that random shocks have great persistence in the long run. To see this, note that, in equation (8), the expression in the integral inside the brackets gives the "term structure of shocks". A shock that occurred at t has a residual impact on $s_{t+\tau}$ of $1 - \frac{\phi}{\omega-\phi}\left(1 - e^{-(\omega-\phi)\tau}\right)$. As τ grows without bound, this residual impact converges to:

$$1 + \frac{\phi}{\omega-\phi} = \frac{\omega}{\omega-\phi} \geq 1. \qquad (10)$$

Assume $\omega > 0$. If $\phi = 0$, the stock price is a random walk. In this case, shocks have permanent effects, and their residual impact is exactly 1. In contrast, when $\phi > 0$, the residual impact of a shock experienced at t, as τ grows without bound, is $\frac{\omega}{\omega-\phi} > 1$. This means that shocks further propagate in the long run. This is the case of momentum.

The financial market is naturally complete through the dependence of m_t on W_t, the stock source of risk. Assume additionally that there are no arbitrage opportunities. Then, there exists a unique probability measure Q, equivalent to Π, such that the discounted prices of the stock (cum dividend) and of other traded assets are martingales under Q (Harrison and Kreps (1979)). In the next section I obtain the stock price process under the Q-measure, and derive formulas for futures prices. It turns out that these formulas are preference-free, an important feature of the model that allows to price derivative contracts without the need to estimate the risk premium.

3 The price process under the Q-measure

In this section I obtain the risk-neutral stock price process and also derive a closed form solution for forward prices. A stochastic dividend yield affects not only options, but forward and futures prices as well. As a portfolio containing a long call and a short put, both on the same stock, and with the same strike and maturity, is equivalent to a forward contract, the impact of the stochastic dividend yield on the forward price explains perceived violations to the put-call parity calculated under the assumption that the dividend yield is constant.

I show that, under the Q-measure, the stock price does not depend on the risk premium. This implies that forward and European options[3] formulas are preference-free. Market completeness is not the only source of this result. Lioui (2006) has studied the general problem of pricing and hedging derivative securities when the underlying asset pays a stochastic dividend yield, and concluded that a risk premium has to be specified, even when the stock and the dividend yield are driven by the same source of risk. As it is shown below, what drives the result in the model studied in this paper is the special structure of equation (4), in which the dividend yield is characterized as an affine function of stock past performance.

Equation (1) defines μ as the total expected return on the stock (capital gains plus dividend yield). μ is assumed constant. Define now r as the constant instantaneous risk-free interest rate, and λ as the risk premium. Then, the total expected return can be decomposed as:

$$\mu \equiv r + \lambda \tag{11}$$

[3] European options are discussed in the next section.

Plugging (11) back in (1) gives the risk-neutralized stock price process:

$$\begin{aligned}\frac{dS_t}{S_t} &= (r-\delta)\,dt + \phi m_t dt + \sigma\left(\frac{\lambda}{\sigma}dt + dW_t\right) \\ &= (r-\delta)\,dt + \phi m_t dt + \sigma dB_t, \end{aligned} \tag{12}$$

where $B_t = \frac{\lambda}{\sigma}t + W_t$ is a Brownian motion under the Q-measure. Thus, the total expected return on the stock under Q is r. More importantly, the risk-neutralized process for m_t does not depend on the risk premium either. To see this, plug (11) in (7) to get:

$$\begin{aligned} dm_t &= -(\omega-\phi)\,(m_t-\theta^*)\,dt + \sigma\left(\frac{\lambda}{\sigma}dt + dW_t\right) \\ &= -(\omega-\phi)\,(m_t-\theta^*)\,dt + \sigma dB_t, \end{aligned} \tag{13}$$

where now:

$$\theta^* = \frac{r - \frac{1}{2}\sigma^2 - \delta}{\omega - \phi}. \tag{14}$$

So neither S_t nor m_t depend on λ under Q. As a consequence, the model renders preference-free formulas for contingent claims.

