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Decomposition of Network Communication Games

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Abstract

Using network control structures this paper introduces network communication games as a generalization of vertex games and edge games corresponding to communication situations and studies their decomposition into unanimity games. We obtain a relation between the dividends of the network communication game and the underlying transferable utility game, which depends on the structure of the undirected graph. This relation extends the computational results for tree communication networks to general undirected graphs and is used to derive new characterizations of the Myerson value and the position value. Moreover, network communication games also allow to consider both the vertices and the edges of the graph as players, leading to a new network value.

Keywords: network communication games, network control structures, decomposition theorems, Myerson value, position value
JEL classification: C71

1 Introduction

Cooperative game theory analyzes allocations of joint revenues among cooperating players, taking the economic possibilities of subcoalitions into account. To describe an allocation problem for a set of players, Von Neumann and Morgenstern (1944) introduced the model of a transferable utility game, in which a characteristic function assigns to each subgroup of the cooperating players its worth, a number reflecting the economic possibilities of the coalition if it acts on its own. Shapley (1953) introduced a well-known solution for this model, known as the Shapley value, which divides the dividend of each coalition (cf. Harsanyi (1959)) equally among its members.

In a cooperative game with communication structure the players are subject to cooperation restrictions. Myerson (1977) introduced communication situations in which these cooperation restrictions are modeled by an undirected graph. Vertices of the undirected graph represent the players of the game and there is an edge between two vertices if and only if the corresponding players are able to communicate directly. A coalition can attain its worth if its members are able to communicate, i.e. if their corresponding vertices induce a connected subgraph.

Myerson (1977) introduced the graph-restricted game corresponding to a communication situation in which each coalition of vertices is assigned the sum of the worths of the components of players of its induced subgraph. We refer to this game as the corresponding

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vertex game. Owen (1986) further studied these vertex games for the special case that the undirected graph is a tree. The Myerson value of a communication situation is defined as the Shapley value of the corresponding vertex game.

Borm et al. (1992) introduced a game on the edges corresponding to a communication situation in which each coalition of edges is assigned the sum of the worths of the components of players of its induced subgraph. We refer to this game as the corresponding edge game. The position value of a communication situation assigns to each player half of the sum of the payoffs allocated to its incident edges by the Shapley value of the corresponding edge game.

In this paper we introduce network communication games as a generalization of both vertex games and edge games. A network communication game is a transferable utility game integrating the features of a communication situation and a network control structure on an undirected graph. Here, a network control structure models the way in which the vertices and edges of the graph are controlled. Where Myerson (1977) considers the vertices and Borm et al. (1992) considers the edges as controllers of the network; a network control structure allows any set of controllers to control the graph. In the corresponding network communication game each coalition of controllers is assigned the sum of the worths of the components of players of the part of the network which the members control together.

In order to simultaneously study the decomposition into unanimity games of vertex games and edge games, we focus on this decomposition for network communication games. It turns out that a communication situation with an underlying unanimity game induces a simple network communication game for any network control structure. The minimal winning coalitions in this game play a central role in the decomposition. We obtain a relation between the dividends of the network communication game and the underlying transferable utility game, which depends on the structure of the undirected graph. This relation extends the computational results using dividends from Owen (1986) and Borm et al. (1992) for trees to general undirected graphs. From these dividends a general expression for the Shapley value is derived, which is used for characterizing the Myerson value and the position value.

The main aim of this paper is to develop the decomposition theory for network communication games as a mathematical tool which can be used to derive the vertex game, the edge game and their related solution concepts for communication situations in a structured way. Besides, the notions of a network communication game and the underlying network control structure provide a general framework which allows for existing and new interpretations and can be used to study a wide range of interesting problems. We give one example of its applicability by introducing the vertex-edge game and a corresponding network value as an alternative for the Myerson value and the position value. Future research should investigate further applications and potential approaches to this new framework. Moreover, one could extend the decomposition theory to hypergraph communication situations, as introduced by Myerson (1980) and further studied by Van den Nouweland et al. (1992), or to more general communication structures (cf. Bilbao (2000)).

This paper is organized in the following way. Section 2 provides an overview of the basic game theoretic and graph theoretic notions and notations. Section 3 formally introduces network control structures, defines the corresponding network communication games and studies their decomposition into unanimity games. In Section 4 we discuss the decomposition theory of vertex games and edge games for the special case that the underlying graph is a tree and we introduce vertex-edge games and a corresponding network value. Section 5 illustrates how the decomposition of network communication games can be extended to more general communication networks such as multigraphs and hypergraphs.

2 Preliminaries

Let N be a nonempty and finite set of *players*. The set of all *coalitions* is denoted by $2^N = \{S \mid S \subseteq N\}$. A set of coalitions $\mathcal{B} \subseteq 2^N$ is called a *Sperner family* if $R \not\subset S$ for all $R, S \in \mathcal{B}$. A *transferable utility game* (cf. Von Neumann and Morgenstern (1944)) is a pair (N, v) in which $v : 2^N \to \mathbb{R}$ is a characteristic function assigning to each coalition $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. The worth of a coalition can be considered as the maximal joint revenue of the members which can be obtained without any assistance of a player which is not a member. Let TU^N denote the class of all transferable utility games with player set N. For convenience, we abbreviate $(N, v) \in \mathrm{TU}^N$ to $v \in \mathrm{TU}^N$. A TU-game $v \in \mathrm{TU}^N$ is called *simple* if the following three conditions are satisfied:

(i) $v(S) \in \{0, 1\}$ for all $S \in 2^N$;

(ii) $v(N) = 1$;

(iii) $v(R) \leq v(S)$ for all $R, S \in 2^N$ for which $R \subseteq S$.

