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Bao, Hailong; Ponds, Eduard; Schumacher, Hans

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Hailong Bao* Eduard H.M. Ponds† Johannes M. Schumacher‡

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Abstract

We work with a multi-period system where a finite number of agents need to share multiple monetary risks. We look for the solutions that are both Pareto efficient utility-wise and financially fair value-wise. A buffer enables the inter-temporal capital transfer. Expected utility is used to evaluate the utility, and a risk-neutral measure is essential for determining the risk sharing rules. It can be shown that in the model setting there always exists a unique risk sharing rule that is both Pareto efficient and financially fair. An iterative algorithm is introduced to calculate this rule numerically.

Keywords: Inter-temporal risk sharing, Pareto efficiency, financial fairness, contract design

1 Introduction

This paper explores the inter-temporal risk sharing in a multi-period setting under the concept of *Pareto efficiency and financial fairness* (PEFF). Pareto efficiency means that the utility of nobody can be improved without hurting the utility of some others, while financial fairness indicates that the market values of the risk positions before and after risk sharing should be equal. A risk-sharing system with respect to monetary uncertainties – the stochastic returns from the financial market, for instance – can be viewed as a bargaining system in the form of a financial contract. On the one hand, Pareto efficiency is fundamental in multilateral bargaining systems, while on the other hand it is important to achieve financial fairness in designing financial contracts.

The model is motivated and abstracted from systems that allow for inter-temporal risk sharing. One example is the collective defined-contribution pension systems which can be viewed as a multilateral financial contract among both current and future cohorts. The

*Netspar, CentER, Department of Econometrics and Operations Research, Tilburg University; h.bao@uvt.nl.

†APG and Netspar, CentER, Department of Economics, Tilburg University; eduard.ponds@apg-am.nl.

‡Netspar, CentER, Department of Econometrics and Operations Research, Tilburg University; j.m.schumacher@uvt.nl.

possibility of inter-temporal risk sharing w.r.t. investment risk is due to the incompleteness of the market, i.e. the future generations cannot take positions in the current financial market. A risk sharing system tries to partly fix this problem by allowing later generations to take risks before they become participants. Risk sharing shall result in welfare gains to the generations; meanwhile, the pension contract shall also be fair from a valuation perspective. Another example is the reinsurance market, in which insurance companies re-allocate the risks by way of reinsurance contracts among themselves. A multi-period contract is appropriate for dealing with long-term risks, or simply when companies agree to make multi-period arrangements. A similar example is the design of structured derivatives, for instance, the practice of tranching. In these examples, Pareto efficiency is pertinent for designing the optimal allocation of risks, while financial fairness guarantees that the contract is fairly priced.

The characterization of Pareto efficient solutions in a single-period setting is well studied in quite a lot of papers, which date back to the 1960s with the focus mainly on the field of insurance. For instance, Borch [6] gives a characterization of the Pareto efficient solutions under the situation where expected utility is used to describe the agents' risk preferences, and later DuMouchel [10] gives proof to these results. Similar work also includes Raviv [19] which takes into consideration the existence of market frictions. The fairness criterion is first considered alongside the Pareto efficiency by, amongst others, Gale [11], Bühlmann and Jewell [8] and Balasko [2] in different settings. In these literatures, the risk sharing is built over both a utility basis and a valuation basis.

The risk sharing problem in a multi-period setting is investigated by Barrieu and Scandolo [4] in a general setting; they talk about risk exchanges between two agents over more than one period without taking into consideration any fairness conditions. Other work has been mainly focused on the design of pension systems and the space of intergenerational risk sharing, where risk redistribution can be organized among both the existing and future cohorts. Pareto-efficient risk sharing can be achieved by maximizing the aggregate expected utility of all the generations in the situation where a social planner is present (e.g. Gordon and Varian [13], Gollier [12], Bovenberg and Mehlkopf [7]) or by looking for an equilibrium (see Ball and Mankiw [3], Krueger and Kubler [16]). Financial fairness has been considered by Cui et al. [9]; however, the valuation approach is only used to check afterwards whether the distribution rule is fair for the participants. Kleinow and Schumacher [15] analyze the pension system with conditional indexation from the perspective of market value; they investigate whether the pension contract is financially fair for existing and incoming cohorts as well as the sponsor. Risk-neutral valuation becomes essential in Bovenberg and Mehlkopf [7] to determine a unique risk sharing solution by setting the ex-ante market values of the intergenerational transfers to zero.