Define $\tau = T - t$ as the time to maturity of a contract. To solve for the stock price under Q replace μ with r in equation (8). This shows that the stock price is a lognormal process under the risk-neutral measure[4]. That is:

$$\ln(S_T)\,\tilde{}\,N\left(s_t + \Omega_\tau, \Sigma_\tau\right). \tag{15}$$

From equation (8) we have that (under Q) the expected return, conditional on t, over an interval of length τ is:

$$\Omega_\tau = \frac{\omega}{\omega-\phi}\left(r-\delta-\frac{1}{2}\sigma^2\right)\tau + \frac{\phi}{\omega-\phi}\left(m_t-\theta^*\right)\left(1-e^{-(\omega-\phi)\tau}\right), \tag{16}$$

and the conditional variance:

$$\Sigma_\tau = \frac{\sigma^2}{(\omega-\phi)^2}\left(\omega^2\tau - \frac{2\phi\omega}{\omega-\phi}\left(1-e^{-(\omega-\phi)\tau}\right) + \frac{\phi^2}{2(\omega-\phi)}\left(1-e^{-2(\omega-\phi)\tau}\right)\right). \tag{17}$$

[4] As both μ and r are constant, the stock price is a lognormal process under the statistical measure as well.

Note that if $\phi = 0$, $\Sigma_\tau = \sigma^2\tau$. That is, the variance grows linearly with time to maturity, which corresponds to the random walk case. If $\phi > 0$, an argument similar to lemma 1 shows that $\Sigma_\tau > \sigma^2\tau$. To see this, write:

$$\Sigma_\tau = \sigma^2\tau\left(\frac{(\omega-\phi)^2 + 2\phi\omega k_1 - \phi^2 k_2}{(\omega-\phi)^2}\right), \tag{18}$$

where:

$$k_1 = 1 - \frac{1-e^{-(\omega-\phi)\tau}}{(\omega-\phi)\,\tau},$$

and:

$$k_2 = 1 - \frac{1-e^{-2(\omega-\phi)\tau}}{2\,(\omega-\phi)\,\tau},$$

It can be shown that for $\omega > \phi$ and $\tau > 0$, $2k_1 > k_2$ (see Appendix). Then, it follows that:

$$2\phi\omega k_1 - \phi^2 k_2 > \phi^2\,(2k_1 - k_2) > 0.$$

Therefore, the expression between parenthesis in equation (18) is larger than one.

The forward price for delivery of one share of the stock τ periods ahead is the expected stock price under the risk-neutral measure. Given the normality of $\log(S_t)$ under Q, the forward price is easily obtained in closed form:

$$\begin{aligned} F_\tau &= E_t^Q\,(S_T) \\ &= S_t \times \exp\left(\Omega_\tau + \frac{1}{2}\Sigma_\tau\right). \end{aligned} \tag{19}$$

From equations (16) and (17), this formula does not include the risk premium, and so it is preference-free.

4 Pricing options

The price of a European call option written on the stock, with maturity T and strike K, is the expectation under Q of its payoff at maturity, discounted by the risk-free rate:

$$C_t = e^{-r\tau}E_t^Q\left[Max\ \ (S_T - K, 0)\right]. \tag{20}$$

Equation (20) can be written as:

$$C_t = e^{-r\tau} E_t^Q \left[(S_T \ ) \times \mathbf{1}_{\{S_T > K\}} \right] - e^{-r\tau} K P^Q (S_T > K) , \tag{21}$$

where $\mathbf{1}_{\{S_T > K\}}$ is the indicator function of the event $\{S_T > K\}$, $E_t^Q \left[(S_T \ ) \times \mathbf{1}_{\{S_T > K\}} \right]$ is the Q-expected value of the stock at maturity, conditioned on the event that the option will be exercised at maturity, and $P^Q (S_T > K)$ is the probability under Q of this event. Due to the normality of $\log (S_t)$, the expectation in the first term of (23) can be solved as:

$$E_t^Q \left[(S_T \ ) \times \mathbf{1}_{\{S_T > K\}} \right] = S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N (d_1) , \tag{22}$$

where $N (d_1)$ is the value of the Normal cumulative distribution function at d_1, and:

$$d_1 = \frac{\log \left(\frac{S_t}{K} \right) + \Omega_\tau + \Sigma_\tau}{\sqrt{\Sigma_\tau}} . \tag{23}$$

The probability of the option finishing in -the-money is:

$$P^Q (S_T > K) = N (d_2) , \tag{24}$$

where:

$$d_2 = d_1 - \sqrt{\Sigma_\tau} . \tag{25}$$

So:

$$C_t = \left[S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N (d_1) - K N (d_2) \right] e^{-r\tau} . \tag{26}$$

It is important to note that this formula, as the formula for the forward price (19), does not include preference parameters.