Let SI^N denote the class of all simple games with player set N. A coalition $S \in 2^N$ is called *winning* in $v \in \mathrm{SI}^N$ if $v(S) = 1$ and *losing* if $v(S) = 0$. The set of *minimal winning* coalitions in $v \in \mathrm{SI}^N$ is given by

$$\mathcal{M}(v) = \left\{ S \in 2^N \middle| \; v(S) = 1, \forall_{R \subset S} : v(R) = 0 \right\}. \tag{1}$$

The *maximum game* $\max\{v \mid v \in \mathcal{G}\} \in \mathrm{TU}^N$ of a nonempty and finite set of transferable utility games $\mathcal{G} \subset \mathrm{TU}^N$ is defined by $\max\{v \mid v \in \mathcal{G}\}(S) = \max\{v(S) \mid v \in \mathcal{G}\}$ for all $S \in 2^N$. The *minimum game* is defined analogously. Note that both the maximum game and the minimum game of a nonempty set of simple games are simple. The *unanimity game* $u_R \in \mathrm{SI}^N$ on $R \in 2^N \setminus \{\emptyset\}$ is for all $S \in 2^N$ defined by

$$u_R(S) = \begin{cases} 1 & \text{if } R \subseteq S; \\ 0 & \text{if } R \not\subseteq S. \end{cases}$$

Note that $v \in \mathrm{SI}^N$ and $\mathcal{M}(v) = \mathcal{B}$ if and only if $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is a nonempty Sperner family and $v = \max\{u_R \mid R \in \mathcal{B}\}$.

A TU-game $v \in \mathrm{TU}^N$ can be uniquely decomposed into unanimity games,

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta^v(S) u_S, \tag{2}$$

where $\Delta^v : 2^N \setminus \{\emptyset\} \to \mathbb{R}$ assigns to each nonempty coalition $S \in 2^N \setminus \{\emptyset\}$ its *dividend* (cf. Harsanyi (1959))

$$\Delta^v(S) = \sum_{R \subseteq S} (-1)^{|S|-|R|} v(R). \tag{3}$$

A *solution* for transferable utility games $f : \mathrm{TU}^N \to \mathbb{R}^N$ assigns to any TU-game $v \in \mathrm{TU}^N$ a payoff allocation $f(v) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(v) = v(N)$. The *Shapley value* (cf. Shapley (1953)) $\Phi : \mathrm{TU}^N \to \mathbb{R}^N$ is for all $v \in \mathrm{TU}^N$ and all $i \in N$ given by

$$\Phi_i(v) = \sum_{S \in 2^N : i \in S} \frac{1}{|S|} \Delta^v(S). \tag{4}$$

Let $E \subseteq \{S \in 2^N \mid |S| = 2\}$ be a set of unordered pairs of players. The pair (N, E) represents an *undirected graph* in which N is the set of *vertices* and E is the set of *edges*. A subset $H \in 2^N \times 2^E$ is called a *subnetwork*. For all $i \in N$ we denote $E_i = \{e \in E \mid i \in e\}$. For all $S \in 2^N$ we denote $E[S] = \{e \in E \mid e \subseteq S\}$. A subnetwork $(S, T) \in 2^N \times 2^E$ is called a *subgraph* if $T \subseteq E[S]$. The subgraph *induced by* $S \in 2^N$ is $(S, E[S])$. For all $T \in 2^E$ we denote $N[T] = \{i \in N \mid i \in \bigcup_{e \in T} e\}$. The subgraph *induced by* $T \in 2^E$ is $(N[T], T)$.

A *path* in $(S, T) \in 2^N \times 2^E$ from $i_1 \in S$ to $i_n \in S$ is a sequence $(i_k)_{k=1}^n$ of $n \geq 2$ distinct vertices in S for which $\{i_k, i_{k+1}\} \in T$ for all $k \in \{1, \ldots, n-1\}$. A subnetwork $H \in 2^N \times 2^E$ *connects* $R \in 2^N \setminus \{\emptyset\}$ if for any $i, j \in R$, $i \neq j$ there exists a path in H from i to j. A coalition $C \in 2^N \setminus \{\emptyset\}$ is called a *component* of $H \in 2^N \times 2^E$ if H connects C and H does not connect any $R \supset C$. The set of all components of the subnetwork $H \in 2^N \times 2^E$ is denoted by $\mathcal{K}(H)$. A subgraph $(S, T) \in (2^N \setminus \{\emptyset\}) \times 2^E$ is called *connected* if it connects S. Note that for each connected subgraph $(S, T) \in 2^N \times 2^E$ with $|S| \geq 2$ we have $S = N[T]$. A connected subgraph $(N[T], T) \in 2^N \times (2^E \setminus \{\emptyset\})$ is called a *tree* if for any $i, j \in N[T]$, $i \neq j$ there exists a unique path in $(N[T], T)$ from i to j.

A subgraph $(S, E[S]) \in 2^N \times 2^E$ is called a *minimal R-connecting vertex-induced subgraph* if it connects $R \in 2^N \setminus \{\emptyset\}$ and any $(S', E[S'])$ with $S' \subset S$ does not connect R. Let $\mathcal{V}_E^R \subseteq 2^N \setminus \{\emptyset\}$ denote the set of coalitions of vertices which induce a minimal R-connecting vertex-induced subgraph. A tree $(N[T], T) \in 2^N \times 2^E$ is called a *minimal R-connecting tree* if it connects $R \in 2^N$, $|R| \geq 2$ and any tree $(N[T'], T')$ with $T' \subset T$ does not connect R. Let $\mathcal{T}_N^R \subseteq 2^E \setminus \{\emptyset\}$ denote the set of coalitions of edges which induce a minimal R-connecting tree.

A *communication situation* (cf. Myerson (1977)) is a triple (N, v, E) in which $v \in \mathrm{TU}^N$ is a transferable utility game and (N, E) is an undirected graph representing the communication possibilities between the players. We assume that $v \in \mathrm{TU}^N$ is *zero-normalized*, i.e. $v(\{i\}) = 0$ for all $i \in N$, and that (N, E) is connected in any communication situation (N, v, E). Let $\mathrm{CS}_{N,E}^N$ denote the class of all such communication situations with player set N and communication network (N, E). For convenience, we abbreviate $(N, v, E) \in \mathrm{CS}_{N,E}^N$ to $v \in \mathrm{CS}_{N,E}^N$. A *solution* for communication situations $f : \mathrm{CS}_{N,E}^N \to \mathbb{R}^N$ assigns to any communication situation $v \in \mathrm{CS}_{N,E}^N$ a payoff allocation $f(v) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(v) = v(N)$.