This paper explores the Pareto efficient and financially fair risk sharing in a multi-period environment. Expected utility is adopted to evaluate the welfare, and a risk-neutral measure works for the valuation purpose. We shall show the existence and uniqueness of the PEFF solution, and give a numerical algorithm to find it. This paper can be seen as a direct generalization of the research by Pazdera et al. [18], which explores the Pareto efficient and financially fair risk sharing rule in a single-period case. Compared to Barrieu and Scandolo [4], we restrict ourselves to the case of expected utility as the preference functional, and risk-neutral valuation is built into the system to determine the uniqueness of the solution. Different from Bovenberg and Mehlkopf [7], no parameterization on the risk sharing rules is needed here; the rules are determined totally under the notion of PEFF. Mathematically, our results resemble the famous consumption-savings model for inter-temporal substitution to some extent. The inter-temporal balance equation, as we call it, has a close relationship with the Euler equation in the inter-temporal substitution theory; see Hall [14]. The main difference is that the model here introduces no subjective discount factor for impatience. The characterization of Pareto efficiency leads to a weighted optimization problem where the weights are unknowns to be determined uniquely by the financial fairness constraints, making use of a risk-neutral measure.

The rest of the paper is structured as follows. The model setting will first be set up and we will formulate the problem of finding PEFF solutions mathematically. Next we establish the existence and uniqueness of the solution. Explicit solution exists when we assume exponential utility functions to all the agents; other than that, there's no hope for an explicit solution in general. We then develop an iterative algorithm to numerically find the solution. Some remarks will conclude the paper in the end.

2 Model Framework

We assume a finite discrete-time system in which a finite number of agents gather to share their risks. Here the risks refer to the stochastic cash inflows from the agents. As a result of the risk sharing, the agents expect to receive cash outflows from the system. Each agent is assumed to get one single cash outflow. The term "cash outflow" is general and can have various interpretations in different circumstances. For instance, it can refer to the risk exposure of the insurance company after risk sharing in the case of a reinsurance contract, or the consumption of the agent in the case of life-cycle modeling. Alongside there is also a long-lived buffer which makes the inter-temporal money transfer possible.

The system starts at time t_0. Assume that altogether there are N cash outflows happening at time points $t_1 \leq t_2 \leq \cdots \leq t_N$. C_n will stand for the cash outflow paid out from the system at time t_n. Let F_n be the buffer size at time t_n. X_n denotes the aggregate risk coming into the system from the agents from time t_{n-1} to t_n, that is, it is the sum of all the stochastic cash inflows from the agents from time t_{n-1} to t_n. It can be interpreted as the risk exposure of the insurance companies in the case of a reinsurance agreement, or the stochastic labor income in the case of life-cycle modeling. The risk stream $X = (X_1, \cdots, X_N)$ is defined on a financial market in which prices are given exogenously. The buffer is invested in a risky asset

R which produces stochastic per-dollar gross return R_n from time t_{n-1} to t_n. We further assume that $R_n > 0$. The X_n's and R_n's are random variables defined on a finitely discrete filtered probability space

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q}, \mathbb{F})$$

where $\mathbb{P}$ and $\mathbb{Q}$ are the given objective and risk-neutral measures respectively and $\mathbb{F}$ is the filtration generated by the X's and R's:

$$\mathbb{F} = \{\mathcal{F}_n | n = 1, \cdots N\}, \quad \mathcal{F}_n = \sigma\{(X_1, R_1), \cdots, (X_n, R_n)\}.$$

There is no need to assume the completeness of the market; any given risk-neutral measure $\mathbb{Q}$ will suffice. The only assumption is that the agents have agreed to adopt some probability measures $\mathbb{P}$ and $\mathbb{Q}$, or the measures are simply specified in a situation where a social planner is present. Let

$$\mathbb{E}_n[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\mathcal{F}_n].$$