The price of a European put on the same stock can be found using put-call parity. That is, because buying a call and shorting a put, both with maturity T and strike K, is equivalent to having a long position in a forward contract with maturity T and forward price K, we can express the put price as:

$$P_t = C_t - \left[E_t^Q (S_T) - K \right] e^{-r\tau} \tag{27}$$

Plugging (19) and (26) in (27) we get:

$$P_t = \left[KN\left(-d_2\right) - S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N\left(-d_1\right) \right] e^{-r\tau}. \tag{28}$$

4.1 The riskless hedge

The financial market in this paper is complete, so it is possible to construct a riskless hedge by continuously trading in the stock and a riskless bond. This section shows how to construct such riskless hedge.

Assume that a call has been written on the stock and that a hedging portfolio is started consisting on the shorted call and a long position in the underlying stock. The initial value of the portfolio is:

$$\Pi_t = \Delta S_t - C\left(S_t, m_t, t\right). \tag{29}$$

where Δ is the number of long units of the stock. The change in the value of the portfolio over the next period is:

$$\begin{aligned} d\Pi_t &= \Delta dS_t + \Delta\left(\delta - \phi m_t\right) S_t dt \\ &- \frac{\partial C}{\partial S} dS_t - \frac{\partial C}{\partial m}\frac{1}{S_t}\left[dS_t - S_t\left(\frac{1}{2}\sigma^2 + \omega m_t\right) dt \right] \\ &- \frac{\partial C}{\partial t} dt - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 dt - \frac{1}{2}\frac{\partial^2 C}{\partial m^2}\sigma^2 dt - \frac{\partial^2 C}{\partial S \partial m}\sigma^2 dt, \end{aligned} \tag{30}$$

where in the fourth term of the equation I use the fact that

$$ds_t = \frac{dS_t}{S_t} - \frac{1}{2}\sigma^2 dt. \tag{31}$$

The risk in the portfolio comes from its exposure to W_t. To eliminate this risk, choose:

$$\Delta = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial m}\frac{1}{S_t}. \tag{32}$$

Note that to hedge the position it is necessary to eliminate not only the risk coming from random changes in the stock (the first term in (32)), but also the risk coming from the stochastic dividend yield (the second term in (32)). So plugging (32) in (30) cancels the portfolio's overall exposure to W_t. As the portfolio is now riskless, it must earn the riskless interest rate to preclude arbitrage:

$$\left(\frac{\partial C}{\partial S}+\frac{\partial C}{\partial m}\frac{1}{S_t}\right)(\delta-\phi m_t)\,S_t+\frac{\partial C}{\partial m}\omega m_t-A\,(t)=r\Pi_t, \tag{33}$$

where:

$$A\,(t)=\frac{\partial C}{\partial t}+\frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2+\frac{1}{2}\frac{\partial^2 C}{\partial m^2}\sigma^2 dt+\frac{\partial^2 C}{\partial S\partial m}\sigma^2. \tag{34}$$

Operating on (33) we get:

$$\frac{\partial C}{\partial S}S_t\,(r-\delta+\phi m_t)+\frac{\partial C}{\partial m}\left(-\,(\omega-\phi)\,m_t+r-\delta-\frac{1}{2}\sigma^2\right)+A\,(t)-rC=0, \tag{35}$$

where the term in parenthesis multiplying $\frac{\partial C}{\partial m}$ can be written as: $-\,(\omega-\phi)\,(m_t-\theta^*)\,.$ So (35) is the fundamental partial differential equation that all contingent claims written on the stock must satisfy. The nature of the derivative at hand will be determined by the boundary conditions.