The *vertex game* $r_E^v \in \mathrm{TU}^N$ corresponding to $v \in \mathrm{CS}_{N,E}^N$ (cf. Myerson (1977)) is for all $S \in 2^N$ defined by

$$r_E^v(S) = \sum_{C \in \mathcal{K}(S, E[S])} v(C). \tag{5}$$

The *Myerson value* $\mu : \mathrm{CS}_{N,E}^N \to \mathbb{R}^N$ is for all $v \in \mathrm{CS}_{N,E}^N$ and all $i \in N$ given by

$$\mu_i(v) = \Phi_i(r_E^v). \tag{6}$$

The *edge game* $r_N^v \in \mathrm{TU}^E$ corresponding to $v \in \mathrm{CS}_{N,E}^N$ (cf. Borm et al. (1992)) is for all $T \in 2^E$ defined by

$$r_N^v(T) = \sum_{C \in \mathcal{K}(N[T], T)} v(C). \tag{7}$$

The *position value* $\pi : \mathrm{CS}_{N,E}^N \to \mathbb{R}^N$ is for all $v \in \mathrm{CS}_{N,E}^N$ and all $i \in N$ given by

$$\pi_i(v) = \frac{1}{2} \sum_{e \in E_i} \Phi_e(r_N^v). \tag{8}$$

3 Decomposition of Network Communication Games

In this section we introduce network communication games and study their decomposition into unanimity games. Network communication games are proposed as a generalisation of vertex games (cf. Myerson (1977)) and edge games (cf. Borm et al. (1992)) in order to study their decomposition by means of a unifying approach. We explicitly model the control of the vertices and edges of an undirected graph by a network control structure.

Definition 3.1 (Network Control Structure)**.**
A *network control structure* on the undirected graph (N, E) is a pair (P, G) in which P is a nonempty and finite set of *controllers* and $G : 2^P \to 2^N \times 2^E$ is a *control function* assigning to each coalition of controllers a subnetwork such that

(i) $G(\emptyset) = (\emptyset, \emptyset)$;

(ii) $G(P) = (N, E)$;

(iii) $G(Q) \subseteq G(Q')$ for all $Q, Q' \in 2^P$ for which $Q \subseteq Q'$.

Let $\mathrm{NCS}^P_{N,E}$ denote the class of all network control structures on (N, E) with controller set P. For convenience, we abbreviate $(P, G) \in \mathrm{NCS}^P_{N,E}$ to $G \in \mathrm{NCS}^P_{N,E}$. Controllers in a network control structure can be interpreted as controllers of the vertices and edges of the underlying network. The controllers all together control the full network, but each individual may control some vertices and edges by itself or in a subgroup, which is modeled by $G \in \mathrm{NCS}^P_{N,E}$.

From the viewpoint of Myerson (1977) the vertices of the graph control the network, i.e. $P = N$, such that each vertex controls itself and each edge is controlled by its two endpoints together. In other words, each coalition of vertices controls its induced subgraph. This can be described by the network control structure $G \in \mathrm{NCS}^N_{N,E}$ in which $G(S) = (S, E[S])$ for all $S \in 2^N$.

From the viewpoint of Borm et al. (1992) the edges of the graph control the network, i.e. $P = E$, such that each edge controls itself and its endpoints. In other words, each coalition of edges controls its induced subgraph. This can be described by the network control structure $G \in \mathrm{NCS}^E_{N,E}$ in which $G(T) = (N[T], T)$ for all $T \in 2^E$.

A network communication game combines a communication situation $v \in \mathrm{CS}^N_{N,E}$ and a network control structure $G \in \mathrm{NCS}^P_{N,E}$ into a transferable utility game with player set P in which the worth of a coalition of controllers is measured by the sum of the worths of the components of players of the part of the network which the members control together. To avoid confusion with the transferable utility game underlying its corresponding communication situation, we use the term 'reward' to refer to 'worth' of a coalition of controllers in a network communication game.

Definition 3.2 (Network Communication Game)**.**
Let $v \in \mathrm{CS}^N_{N,E}$ be a communication situation and let $G \in \mathrm{NCS}^P_{N,E}$ be a network control structure. In the corresponding *network communication game* $r^v_G \in \mathrm{TU}^P$ the *reward* of each coalition of controllers $Q \in 2^P$ is given by

$$r^v_G(Q) = \sum_{C \in \mathcal{K}(G(Q))} v(C). \tag{9}$$

The network communication game corresponding to the communication situation $v \in \text{CS}^N_{N,E}$ and the network control structure $G \in \text{NCS}^N_{N,E}$ with $G(S) = (S, E[S])$ for all $S \in 2^N$ coincides with the vertex game (cf. Myerson (1977)) for which we can write the reward of each coalition of vertices $S \in 2^N$ as

$$r^v_G(S) \overset{(9)}{=} \sum_{C \in \mathcal{K}(G(S))} v(C) = \sum_{C \in \mathcal{K}(S, E[S])} v(C) \overset{(5)}{=} r^v_E(S).$$

The network communication game corresponding to the communication situation $v \in \text{CS}^N_{N,E}$ and the network control structure $G \in \text{NCS}^E_{N,E}$ with $G(T) = (N[T], T)$ for all $T \in 2^E$ coincides with the edge game (cf. Borm et al. (1992)) for which we can write the reward of each coalition of edges $T \in 2^E$ as

$$r^v_G(T) \overset{(9)}{=} \sum_{C \in \mathcal{K}(G(T))} v(C) = \sum_{C \in \mathcal{K}(N[T], T)} v(C) \overset{(7)}{=} r^v_N(T).$$

In this paper we study the decomposition of network communication games into unanimity games. We focus on the relation between the decomposition of a network communication game and the decomposition of the transferable utility game underlying its corresponding communication situation. It turns out that a communication situation with an underlying unanimity game corresponds to a simple network communication game for any network control structure.

Lemma 3.1.
Let $G \in \text{NCS}^P_{N,E}$ and let $R \in 2^N \setminus \{\emptyset\}$. Then $r^{u_R}_G \in \text{SI}^P$. Moreover,

$$\mathcal{M}\left(r^{u_R}_G\right) = \left\{Q \in 2^P \mid \exists_{C \in \mathcal{K}(G(Q))} : R \subseteq C, \forall_{Q' \subset Q} \forall_{C \in \mathcal{K}(G(Q'))} : R \nsubseteq C\right\}. \tag{10}$$