It is assumed that the processes X and R are not necessarily independent, but the process $\{(X_n, R_n)\}$ is sequentially independent, i.e. (X_t, R_t) and (X_s, R_s) are independent for $t \neq s$. As we are working on a finite probability space, the sets $X_n(\Omega)$ and $R_n(\Omega)$ are both finite. Every pair (X_n, R_n) can then be totally characterized by

$$\left\{ \left((X_n^{j_n}, R_n^{j_n}), \mathbb{P}(j_n), \mathbb{Q}(j_n) \right) \middle| j_n = 1, \cdots m_n \right\}$$

where $(X_n^{j_n}, R_n^{j_n})$ represents all the possible and distinct values of (X_n, R_n) and $\mathbb{P}(j_n), \mathbb{Q}(j_n)$ are the corresponding $\mathbb{P}$- and $\mathbb{Q}$-probabilities. Here $\mathbb{P}(\cdot)$ and $\mathbb{Q}(\cdot)$ stand for the probabilities of any given cases of interest. A technical requirement is that for any $n = 1, \cdots, N$

$$\mathbb{Q}(X_n = \max X_n, R_n = \max R_n) > 0 \tag{2.1}$$

which means that X_n and R_n can attain their maximum under $\mathbb{Q}$ simultaneously.

Write $J_n = j_1 j_2 \cdots j_n$ as the trajectory $\left((X_1^{j_1}, R_1^{j_1}), \cdots, (X_n^{j_n}, R_n^{j_n}) \right)$. Let $\mathcal{J}_n$ be the set of all the possible trajectories of (X, R) up to time t_n. $J_n j_{n+1}$ will denote any trajectory whose up-to-time-t_n part is J_n. In such a situation we write $j_{n+1} \in \mathcal{J}_n^{n+1}$ where $\mathcal{J}_n^{n+1}$ denotes the set of all the possible cases of (X_{n+1}, R_{n+1}).

The risk-neutral measure $\mathbb{Q}$ is used to price the risks X as well as the investment returns R. In this generic setting, write

$$x_n := \mathbb{E}^{\mathbb{Q}} X_n, \quad 1 + r_n := \mathbb{E}^{\mathbb{Q}} R_n, \quad n = 1, \cdots, N.$$

The x_n's are the market prices of the risks X and the r_n's are the risk-free returns implied by the pricing measure $\mathbb{Q}$. Please note that now and later we consider no discounting for the market values, i.e. prices are stated in terms of a numéraire.

The agents expect to receive stochastic cash outflows C_n after risk sharing, which are the decision variables. The utilities of the agents depend solely on the cash outflows they receive. As we have assumed, at each time point $t_1, \cdots, t_N$ there can be only one cash flow paid out. In case there will be more than one cash flow paid out at the same time, say at time $t_{n-1} = t_n$ for some n, we let $X_n \equiv 0$ and $R_n \equiv 1$, i.e. there will be no risks coming in and the buffer will not evolve. Furthermore we assume that $R_n > 0$ for all n as the R's have the interpretation as the gross return of the asset R.

Utility function $u_n(\cdot)$ will be used to evaluate cash flow C_n which is defined on $x \in (b_n, +\infty)$. b_n can either be finite (e.g. shifted power utility) or equal to $-\infty$ (e.g. exponential utility). These utility functions are *stereotype utility functions* defined as follows:

1. it is continuous and differentiable;
2. it is strictly concave;
3. the marginal utility satisfies the *Inada conditions*

$$\lim_{x \downarrow b_n} u_n'(x) = +\infty, \qquad \lim_{x \to \infty} u_n'(x) = 0.$$

For any agent, define $I_n = (u_n')^{-1}$, which is the inverse function of the marginal utility function. Since u_n' satisfies the Inada conditions, we know that I_n is a strictly decreasing function mapping $(0, +\infty)$ into $(b_n, +\infty)$ and is a bijection.