It is possible to calculate the Δ of the call in closed form using equations (26) and (32). We have:

$$\frac{\partial C}{\partial S}=e^{\Omega_\tau+\frac{1}{2}\Sigma_\tau}N\,(d_1)\,e^{-r\tau}, \tag{36}$$

and:

$$\frac{\partial C}{\partial m}=S_t e^{\Omega_\tau+\frac{1}{2}\Sigma_\tau}N\,(d_1)\,\frac{\phi}{\omega-\phi}\left(1-e^{-(\omega-\phi)\tau}\right)e^{-r\tau}. \tag{37}$$

Therefore:

$$\Delta=\frac{\partial C}{\partial S}+\frac{\partial C}{\partial m}\frac{1}{S_t}=\frac{\partial C}{\partial S}\left[1+\frac{\phi}{\omega-\phi}\left(1-e^{-(\omega-\phi)\tau}\right)\right]. \tag{38}$$

Note that $\Delta\geq 0$. Also, as expected, $\Delta\rightarrow\left\{\begin{smallmatrix}1 \text{ if } S_t>K\\ 0 \text{ if } S_t\leq K\end{smallmatrix}\right.$ as $\tau\rightarrow 0$.

For details about these formulas and their derivation, see the Appendix.

5 Pricing implications

In this section I show that even a modest amount of momentum induced by the dividend yield can have noticeable consequences on option prices, and that these consequences are consistent with the empirical findings in Amin et al. (2004).

To assess the empirical relevance of a momentum-inducing dividend yield I calibrate the model in equation (8) to reproduce the behavior of a hypothetical stock or index, and compare the pricing results to Black-Scholes prices under the random walk hypothesis. The stock has total return $\mu = 0.11$, and instantaneous volatility $\sigma = 0.20$. The annual risk-free interest rate is $r = 0.05$. Results should not depend on an artificially inflated momentum, or on a dividend yield likely to become negative, so a main concern in choosing values for ϕ and ω is to guarantee that the first autocorrelation of returns and the probability of a negative dividend yield are sufficiently low. I chose the parameters in such a way that the probability of a negative dividend yield is 0.12% (that is, you will observe a negative dividend once every 834 years), the first autocorrelation of monthly returns is 1.17%, and the first autocorrelation of annual returns is 0.46%. These values seem low enough to conduct the exercise. The values of the parameters are shown in Table 1.

Table 1: Parameter values

Parameters	
μ	0.11
δ	0.0417
σ	0.20
ϕ	0.25
ω	7.50
r	0.05
S_0	40

These parameters also imply an annual return volatility of 20.59% (equation 17), a risk premium of 6%, and an average dividend yield of:

$$\delta \times \frac{\omega}{\omega - \phi} - \left(\mu - \frac{1}{2}\sigma^2\right) \times \frac{\phi}{\omega - \phi} = 0.04, \tag{39}$$

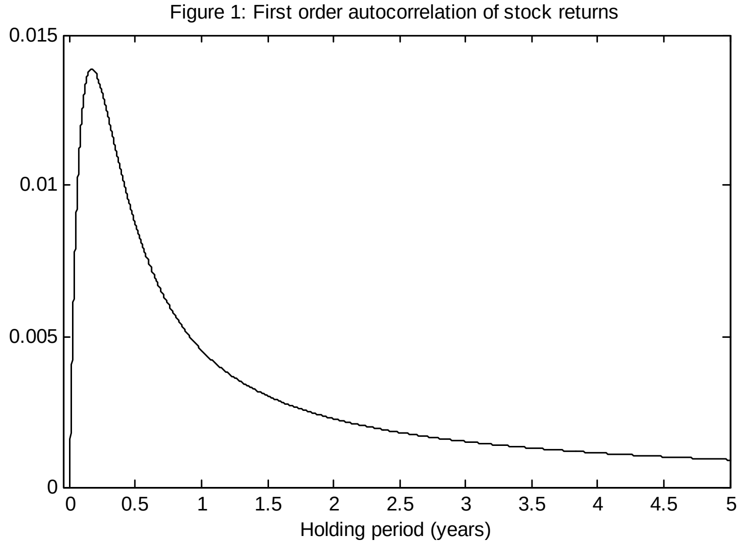
or 4%, and an uncoditional volatility of the dividend yield of:

$$\frac{\phi\sigma}{\sqrt{2\left(\omega - \phi\right)}} = 0.013.$$

These values, together with the autocorrelations discussed above, are roughly in line with data and with values used in similar studies.

The first autocorrelation of stock returns as a function of the holding period depends only on ϕ and ω (see equation (9)). Figure 1 shows that for the values reported in table 1, this autocorrelation is never larger that 1.5%, and that it is close to zero for holding periods above two years.

Figure 1: First order autocorrelation of stock returns



The values of ϕ and ω may underestimate the true autocorrelations, but they are chosen to illustrate that even this modest momentum has the potential to generate sizable pricing differences with the standard Black-Scholes model.