Proof. Using that for any coalition of controllers $Q \in 2^P$ there is at most one component $C \in \mathcal{K}(G(Q))$ for which $R \subseteq C$, we can write for each $Q \in 2^P$

$$\begin{aligned} r^{u_R}_G(Q) &\overset{(9)}{=} \sum_{C \in \mathcal{K}(G(Q))} u_R(C) &&= |\{C \in \mathcal{K}(G(Q)) \mid R \subseteq C\}| \\ &= \begin{cases} 1 & \text{if } \exists_{C \in \mathcal{K}(G(Q))} : R \subseteq C; \\ 0 & \text{if } \forall_{C \in \mathcal{K}(G(Q))} : R \nsubseteq C \end{cases} &&= \begin{cases} 1 & \text{if } G(Q) \text{ connects } R; \\ 0 & \text{if } G(Q) \text{ does not connect } R. \end{cases} \end{aligned} \tag{11}$$

Since (N, E) is connected, $G(P) = (N, E)$ connects R, so $r^{u_R}_G(P) = 1$. If $G(Q) \in 2^N \times 2^E$ connects R for some $Q \in 2^P$, then $G(Q') \supseteq G(Q)$ connects R for all $Q' \in 2^P$ for which $Q \subseteq Q'$, so $r^{u_R}_G(Q) \leq r^{u_R}_G(Q')$ for all $Q, Q' \in 2^P$ for which $Q \subseteq Q'$. This means that $r^{u_R}_G(Q) \in \{0, 1\}$ for all $Q \in 2^P$, $r^{u_R}_G(P) = 1$ and $r^{u_R}_G(Q) \leq r^{u_R}_G(Q')$ for all $Q, Q' \in 2^P$ for which $Q \subseteq Q'$. Hence, $r^{u_R}_G \in \text{SI}^P$. Moreover, equation (10) is a direct consequence of equation (1) and equation (11). □

The set of minimal winning coalitions in the vertex game $r^{u_R}_E \in \text{SI}^N$ with $R \in 2^N \setminus \{\emptyset\}$ is given by $\mathcal{M}(r^{u_R}_E) = \mathcal{V}^R_E$, the set of coalitions of vertices which induce a minimal R-connecting vertex-induced subgraph. The set of minimal winning coalitions in the edge game $r^{u_R}_N \in \text{SI}^E$ with $R \in 2^N$, $|R| \geq 2$ is given by $\mathcal{M}(r^{u_R}_N) = \mathcal{T}^R_N$, the set of coalitions of edges which induce a minimal R-connecting tree.

Lemma 3.2.
Let $v \in \mathrm{SI}^N$. Then

$$v = \sum_{\mathcal{B} \subseteq \mathcal{M}(v) : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{R \in \mathcal{B}} R)}. \tag{12}$$

Moreover, for each $S \in 2^N \setminus \{\emptyset\}$ we have

$$\Delta^v(S) = \sum_{\mathcal{B} \subseteq \mathcal{M}(v) : \bigcup_{R \in \mathcal{B}} R = S} (-1)^{|\mathcal{B}|+1}. \tag{13}$$

Proof. Since equation (13) is a direct consequence of equation (12), it suffices to show equation (12). We first show that for each $R' \in 2^N \setminus \{\emptyset\}$ we have

$$\min\{v, u_{R'}\} = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_{R \cup R'}. \tag{14}$$

We know $v = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_R$. Let $S \in 2^N$. Then $v(S) \in \{0, 1\}$. Let $R' \in 2^N \setminus \{\emptyset\}$ and suppose we have $R' \not\subseteq S$. Then we have $u_{R'}(S) = 0$ and $R \cup R' \not\subseteq S$ for any $R \in 2^N \setminus \{\emptyset\}$, which implies that $u_{R \cup R'}(S) = 0$ for any $R \in 2^N \setminus \{\emptyset\}$. Consequently,

$$\min\{v, u_{R'}\}(S) = \min\{v(S), u_{R'}(S)\} = \min\{v(S), 0\} = 0 = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_{R \cup R'}(S).$$

Next suppose we have $R' \subseteq S$. Then we have $u_{R'}(S) = 1$, and $R \cup R' \subseteq S$ if and only if $R \subseteq S$ for any $R \in 2^N \setminus \{\emptyset\}$, which implies that $u_{R \cup R'}(S) = u_R(S)$ for any $R \in 2^N \setminus \{\emptyset\}$. Consequently,

$$\begin{aligned} \min\{v, u_{R'}\}(S) = \min\{v(S), u_{R'}(S)\} \quad &= \min\{v(S), 1\} \qquad\qquad = v(S) \\ = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_R(S) &= \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_{R \cup R'}(S). \end{aligned}$$

Hence, equation (14) applies.

Next, we prove equation (12) by induction on $|\mathcal{M}(v)|$. Suppose we have $|\mathcal{M}(v)| = 1$ and denote $\mathcal{M}(v) = \{R_1\}$. Then we can write

$$v = \max\{u_R \mid R \in \mathcal{M}(v)\} = \max\{u_{R_1}\} = u_{R_1} = \sum_{\mathcal{B} \subseteq \mathcal{M}(v) : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{R \in \mathcal{B}} R)}.$$

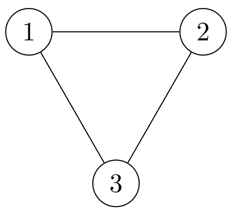
Let $n \in \mathbb{N}$ and assume that for any simple game $v' \in \mathrm{SI}^N$ for which $|\mathcal{M}(v')| = n$ we have $v' = \sum_{\mathcal{B} \subseteq \mathcal{M}(v') : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{R \in \mathcal{B}} R)}$. Suppose we have $|\mathcal{M}(v)| = n + 1$. Denote $\mathcal{M}(v) = \{R_1, \ldots, R_{n+1}\}$.

Then we can write

$$
\begin{aligned}
v &= \max\{u_R \mid R \in \mathcal{M}(v)\} \\
&= \max\{u_{R_1}, \dots, u_{R_{n+1}}\} \\
&= \max\{\max\{u_{R_1}, \dots, u_{R_n}\}, u_{R_{n+1}}\} \\
&= \max\{u_{R_1}, \dots, u_{R_n}\} + u_{R_{n+1}} - \min\{\max\{u_{R_1}, \dots, u_{R_n}\}, u_{R_{n+1}}\} \\
&\overset{(14)}{=} \sum_{\mathcal{B} \subseteq \{R_1, \dots, R_n\} : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{R \in \mathcal{B}} R)} + u_{R_{n+1}} - \sum_{\mathcal{B} \subseteq \{R_1, \dots, R_n\} : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{R \in \mathcal{B}} R) \cup R_{n+1}} \\
&= \sum_{\mathcal{B} \subseteq \{R_1, \dots, R_{n+1}\} : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{R \in \mathcal{B}} R)} \\
&= \sum_{\mathcal{B} \subseteq \mathcal{M}(v) : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{R \in \mathcal{B}} R)}.
\end{aligned}
$$

□

Example 1.
Let $N = \{1, 2, 3\}$, let $E = \big\{\{1, 2\}, \{1, 3\}, \{2, 3\}\big\}$ and consider the communication situation $u_{\{1,2\}} \in \mathrm{CS}^N_{N,E}$. The graph (N, E) is depicted below.