The budget constraints of the system are then rather straightforward: at each time point, the invested capital will be distributed between the buffer and the current cash flow, i.e.

$$F_n + C_n = X_n + F_{n-1} R_n \quad n = 1, \cdots, N. \tag{2.2}$$

It is assumed without loss of generality that

$$F_0 = 0$$

and the buffer can be both positive and negative. Note that the budget constraint is

$$C_1 + F_1 = X_1 + F_0 R_1 := \tilde{X}_1,$$

which suggests that the situation when F_0 is nonzero or even a random variable can always be dealt with by regarding $X_1 + F_0 R_1$ as a new random variable $\tilde{X}_1$. Hence, the variable R_1 doesn't really play a role, and we will ignore it from now.

For the end-phase buffer we may have the following two cases:

- *Closed end-buffer (CEB)* case: F_N will be a constant. Without loss of generality we assume

  $$F_N = 0.$$

  When F_N is supposed to be a nonzero constant, we can then redefine a new random variable $\tilde{X}_N$ such that

  $$C_N = (X_N - F_N) + F_{N-1} R_N := \tilde{X}_N + F_{N-1} R_N.$$

- *Open end-buffer (OEB)* case: F_N will be a decision variable just as the C's. This means that the buffer provider will also participate in the risk sharing. In this case, a stereotype utility function u_p will be employed to evaluate the utility of F_N.

It can be argued as follows that any OEB case can always be converted into a CEB case. For any OEB case $(C_1, \cdots, C_N, F_N)$ with utility functions $(u_1, \cdots, u_N, u_p)$, we add in a new time point $t_{N+1} := t_N$ with $X_{N+1} := 0$ and $R_{N+1} := 1$. Then the OEB setting is thus formulated into a CEB one with an extra cash outflow C_{N+1} with utility u_p

$$C_{N+1} = X_{N+1} + F_N R_{N+1} = F_N.$$

On the other hand, any CEB setting can be turned into an OEB setting in the sense of Pareto efficiency as we shall see later. In this paper we will proceed mainly with the OEB setting. The utility of the final buffer F_N will be evaluated according to the utility function u_p defined on $(b_p, +\infty)$.

We will try to determine the C's and the F's. For any $n = 1, \cdots, N$, both F_n and C_n are by nature $\mathcal{F}_n$-measurable random variables. We then have the following important definition.

DEFINITION 2.1 A vector of random variables $(C_1, C_2, \cdots C_N, F_N)$ is called a *risk-sharing rule* if it satisfies

- the measurability condition: $C_n \in \mathcal{F}_n$ for $n = 1, \cdots, N$ and $F_N \in \mathcal{F}_N$,
- the budget constraints (2.2), and
- the domain requirements of the utility functions, i.e. $C_n > b_n$ for all n and $F_N > b_p$ along any trajectory.

One last thing to mention is that the budget constraints (2.2) will imply a single *global budget constraint* by eliminating the F's:

$$\sum_{n=1}^{N-1} \left[C_n \left(\prod_{i=n+1}^{N} R_i \right) \right] + C_N + F_N = \sum_{n=1}^{N-1} \left[X_n \left(\prod_{i=n+1}^{N} R_i \right) \right] + X_N. \tag{2.3}$$

This will imply that in order to make the problem well-posed, one needs to have that, for any realizations of X and R

$$\sum_{n=1}^{N-1} \left[b_n \left(\prod_{i=n+1}^{N} R_i \right) \right] + b_N + b_p < \sum_{n=1}^{N-1} \left[X_n \left(\prod_{i=n+1}^{N} R_i \right) \right] + X_N.$$

Otherwise there will be no possible risk sharing rules as the domain requirements of the utility functions can never be satisfied.

EXAMPLE 2.2 *(The autarky.)* A trivial solution to the risk sharing problem is the *autarky* where there is no risk-sharing effect: all agents will be on their own and the buffer will be left unused.

Example 2.3 *(Possible variations of the model.)* The budget constraint (2.2) shows that the model is very general and can handle different risk sharing systems. Examples are

- if we let

$$\begin{aligned} t_1 = t_2 = \cdots = t_N \\ X_2 = \cdots = X_N \equiv 0 \\ R_2 = \cdots = R_N \equiv 1 \end{aligned}$$

  then the system degenerates to a single-period problem as in Pazdera et al. [18] and the budget constraint becomes

$$\sum_{n=1}^{N} C_n + F_N = X_1$$

  where X_1 represents the aggregate risk to be shared.