As noted above, the benchmark case is the Black-Scholes price, computed under the assumption that the stock is a random walk. In the benchmark case, the dividend yield is constant and equal to 4%. Note that the benchmark dividend yield is different from δ (see table 1), which is the intercept in a regression on the stochastic dividend yield against the m_t measure of stock performance. Given the values of the other parameters, δ is calibrated to make the mean dividend yield equal to 4%. Also, the instantaneous volatility of the benchmark case is $\sigma = 0.20$. The volatility of longer holding periods is computed as $\sigma\sqrt{T}$, which is consistent with the practice of a trader who, ignoring time variation in the dividend yield, estimates the volatility using daily or weekly data and then extrapolates to longer horizons using the rule of the square root[5].

Table 2 compares Black-Scholes prices and prices obtained from equation (26) and

[5] Alternatively, the volatility can be computed using the adjustment suggested in Lo and Wang (1995) to account for predictability in stock returns. With positive autocorrelation, this adjustment implies an instantaneous volatility lower than σ, and therefore option prices lower than Black-Scholes prices obtained under the random walk assumption. Note that option prices under the Lo and Wang (1995) adjustment do not depend on past stock performance.

(28) for various holding periods (one week to one year) and strikes, and for three different states of the variable m_t: -0.05, 0, 0.05, corresponding to negative, zero and positive past performance of returns, respectively.

There are two forces explaining the differences between Black-Scholes prices and prices obtained from equations (26) and (28) reported in Table 2. On the one hand, there is the volatility effect, arising from the fact that $Var\left(\rho_{t+\tau}\right) > \sigma^2\tau$. On the other hand there is a level effect, stemming from the influence of past stock returns on the dividend yield. Note that the level effect is not affected by the risk premium, because formulas (26) and (28) are preference-free. The volatility effect increases the prices of options relative to Black-Scholes prices for all maturities and across all strikes, although this effect is relatively more pronounced for out-of-the money options. The level effect is more complicated: it increases the prices of calls and reduces the prices of puts after a stock rally ($m_t = 0.05$), because momentum implies that current good performance raises the probability of good performance in the future (and thus of a lower dividend yield), and it reduces the prices of calls and increases the prices of puts after a stock decline ($m_t = -0.05$), as momentum implies that current bad performance raises the probability of bad performance in the future (and thus of a larger dividend yield). The level effect applies also for all maturities and across strikes.

Results reported in table 2 can be summarized as follows. At-the-money and in-the-money call prices are lower than the corresponding Black-Scholes prices when $m_t = -0.05$, but increase with m_t and eventually become higher than Black-Scholes prices as $m_t = 0.05$. Out-of-the money call prices are uniformly higher that Black-Scholes prices, and the price differences increase with m_t. As an example, the price of a 3-month at-the-money call struck at 45 is 2.6% higher than the Black-Scholes price when $m_t = -0.05$, while it is 8.7% higher when $m_t = 0.05$. At the money and in-the-money put prices are higher than the corresponding Black-Scholes prices for $m_t = -0.05$, but decline as m_t increases to become lower than Black-Scholes as $m_t = 0.05$. Out-of-the money put prices are uniformly higher than Black-Scholes prices, and the price differences decline as m_t increases. As an example, the price of a 3-month put struck at 35 is 11.4% higher than the Black-Scholes price when $m_t = 0.05$, while it is 1% higher when $m_t = 0.05$.

Call prices increase with m_t for all maturities and across all strikes. Put prices decline with m_t for all maturities and across all strikes. In particular, if prices are measured under the incorrect assumption that the dividend yield is constant, the realignment of option prices after a row of positive or negative returns will be seen as a violation of put-call parity, consistent with Amin et al (2004). Suppose options are at-the-money, and maturity time is three months. Black-Scholes implies that $C - P = 0.0989$. Suppose instead that the correct model has a stochastic dividend yield and that $m_t = 0.05$, then $C - P = 0.1445$, an increase of more than 45%. So violations of the put-call parity depend on the past performance of returns through the stochastic dividend yield. The same pattern appears for put-call volatility spreads. For the values in the example, the put-call volatility spread is -0.0087 ($m_t = -0.05$), -0.0015 ($m_t = 0$), and 0.0058 ($m_t = 0.05$).