The set of coalitions of vertices which induce a minimal $\{1, 2\}$-connecting vertex-induced subgraph is given by $\mathcal{V}^{\{1,2\}}_E = \big\{\{1, 2\}\big\}$. Using $\mathcal{M}(r^{u_{\{1,2\}}}_E) = \mathcal{V}^{\{1,2\}}_E$, we know from Lemma 3.2 that

$$r^{u_{\{1,2\}}}_E = u_{\{1,2\}}.$$

The set of coalitions of edges which induce a minimal $\{1, 2\}$-connecting tree is given by $\mathcal{T}^{\{1,2\}}_N = \Big\{\big\{\{1, 2\}\big\}, \big\{\{1, 3\}, \{2, 3\}\big\}\Big\}$. Using $\mathcal{M}(r^{u_{\{1,2\}}}_N) = \mathcal{T}^{\{1,2\}}_N$, we know from Lemma 3.2 that

$$r^{u_{\{1,2\}}}_N = u_{\{\{1,2\}\}} + u_{\{\{1,3\},\{2,3\}\}} - u_{\{\{1,2\},\{1,3\},\{2,3\}\}}.$$

△

The dividends in general network communication games can be derived from the dividends in the underlying transferable utility game and the dividends in network communication games with an underlying unanimity game.

Lemma 3.3.
Let $v \in \mathrm{CS}^N_{N,E}$, let $G \in \mathrm{NCS}^P_{N,E}$ and let $Q \in 2^P \setminus \{\emptyset\}$. Then

$$\Delta^{r^v_G}(Q) = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \Delta^{r^{u_R}_G}(Q). \tag{15}$$

Proof. We can write

$$\begin{aligned}
\Delta^{r_G^v}(Q) &\overset{(3)}{=} \sum_{Q' \subseteq Q} (-1)^{|Q|-|Q'|} r_G^v(Q') \\
&\overset{(9)}{=} \sum_{Q' \subseteq Q} (-1)^{|Q|-|Q'|} \sum_{C \in \mathcal{K}(G(Q'))} v(C) \\
&\overset{(2)}{=} \sum_{Q' \subseteq Q} (-1)^{|Q|-|Q'|} \sum_{C \in \mathcal{K}(G(Q'))} \left(\sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_R(C) \right) \\
&= \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{Q' \subseteq Q} (-1)^{|Q|-|Q'|} \sum_{C \in \mathcal{K}(G(Q'))} u_R(C) \\
&\overset{(9)}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{Q' \subseteq Q} (-1)^{|Q|-|Q'|} r_G^{u_R}(Q') \\
&\overset{(3)}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \Delta^{r_G^{u_R}}(Q).
\end{aligned}$$

□

Using Lemma 3.3 we can extend the results for the decomposition of network communication games with an underlying unanimity game to the decomposition of general network communication games.

Theorem 3.4 (Decomposition of Network Communication Game)**.**
Let $v \in \mathrm{CS}_{N,E}^N$ *be a communication situation and let* $G \in \mathrm{NCS}_{N,E}^P$ *be a network control structure. Then*

$$r_G^v = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{M}(r_G^{u_R}) : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{Q \in \mathcal{B}} Q)}.$$

Proof. From Lemma 3.1 we know $r_G^{u_R} \in \mathrm{SI}^P$ for any $R \in 2^N \setminus \{\emptyset\}$. Using Lemma 3.2 and Lemma 3.3 we can write

$$\begin{aligned}
r_G^v &\overset{(2)}{=} \sum_{Q \in 2^P \setminus \{\emptyset\}} \Delta^{r_G^v}(Q) u_Q \\
&\overset{(15)}{=} \sum_{Q \in 2^P \setminus \{\emptyset\}} \left(\sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \Delta^{r_G^{u_R}}(Q) u_Q \right) \\
&= \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{Q \in 2^P \setminus \{\emptyset\}} \Delta^{r_G^{u_R}}(Q) u_Q \\
&\overset{(2)}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) r_G^{u_R} \\
&\overset{(12)}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{M}(r_G^{u_R}) : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{Q \in \mathcal{B}} Q)}.
\end{aligned}$$

□

Using Theorem 3.4 we find the decomposition into unanimity games of vertex games and edge games in terms of the dividends of the transferable utility game underlying the corresponding communication situation.

Corollary 3.5.
Let $v \in \mathrm{CS}_{N,E}^{N}$*. Then*

$$\begin{aligned} r_E^v &= \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{V}_E^R : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{S \in \mathcal{B}} S)} \\ \textit{and} \quad r_N^v &= \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{T}_N^R : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{(\bigcup_{T \in \mathcal{B}} T)}. \end{aligned}$$

Using Lemma 3.3 we can also derive an explicit formula for the Shapley value (cf. Shapley (1953)) of any network communication game.

Theorem 3.6 (Shapley Value of Network Communication Game)**.**
Let $v \in \mathrm{CS}_{N,E}^{N}$ *be a communication situation, let* $G \in \mathrm{NCS}_{N,E}^{P}$ *be a network control structure and let* $i \in P$ *be a controller. Then*

$$\Phi_i(r_G^v) = \sum_{Q \in 2^P : i \in Q} \frac{1}{|Q|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{M}(r_G^{u_R}) : \bigcup_{Q' \in \mathcal{B}} Q' = Q} (-1)^{|\mathcal{B}|+1}.$$

Proof. From Lemma 3.1 we know $r_G^{u_R} \in \mathrm{SI}^P$ for any $R \in 2^N \setminus \{\emptyset\}$. Using Lemma 3.2 and Lemma 3.3 we can write

$$\begin{aligned} \Phi_i(r_G^v) &\overset{(4)}{=} \sum_{Q \in 2^P : i \in Q} \frac{1}{|Q|} \Delta^{r_G^v}(Q) \\ &\overset{(15)}{=} \sum_{Q \in 2^P : i \in Q} \frac{1}{|Q|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \Delta^{r_G^{u_R}}(Q) \\ &\overset{(13)}{=} \sum_{Q \in 2^P : i \in Q} \frac{1}{|Q|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{M}(r_G^{u_R}) : \bigcup_{Q' \in \mathcal{B}} Q' = Q} (-1)^{|\mathcal{B}|+1}. \end{aligned}$$

□

Using Theorem 3.6 we find new characterizations of the Myerson value (cf. Myerson (1977)) and the position value (cf. Borm et al. (1992)) in terms of the dividends of the transferable utility game underlying the corresponding communication situation.