- If we only let

$$X_2 = \cdots = X_N \equiv 0$$

  then this represents a *decumulation system* where the only cash inflow X_1 will be distributed into several cash outflows in the future.

- A *defined-contribution pension fund* in the form of a successive generations model can be modeled by modifying the budget constraint

$$F_n + C_n = (Y_{n-1} + F_{n-1})R_n \quad n = 1, \cdots, N,$$

  where the Y's are the contributions paid into the system by the beginning of each period, the C's are the benefits paid out from the system by the end of each period and the R's now represent the fixed asset mix where the fund would invest its capital.

- The life-cycle modeling is when we let the timeline $\{t_n\}$ be equispaced in a CEB setting and let a representative agent own all the cash flows. The X's stand for the stochastic income and the C's are the consumptions. The buffer F now is interpreted as the savings account.

3 Pareto Efficiency in the Multi-Period Setting

This section deals with the concept of Pareto efficiency in this multi-period setting, which is the first step to look for a PEFF risk sharing rule. We shall characterize parametrically all the PE solutions among which we look for the one that is also financially fair in the following sections.

It may be convenient to introduce first some notations. Let $\mathbb{R}_+^{N+1}$ be the nonnegative cone in $\mathbb{R}^{N+1}$: $\{\theta \in \mathbb{R}^{N+1} | \theta_i \geq 0\}$, and define $\mathbb{R}_{++}^{N+1} := \{\theta \in \mathbb{R}^{N+1} | \theta_i > 0\}$ as the strictly positive cone. For simplicity we write $X := (X_1, \cdots, X_N)$ and $R := (R_2, \cdots, R_N)$ which are functions from Ω to the discrete sets $X(\Omega) \subset \mathbb{R}^N$ and $R(\Omega) \subset \mathbb{R}^{N-1}_{++}$. Write $\rho := (C_1, C_2, \cdots C_N, F_N) : X(\Omega) \times R(\Omega) \mapsto \mathbb{R}^{N+1}$ as the generic notation for a risk-sharing rule and the set of all the possible ρ's is denoted as $\mathcal{RS}$. We will be particularly interested in the subset $\mathcal{P} \subset \mathcal{RS}$ which is the set of all Pareto-efficient risk sharing rules. First we need the following definition.

Definition 3.1 *(Multi-period Pareto efficiency.)* A risk-sharing rule $(C_1, C_2, \cdots C_N, F_N)$ is called *Pareto efficient*, or *Pareto optimal*, if there does not exist another risk-sharing rule $(\tilde{C}_1, \tilde{C}_2, \cdots \tilde{C}_N, \tilde{F}_N)$ such that

$$\left(\mathbb{E}^{\mathbb{P}} u_1(\tilde{C}_1), \cdots, \mathbb{E}^{\mathbb{P}} u_N(\tilde{C}_N), \mathbb{E}^{\mathbb{P}} u_p(\tilde{F}_N)\right) \gneqq \left(\mathbb{E}^{\mathbb{P}} u_1(C_1), \cdots, \mathbb{E}^{\mathbb{P}} u_N(C_N), \mathbb{E}^{\mathbb{P}} u_p(F_N)\right).$$

We then have the following important theorem in this discrete probability space, which can be seen as a generalization of the Borch-type characterization of the Pareto efficiency: every Pareto-efficient risk-sharing rule can be totally characterized by optimizing a weighted time-additive aggregate utility.

Theorem 3.2 *(Characterization of Pareto efficiency.) For a risk-sharing rule $(C_1, C_2, \cdots, C_N, F_N)$, the following statements are equivalent.*

1. *The risk-sharing rule is Pareto efficient.*
2. *The risk-sharing rule maximizes*

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N} \theta_n u_n(C_n) + \theta_p u_p(F_N)\right] \tag{3.1}$$

for some positive constants $\theta = (\theta_1, \cdots, \theta_N, \theta_p)$.