Table 2: Call and Put Prices

Table 2 compares Black-Scholes call (BS call) and put (BS put) option prices under geometric Brownian motion to call and put prices from equations (26) and (28). Parameters are as in table 1. The stock on which the options are written has a current value of \$40. In the case of Black-Scholes, the dividend yield is assumed constant and equal to 4%. In the case of equations (26) and (28), the average dividend yield is 4%. Prices are compared for three values of the state variable $m_t : -0.05, 0$, and 0.05, corresponding to negative, constant, and positive performance of the stock, respectively. Please find the table in the next page.

Table 2: Call and Put Prices

	BS	Call (Eq. 26)			BS	Put (Eq. 28)		
Strike	Call	-0.05	0.00	0.05	Put	-0.05	0.00	0.05
Time to maturity: 7 days ($T-t=7/364$)								
30	9.9981	9.9879	9.9968	10.006	0.000	0.000	0.000	0.000
35	5.0029	4.9927	5.0016	5.0106	0.000	0.000	0.000	0.000
40	0.4460	0.4419	0.4464	0.4510	0.4384	0.4444	0.4400	0.4356
45	0.000	0.000	0.000	0.000	4.9875	4.9977	4.9888	4.9798
50	0.000	0.000	0.000	0.000	9.9827	9.9929	9.9840	9.9750
Time to maturity: 91 days ($T-t=91/364$)								
30	9.9765	9.9084	9.9654	10.0224	0.0018	0.0024	0.0023	0.0022
35	5.1846	5.1328	5.1851	5.2375	0.1478	0.1647	0.1598	0.1552
40	1.6272	1.6203	1.6504	1.6808	1.5283	1.5901	1.5630	1.5363
45	0.2588	0.2656	0.2734	0.2815	5.0978	5.1732	5.1240	5.0749
50	0.0212	0.0230	0.0240	0.0250	9.7981	9.8686	9.8124	9.7563
Time to maturity 182 days ($T-t=182/364$)								
30	9.9823	9.9051	9.9695	10.0342	0.0336	0.0412	0.0399	0.0387
35	5.5195	5.4803	5.5359	5.5918	0.4474	0.4930	0.4829	0.4729
40	2.3039	2.3137	2.3491	2.3849	2.1083	2.2029	2.1726	2.1426
45	0.7073	0.7303	0.7459	0.7618	5.3883	5.4961	5.4460	5.3960
50	0.1641	0.1772	0.1821	0.1871	9.7216	9.8195	9.7587	9.6979
Time to maturity: 273 days ($T-t=273/364$)								
30	10.0267	9.9540	10.0178	10.0818	0.1047	0.1243	0.1216	0.1189
35	5.8388	5.8150	5.8688	5.9229	0.7328	0.8013	0.7886	0.7760
40	2.8164	2.8407	2.8772	2.9140	2.5264	2.6430	2.6129	2.5831
45	1.1261	1.1642	1.1838	1.2037	5.6520	5.7824	5.7355	5.6887
50	0.3818	0.4094	0.4180	0.4267	9.7237	9.8436	9.7856	9.7277
Time to maturity: 364 days ($T-t=364/364$)								
30	10.0933	10.0276	10.0897	10.1519	0.1986	0.2313	0.2272	0.2232
35	6.1277	6.1161	6.1681	6.2203	0.9891	1.0759	1.0618	1.0478
40	3.2411	3.2764	3.3132	3.3503	2.8587	2.9924	2.9631	2.9340
45	1.5053	1.5556	1.5777	1.6000	5.8791	6.0277	5.9837	5.9398
50	0.6256	0.6667	0.6781	0.6898	9.7555	9.8949	9.8403	9.7857

6 Conclusions

This paper studies the pricing of options with a stochastic dividend yield using a new model in which the dividend yield is an affine function of past stock performance. The model accommodates momentum in stock returns under complete markets, and it has the standard Black-Scholes (1973) model as a special case. Moreover, the model exhibits dependence of expected returns on past stock performance under the risk-neutral measure, and renders preference-free formulas for European options.

The model shows that when stock returns and changes in the dividend yield are negatively correlated, stock returns will exhibit positive autocorrelation. In this realistic case, the model predicts that calls will be overpriced (underpriced) relative to puts after stock market increases (declines), and that put-call volatility spreads will be affected likewise by stock price performance. These predictions are in line with the findings of recent empirical research conducted by Amin et al. (2004).