Corollary 3.7.
Let $v \in \mathrm{CS}_{N,E}^{N}$ *and let* $i \in N$*. Then*

$$\begin{aligned} \mu_i(v) &= \sum_{S \in 2^N : i \in S} \frac{1}{|S|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{V}_E^R : \bigcup_{S' \in \mathcal{B}} S' = S} (-1)^{|\mathcal{B}|+1} \\ \textit{and} \quad \pi_i(v) &= \frac{1}{2} \sum_{e \in E_i} \sum_{T \in 2^E : e \in T} \frac{1}{|T|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{T}_N^R : \bigcup_{T' \in \mathcal{B}} T' = T} (-1)^{|\mathcal{B}|+1}. \end{aligned}$$

4 Tree Communication Networks

In this section we discuss the decomposition theory of vertex games (cf. Myerson (1977)) and edge games (cf. Borm et al. (1992)) as presented in Section 3 for the special case that the undirected graph (N, E) is a tree. Note that a tree (N, E) contains a unique minimal R-connecting vertex-induced subgraph, i.e. $|\mathcal{V}_E^R| = 1$, for any $R \in 2^N \setminus \{\emptyset\}$. Moreover, (N, E) is a tree if and only if it contains a unique minimal R-connecting tree, i.e. $|\mathcal{T}_N^R| = 1$, for each $R \in 2^N$ with $|R| \geq 2$. These unique minimal R-connecting vertex-induced subgraph and unique minimal R-connecting tree coincide, i.e. for any $R \in 2^N$, $|R| \geq 2$ we have $(S, E[S]) = (N[T], T)$ for $S \in \mathcal{V}_E^R$ and $T \in \mathcal{T}_N^R$. If (N, E) is a tree, let $T_N^R \in 2^E \setminus \{\emptyset\}$ denote for any $R \in 2^N$ with $|R| \geq 2$ the unique set of edges such that $\mathcal{T}_N^R = \{T_N^R\}$ and $\mathcal{V}_E^R = \{N[T_N^R]\}$.

If the underlying network is a tree, any vertex game or edge game for which a unanimity game underlies its corresponding communication situation is a unanimity game too. If (N, E) is a tree, we know from Lemma 3.2 that $r_E^{u_R} = u_{N[T_N^R]}$ and $r_N^{u_R} = u_{T_N^R}$ for any $R \in 2^N$ with $|R| \geq 2$. Combining these observations with Lemma 3.3 we find the following relations.

Corollary 4.1.
Let $v \in \mathrm{CS}_{N,E}^N$. If (N, E) is a tree, then

$$\Delta^{r_E^v}(S) = \sum_{R \in 2^N \setminus \{\emptyset\} : N[T_N^R] = S} \Delta^v(R) \text{ for all } S \in 2^N \setminus \{\emptyset\}$$

$$\text{and} \quad \Delta^{r_N^v}(T) = \sum_{R \in 2^N \setminus \{\emptyset\} : T_N^R = T} \Delta^v(R) \quad \text{for all } T \in 2^E \setminus \{\emptyset\}.$$

Corollary 4.1 offers results which were also found by Owen (1986) and Borm et al. (1992). The following results are derived from Corollary 3.5 and Corollary 3.7, respectively.

Corollary 4.2.
Let $v \in \mathrm{CS}_{N,E}^N$. If (N, E) is a tree, then

$$r_E^v = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_{N[T_N^R]}$$

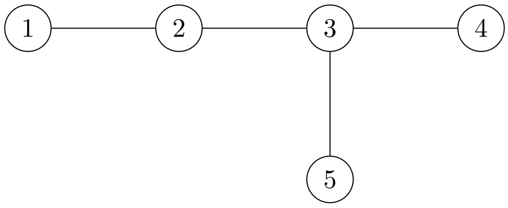
$$\text{and} \quad r_N^v = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_{T_N^R}.$$

Corollary 4.3.
Let $v \in \mathrm{CS}_{N,E}^N$ and let $i \in N$. If (N, E) is a tree, then

$$\mu_i(v) = \sum_{R \in 2^N \setminus \{\emptyset\} : i \in N[T_N^R]} \frac{1}{|N[T_N^R]|} \Delta^v(R)$$

$$\text{and} \quad \pi_i(v) = \sum_{e \in E_i} \sum_{R \in 2^N \setminus \{\emptyset\} : e \in T_N^R} \frac{1}{2|T_N^R|} \Delta^v(R).$$

Example 2.
Let $N = \{1,2,3,4,5\}$ and let $E = \big\{\{1,2\},\{2,3\},\{3,4\},\{3,5\}\big\}$. The graph (N,E) is depicted below.



Let $v \in \mathrm{CS}^N_{N,E}$ be the communication situation in which

$$v = u_{\{1,3\}} + 2u_{\{1,2,3\}} + 2u_{\{1,4,5\}} + 4u_{\{2,3,4,5\}} + 3u_{\{1,2,3,4,5\}}.$$

Using Corollary 4.2 we can write the corresponding vertex game as

$$\begin{aligned} r^v_E &= u_{\{1,2,3\}} + 2u_{\{1,2,3\}} + 2u_{\{1,2,3,4,5\}} + 4u_{\{2,3,4,5\}} + 3u_{\{1,2,3,4,5\}} \\ &= 3u_{\{1,2,3\}} + 4u_{\{2,3,4,5\}} + 5u_{\{1,2,3,4,5\}} \end{aligned}$$

and we can write the corresponding edge game as

$$\begin{aligned} r^v_N &= u_{\{\{1,2\},\{2,3\}\}} + 2u_{\{\{1,2\},\{2,3\}\}} + 2u_{\{\{1,2\},\{2,3\},\{3,4\},\{3,5\}\}} + 4u_{\{\{2,3\},\{3,4\},\{3,5\}\}} \\ &\quad + 3u_{\{\{1,2\},\{2,3\},\{3,4\},\{3,5\}\}} \\ &= 3u_{\{\{1,2\},\{2,3\}\}} + 4u_{\{\{2,3\},\{3,4\},\{3,5\}\}} + 5u_{\{\{1,2\},\{2,3\},\{3,4\},\{3,5\}\}}. \end{aligned}$$