3. *The risk-sharing rule will satisfy the following which are hereafter called the inter-temporal balance equations (IBEs) for some positive constants $\theta = (\theta_1, \cdots, \theta_N, \theta_p)$:*

$$\begin{aligned} \theta_n u_n'(C_n) &= \theta_{n+1} \mathbb{E}_n^{\mathbb{P}}\left[u_{n+1}'(C_{n+1}) R_{n+1}\right] \quad \forall n = 1, \cdots N-1, \\ \theta_N u_N'(C_N) &= \theta_p u_p'(F_N). \end{aligned}$$

Proof See appendix. □

Remark 3.3 *(Link to Borch [6].)* Consider $t_n = t_{n+1}$ for some n. Then the model assumes that $X_{n+1} \equiv 0$ and $R_{n+1} \equiv 1$. Thus $\mathcal{F}_n = \mathcal{F}_{n+1}$ and the IBE becomes

$$\theta_n u_n'(C_n) = \theta_{n+1} \mathbb{E}_n^{\mathbb{P}}\left[u_{n+1}'(C_{n+1}) R_{n+1}\right] = \theta_{n+1} u_{n+1}'(C_{n+1}).$$

This means that in a single period setting, the IBEs will coincide with the characterization of PE risk sharing rules by Borch [6].

Remark 3.4 *(Comparison to the Euler equation.)* The IBEs look very similar to the famous Euler equation derived amongst others by Hall [14] for solving the consumption-savings model. In fact, the model setting in this paper can be definitely interpreted as a life-cycle model. If the cash outflows will be received at different stages/periods of an individual and we set $R_n = 1 + r$ and $u_n = u$ for all n, then the model setting is also similar to Hall's: every period there is a stochastic earning and a consumption, which correspond to the incoming "risks" and the "cash outflows" in this setting.

The optimization targets are different regarding weighing inter-temporally the utilities: Hall assumed a single rate of subjective time preference δ while the IBEs are parameterized by weight vector θ.

Formula-wise, Hall gave

$$\mathbb{E}_n u'(C_{n+1}) = \left(\frac{1+\delta}{1+r}\right) u'(C_n),$$

while the IBE gives

$$\mathbb{E}_n u'(C_{n+1}) = \left(\frac{\frac{\theta_n}{\theta_{n+1}}}{1+r}\right) u'(C_n).$$

It is obvious that Hall adopts a specific set of weights in the scope of Theorem 3.2. As we shall see later, the weights θ can be seen as unknowns within the framework here and will be determined endogenously by the financial fairness constraint.

The theorem shows that it is equivalent to solve the optimization problem (3.1) subject to the budget constraints when one wants to find the corresponding PE risk sharing rule given any $\theta \in \mathbb{R}^{N+1}_{++}$. We can then construct a mapping to compute the PE solution given any $\theta \in \mathbb{R}^{N+1}_{++}$, which we will call $\Phi : \mathbb{R}^{N+1}_{++} \to \mathcal{P}$. This can be done by solving the corresponding parameterized optimization problem of time-additive utility functions:

$$\begin{aligned}
&\max_{C_1,\cdots,C_N} \mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N} \theta_n u_n(C_n) + \theta_p u_p(F_N)\right] \\
s.t.\quad & F_n + C_n = X_n + F_{n-1}R_n \quad n = 1, \cdots, N, \\
& F_0 = 0.
\end{aligned}$$

This optimization problem can be solved by dynamic programming. Add in a new time point $t_{N+1} = t_N$, and

$$X_{N+1} \equiv 0, \quad R_{N+1} \equiv 1.$$

Define

$$A_n := X_n + F_{n-1}R_n \quad n = 1, \cdots, N+1,$$

which has the interpretation as the total available asset at time t_n to be divided into the current cash flow and the buffer for later use. Note that by definition $A_{N+1} = F_N$. The

A's are the state variables, the C's are the decision variables and the X's and R's are the risks. Then we shall have the optimization problem formulated as, in line with the routine by Bertsekas [5]

$$\begin{aligned}
&\max_{C_1,\cdots,C_N} \mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N}\theta_n u_n(C_n)+\theta_p u_p(A_{N+1})\right] \\
s.t.\quad & A_{n+1}=X_{n+1}+(A_n-C_n)R_{n+1}, \quad n=1,\cdots,N, \\
& A_1 = X_1.
\end{aligned}$$