Results obtained in this paper are economically significant. Interestingly, they are derived under the assumption of a positive autocorrelation of monthly returns as low as 1.7%. This suggests that a momentum inducing dividend yield, and return predictability in general, may have a larger impact on option prices than previously thought.

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7 Appendix

In this Appendix I provide an overview of the derivation of second moments of returns, and show how to obtain the delta of a call.

7.1 Second moments of returns

First, define $q_t = m_t - \theta$ and $q_{t-\tau} = m_{t-\tau} - \theta$. Then, from equation (7) in the main text we have:

$$q_t = q_{t-\tau} e^{-(\omega-\phi)\tau} + \int_{t-\tau}^{t} e^{-(\omega-\phi)(t-u)} dW_u,$$

and:

$$E\left(q_t q_{t-\tau}\right) = e^{-(\omega-\phi)\tau} Var\left(q_{t-\tau}\right) = e^{-(\omega-\phi)\tau} \frac{\sigma^2}{2\left(\omega-\phi\right)}. \tag{40}$$

From equation (8), the unconditional variance of returns is:

$$\begin{aligned} Var\left(\rho_{t+\tau}\right) &= E\left(\frac{\phi}{\omega-\phi}\left(m_t-\theta\right)\left(1-e^{-(\omega-\phi)\tau}\right) + \sigma\int_t^{t+\tau}\left[1+\frac{\phi}{\omega-\phi}\left(1-e^{-(\omega-\phi)(t+\tau-u)}\right)\right]dW_u\right)^2 \\ &= \left(\frac{\phi}{\omega-\phi}\right)^2 \frac{\sigma^2}{2\left(\omega-\phi\right)}\left(1-e^{-(\omega-\phi)\tau}\right)^2 + \frac{\sigma^2}{\left(\omega-\phi\right)^2}\int_t^{t+\tau}\left[1-\phi e^{-(\omega-\phi)(t+\tau-u)}\right]^2 du. \end{aligned} \tag{41}$$

Solving the integral, and after some messy algebra, we get:

$$Var\left(\rho_{t+\tau}\right) = \frac{\sigma^2}{\left(\omega-\phi\right)^2}\left[\omega^2\tau - \frac{2\phi}{\omega-\phi}\left(\omega-\frac{\phi}{2}\right)\left(1-e^{-(\omega-\phi)\tau}\right)\right]. \tag{42}$$

The difference between (42) and (17) is that (17) is a conditional variance, so only the second term in (41) is used in the computation.

The formula for $Cov\left(\rho_t, \rho_{t+\tau}\right)$ is calculated in the same way, using now equation (40) with $\tau > 0$ and taking care that the cross-products overlap.

7.2 $2k_1 - k_2 > 0.$

Proposition: For $\tau > 0$, $2k_1 - k_2 > 0$.

First I prove the following lemma:

Lemma: Define $f(\tau) = (\omega - \phi)\tau$, and $g(\tau) = \frac{3 - 4e^{-(\omega-\phi)\tau} + e^{-2(\omega-\phi)\tau}}{2}$. Then, for $\tau > 0$:

$$f(\tau) > g(\tau)$$

Proof: First note that:

$$f(0) = g(0) = 0,$$

and that:

$$f'(\tau) = \omega - \phi.$$

Also:

$$g'(\tau) = 2(\omega - \phi)e^{-(\omega-\phi)\tau} - (\omega - \phi)e^{-2(\omega-\phi)\tau}.$$

Adding and subtracting $\omega - \phi$, this last equation can be written as:

$$g'(\tau) = (\omega - \phi)\left[1 - \left(1 - e^{-(\omega-\phi)\tau}\right)^2\right] < (\omega - \phi).$$

Now define:

$$h(\tau) = f(\tau) - g(\tau).$$

Then:

$$h(0) = f(0) - g(0) = 0,$$

and that for $\tau > 0$:

$$h'(\tau) = f'(\tau) - g'(\tau) > 0,$$

which implies $h(\tau) > 0$. Therefore, it must be that:

$$f(\tau) > g(\tau),$$

for $\tau > 0$, and the lemma is proved.