Using Corollary 4.3 we can derive the Myerson value and the position value.

$$\begin{aligned} \mu(v) &= (1+1, 1+1+1, 1+1+1, 1+1, 1+1) &&= (2,3,3,2,2) \\ \pi(v) &= \left(\frac{3}{4}+\frac{5}{8}, \frac{3}{2}+\frac{4}{6}+\frac{5}{4}, \frac{3}{4}+\frac{4}{2}+\frac{15}{8}, \frac{4}{6}+\frac{5}{8}, \frac{4}{6}+\frac{5}{8}\right) &&= \left(1\frac{9}{24}, 3\frac{10}{24}, 4\frac{15}{24}, 1\frac{7}{24}, 1\frac{7}{24}\right) \end{aligned}$$

△

Network control structures provide a framework to introduce a new, third type of communication game in which the existing vertex game and edge game are combined. Instead of regarding either the vertices or the edges of the graph as controllers of the network, one could consider each vertex and each edge as a controller of itself, i.e. $P = N \cup E$. For an undirected graph (N,E) this can be described by the network control structure $G \in \mathrm{NCS}^{N\cup E}_{N,E}$ in which $G(Z) = (Z \cap N, Z \cap E)$ for all $Z \in 2^{N\cup E}$. For any communication situation $v \in \mathrm{CS}^N_{N,E}$, the *vertex-edge game* $r^v \in \mathrm{TU}^{N\cup E}$ is the corresponding network communication game in which the reward of each coalition of vertices and edges $Z \in 2^{N\cup E}$ is given by

$$r^v(Z) = r^v_G(Z) \overset{(9)}{=} \sum_{C\in\mathcal{K}(G(Z))} v(C) = \sum_{C\in\mathcal{K}(Z\cap N, Z\cap E)} v(C).$$

The corresponding value $\psi : \mathrm{CS}^N_{N,E} \to \mathbb{R}^N$ is for all $v \in \mathrm{CS}^N_{N,E}$ and all $i \in N$ defined by

$$\psi_i(v) = \Phi_i(r^v) + \frac{1}{2}\sum_{e\in E_i} \Phi_e(r^v). \tag{16}$$

The set of minimal winning coalitions in the vertex-edge game $r^{u_R} \in \mathrm{SI}^{N \cup E}$ with $R \in 2^N$, $|R| \geq 2$ is given by $\{N[T] \cup T \mid T \in \mathcal{T}_N^R\}$, the set of coalitions of vertices and edges which induce a minimal R-connecting tree. If (N, E) is a tree, we know from Lemma 3.2 that $r^{u_R} = u_{N[T_N^R] \cup T_N^R}$ for any $R \in 2^N$ with $|R| \geq 2$. Using this observation we derive that the value ψ of a communication situation with an underlying unanimity game is a specific convex combination of the Myerson value μ and the position value π if the graph is a tree.

Theorem 4.4.
Let $R \in 2^N$ with $|R| \geq 2$. If (N, E) is a tree, then

$$\psi(u_R) = \frac{|T_N^R| + 1}{2|T_N^R| + 1}\mu(u_R) + \frac{|T_N^R|}{2|T_N^R| + 1}\pi(u_R).$$

Proof. Let $i \in N$ and assume (N, E) is a tree. If $i \notin N[T_N^R]$, then $e \notin T_N^R$ for all $e \in E_i$, so $\psi_i(u_R) = \mu_i(u_R) = \pi_i(u_R) = 0$ and the statement follows. Now suppose $i \in N[T_N^R]$. Note that $|N[T_N^R]| = |T_N^R| + 1$. Then we can write

$$\begin{aligned}
\psi_i(u_R) &\overset{(16)}{=} \Phi_i(r^{u_R}) + \frac{1}{2}\sum_{e \in E_i} \Phi_e(r^{u_R}) \\
&\overset{(4)}{=} \sum_{Z \in 2^{N \cup E}: i \in Z} \frac{1}{|Z|}\Delta^{r^{u_R}}(Z) + \frac{1}{2}\sum_{e \in E_i}\sum_{Z \in 2^{N \cup E}: e \in Z} \frac{1}{|Z|}\Delta^{r^{u_R}}(Z) \\
&= \frac{1}{|N[T_N^R] \cup T_N^R|} + \frac{1}{2}\sum_{e \in E_i \cap T_N^R} \frac{1}{|N[T_N^R] \cup T_N^R|} \\
&= \frac{1}{2|T_N^R| + 1} + \frac{1}{2}\sum_{e \in E_i \cap T_N^R} \frac{1}{2|T_N^R| + 1} \\
&= \frac{|T_N^R| + 1}{2|T_N^R| + 1}\left(\frac{1}{|N[T_N^R]|}\right) + \frac{|T_N^R|}{4|T_N^R| + 2}\sum_{e \in E_i \cap T_N^R} \frac{1}{|T_N^R|} \\
&= \frac{|T_N^R| + 1}{2|T_N^R| + 1}\sum_{S \in 2^N: i \in S} \frac{1}{|S|}\Delta^{r_E^{u_R}}(S) + \frac{|T_N^R|}{4|T_N^R| + 2}\sum_{e \in E_i}\sum_{T \in 2^E: e \in T} \frac{1}{|T|}\Delta^{r_N^{u_R}}(T) \\
&\overset{(4)}{=} \frac{|T_N^R| + 1}{2|T_N^R| + 1}\Phi_i(r_E^{u_R}) + \frac{|T_N^R|}{2|T_N^R| + 1}\left(\frac{1}{2}\sum_{e \in E_i} \Phi_e(r_N^{u_R})\right) \\
&\overset{(6)\&(8)}{=} \frac{|T_N^R| + 1}{2|T_N^R| + 1}\mu_i(u_R) + \frac{|T_N^R|}{2|T_N^R| + 1}\pi_i(u_R).
\end{aligned}$$

□

Example 3.
Let $N = \{1, 2, 3, 4, 5\}$ and let $E = \big\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\big\}$ as in Example 2. Consider the communication situation $u_{\{1,5\}} \in \mathrm{CS}_{N,E}^N$. We have $T_N^{\{1,5\}} = \big\{\{1, 2\}, \{2, 3\}, \{3, 5\}\big\}$. Then the vertex game, the edge game and the vertex-edge game are given by

$$\begin{aligned}
r_E^{u_{\{1,5\}}} &= u_{\{1,2,3,5\}}, \\
r_N^{u_{\{1,5\}}} &= u_{\{\{1,2\},\{2,3\},\{3,5\}\}} \\
\text{and} \quad r^{u_{\{1,5\}}} &= u_{\{1,2,3,5,\{1,2\},\{2,3\},\{3,5\}\}},
\end{aligned}$$

respectively. From the first two expressions it readily follows that the Myerson value and the position value are given by

$$\mu(u_{\{1,5\}}) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}\right)$$
$$\text{and} \quad \pi(u_{\{1,5\}}) = \left(\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, 0, \frac{1}{6}\right).$$

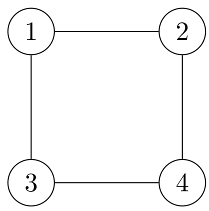
From Theorem 4.4 we know that

$$\psi(u_{\{1,5\}}) = \frac{4}{7}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}\right) + \frac{3}{7}\left(\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, 0, \frac{1}{6}\right) = \left(\frac{3}{14}, \frac{4}{14}, \frac{4}{14}, 0, \frac{3}{14}\right).$$

△

The following example illustrates that the value ψ is not necessarily a convex combination of the Myerson value μ and the position value π if the underlying network is not a tree.

Example 4.
Let $N = \{1,2,3,4\}$, let $E = \big\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\big\}$ and consider the communication situation $u_{\{1,2,3\}} \in \mathrm{CS}^N_{N,E}$. The graph (N,E) is depicted below.



Since

$$\mathcal{V}_E^{\{1,2,3\}} = \big\{\{1,2,3\}\big\}$$
$$\text{and} \quad \mathcal{T}_N^{\{1,2,3\}} = \Big\{\big\{\{1,2\},\{1,3\}\big\},\big\{\{1,2\},\{2,4\},\{3,4\}\big\},\big\{\{1,3\},\{2,4\},\{3,4\}\big\}\Big\},$$

the vertex game, the edge game and the vertex-edge game are given by

$$r_E^{u_{\{1,2,3\}}} = u_{\{1,2,3\}},$$
$$r_N^{u_{\{1,2,3\}}} = u_{\{\{1,2\},\{1,3\}\}} + u_{\{\{1,2\},\{2,4\},\{3,4\}\}} + u_{\{\{1,3\},\{2,4\},\{3,4\}\}} - 2u_{\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}}$$
$$\text{and} \quad r^{u_{\{1,2,3\}}} = u_{\{1,2,3,\{1,2\},\{1,3\}\}} + u_{\{1,2,3,4,\{1,2\},\{2,4\},\{3,4\}\}} + u_{\{1,2,3,4,\{1,3\},\{2,4\},\{3,4\}\}} - 2u_{\{1,2,3,4,\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}},$$

respectively. Consequently,

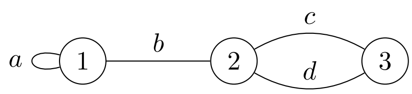
$$\mu(u_{\{1,2,3\}}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right),$$
$$\pi(u_{\{1,2,3\}}) = \left(\frac{3}{12}, \frac{4}{12}, \frac{3}{12}, \frac{2}{12}\right),$$
$$\psi(u_{\{1,2,3\}}) = \left(\frac{21}{70}, \frac{23}{70}, \frac{21}{70}, \frac{5}{70}\right),$$

and $\psi(u_{\{1,2,3\}})$ is not a convex combination of $\mu(u_{\{1,2,3\}})$ and $\pi(u_{\{1,2,3\}})$. △

5 Concluding Remarks

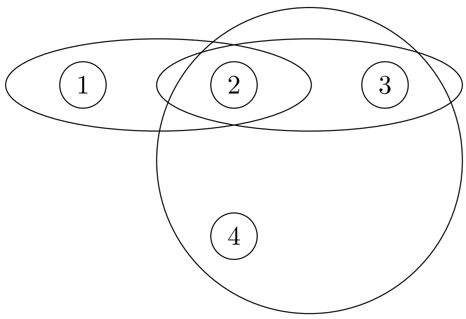
We conclude this paper with two examples of possible extensions of the decomposition theory studied in Section 3 to more general communication networks: undirected multigraphs and hypergraphs. For convenience, we restrict ourselves in these examples to an outline of the edge game and the corresponding position value.

Example 5.
Let $\{1,2,3\}$ be the set of vertices and let $\{a,b,c,d\}$ be the set of edges of the multigraph depicted below, and consider the communication situation with underlying game $u_{\{1,3\}}$.



The set of coalitions of edges which induce a minimal $\{1,3\}$-connecting tree is given by $\big\{\{b,c\},\{b,d\}\big\}$. The corresponding edge game can then be written as $u_{\{b,c\}}+u_{\{b,d\}}-u_{\{b,c,d\}}$. The position value of this communication situation is given by $(\frac{2}{6},\frac{3}{6},\frac{1}{6})$. △

Example 6.
Let $\{1,2,3,4\}$ be the set of vertices and let $\big\{\{1,2\},\{2,3\},\{2,3,4\}\big\}$ be the set of (hyper)edges of the hypergraph depicted below, and consider the communication situation with underlying game $u_{\{1,3\}}$.



The set of coalitions of edges which induce a minimal $\{1,3\}$-connecting tree is given by $\Big\{\big\{\{1,2\},\{2,3\}\big\},\big\{\{1,2\},\{2,3,4\}\big\}\Big\}$. The corresponding edge game can then be written as $u_{\{\{1,2\},\{2,3\}\}}+u_{\{\{1,2\},\{2,3,4\}\}}-u_{\{\{1,2\},\{2,3\},\{2,3,4\}\}}$. The position value of this communication situation is given by $(\frac{12}{36},\frac{17}{36},\frac{5}{36},\frac{2}{36})$. △

Future research could formalize these and other extensions of the decomposition theory for network communication games studied in this paper to more general communication networks with or without a specific type of network control structure.

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