■

Proof of the proposition: Assume, on the contrary, that $2k_1 - k_2 \leq 0$. Then:

$$2\left(1 - \frac{1 - e^{-(\omega-\phi)\tau}}{(\omega-\phi)\,\tau}\right) \leq 1 - \frac{1 - e^{-2(\omega-\phi)\tau}}{2\,(\omega-\phi)\,\tau}.$$

Operating on both sides:

$$\frac{2\,(\omega-\phi)\,\tau - 2\left(1 - e^{-(\omega-\phi)\tau}\right)}{(\omega-\phi)\,\tau} \leq \frac{2\,(\omega-\phi)\,\tau - \left(1 - e^{-2(\omega-\phi)\tau}\right)}{2\,(\omega-\phi)\,\tau}$$

.

Multiplying both sides by $(\omega-\phi)\,\tau$:

$$2\,(\omega-\phi)\,\tau - 2\left(1 - e^{-(\omega-\phi)\tau}\right) \leq \frac{2\,(\omega-\phi)\,\tau - \left(1 - e^{-2(\omega-\phi)\tau}\right)}{2}$$

.

Operating again:

$$(\omega-\phi)\,\tau \leq \frac{3 - 4e^{-(\omega-\phi)\tau} + e^{-2(\omega-\phi)\tau}}{2}.$$

But this contradicts the previous lemma. So it must be that:

$$2k_1 - k_2 > 0,$$

completing the proof.

■

7.3 Derivation of delta

The following lemma will be useful in the derivation of delta:

Lemma: Define $F_\tau = S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau}$. Then:

$$F_\tau N'(d_1) - KN'(d_2) = 0,$$

where d_1 and d_2 are as in equations (23) and (25).

Proof: Recall that:

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and write $d_1 = d_2 + \sqrt{\Sigma_\tau}$. Then:

$$\begin{aligned} N'(d_1) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_2^2 + 2d_2\sqrt{\Sigma_\tau} + \Sigma_\tau}{2}\right) \\ &= N'(d_2) \exp\left(-d_2\sqrt{\Sigma_\tau} - \frac{1}{2}\Sigma_\tau\right) \\ &= N'(d_2) \frac{K}{F_\tau}. \end{aligned}$$

So:

$$F_\tau N'(d_1) - KN'(d_2) = F_\tau N'(d_2) \frac{K}{F_\tau} - KN'(d_2) = 0,$$

and the lemma is proved.

■

Now it is straightforward to derive delta.

Proposition:

$$\Delta = \frac{\partial C}{\partial S}\left[1 + \frac{\phi}{\omega - \phi}\left(1 - e^{-(\omega-\phi)\tau}\right)\right]$$

Proof: Recall that:

$$\Delta = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial m}\frac{1}{S_t}.$$

Deriving equation (26) with respecto to S_t gives:

$$\frac{\partial C}{\partial S} = \left[\frac{\partial F_\tau}{\partial S} N(d_1) + F_\tau N'(d_1)\frac{\partial d_1}{\partial S} - KN'(d_2)\frac{\partial d_2}{\partial S}\right] e^{-r\tau}.$$

Noting that $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$,

$$\frac{\partial C}{\partial S} = \left[\frac{\partial F_\tau}{\partial S} N(d_1) + \left[F_\tau N'(d_1) - KN'(d_2)\right]\frac{\partial d_2}{\partial S}\right] e^{-r\tau},$$

which, from the previous lemma, is:

$$\frac{\partial C}{\partial S}=\frac{\partial F_\tau}{\partial S}N\left(d_1\right)e^{-r\tau}.$$

Now,

$$\frac{\partial C}{\partial m}=\left[\frac{\partial F_\tau}{\partial m}N\left(d_1\right)+F_\tau N'\left(d_1\right)\frac{\partial d_1}{\partial S}\frac{\partial S}{\partial m}-KN'\left(d_2\right)\frac{\partial d_2}{\partial S}\frac{\partial S}{\partial m}\right]e^{-r\tau}.$$

Proceeding as before, we get:

$$\frac{\partial C}{\partial m}=\frac{\partial F_\tau}{\partial m}N\left(d_1\right)e^{-r\tau}.$$

Noting that:

$$\frac{\partial F_\tau}{\partial m}=S\frac{\partial C}{\partial S}\frac{\phi}{\omega-\phi}\left(1-e^{-(\omega-\phi)\tau}\right),$$

the result follows.

■