Proposition 1.3.1 in [5] tells that in order to solve the problem one needs to define the value functions (indirect utility): first for the last period

$$V_{N+1}(A_{N+1})=\theta_p u_p(A_{N+1}),$$

and then define backwards, for $n=1,\cdots,N$

$$V_n(A_n)=\max_{C_n}\mathbb{E}_n^{\mathbb{P}}\left[\theta_n u_n(C_n)+V_{n+1}(X_{n+1}+(A_n-C_n)R_{n+1})\right]. \tag{3.2}$$

The final result is presented below. This mapping Φ gives an explicit expression of the risk sharing rule ρ as a function of the weights θ, which makes it possible to express the financial fairness condition in terms of the weights later in the paper.

THEOREM 3.5 *(The construction of Φ.) For any given $\theta=(\theta_1,\cdots,\theta_N,\theta_p)\in\mathbb{R}_{++}^{N+1}$, the corresponding PE solution is given by*

$$A_n = X_n + F_{n-1}R_n \quad n=1,\cdots,N, \tag{3.3}$$

$$C_n = I_n\left(\frac{g_n(A_n)}{\theta_n}\right) \quad n=1,\cdots,N, \tag{3.4}$$

$$F_n = H_n\left(\frac{g_n(A_n)}{\theta_{n+1}}\right) \quad n=1,\cdots,N-1, \tag{3.5}$$

$$F_N = I_p\left(\frac{g_N(A_N)}{\theta_p}\right), \tag{3.6}$$

where the functions are defined recursively by

$$\begin{aligned}
G_N(x) &:= I_N\left(\frac{x}{\theta_N}\right)+I_p\left(\frac{x}{\theta_p}\right), \\
g_N(x) &:= G_N^{-1}(x),
\end{aligned}$$

and for $n = 1, \cdots, N-1$

$$h_n(x) = \mathbb{E}_n^{\mathbb{P}}\left[\frac{1}{\theta_{n+1}} g_{n+1}(X_{n+1} + xR_{n+1})R_{n+1}\right] \tag{3.7}$$

$$= \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\theta_{n+1}} g_{n+1}(X_{n+1} + xR_{n+1})R_{n+1}\right], \tag{3.8}$$

$$H_n = h_n^{-1},$$

$$G_n(x) := I_n\left(\frac{x}{\theta_n}\right) + H_n\left(\frac{x}{\theta_{n+1}}\right),$$

$$g_n(x) := G_n^{-1}.$$

The mapping (3.3) - (3.6) will be denoted as $\Phi : \mathbb{R}_{++}^{N+1} \to \mathcal{P}$.

Proof See appendix. Please note that from expression (3.7) to (3.8) we utilized the assumption that the processes X and R are sequentially independent. □

The functions above have the following interpretation. While u_n' is the marginal utility function of the cash outflow C_n, the function h_n is the implied marginal utility of the buffer F_n and g_n the implied marginal utility of the total available asset A_n. The capital-letter functions I, H, G are the corresponding inverse functions. The following relationship will always hold:

$$g_n(A_n) = \theta_n u_n'(C_n) = \theta_{n+1} h_n(F_n), \quad n = 1, \cdots, N-1,$$
$$g_N(A_N) = \theta_N u_N'(C_N) = \theta_p u_p'(F_N).$$

The function g's are also the derivatives of the value functions. The proof in the appendix shows that for any n

$$\frac{\mathrm{d}V_n}{\mathrm{d}A_n}(A_n) = g_n(A_n).$$

Write

$$L_n := g_n(A_n),$$

which is interpreted as the weighted marginal utility of the cash outflows. Furthermore, the IBE will be translated into

$$L_n = \mathbb{E}_n^{\mathbb{P}}[L_{n+1}R_{n+1}].$$

The idea of dynamic programming tells that in each period, the system has to ponder how to distribute the risks between the current cash outflow and all the future cash outflows: for any $n < N$, it compares the marginal utilities of paying out the money now (i.e. C_n) or saving it for the future (i.e. F_n):

$$\theta_n u_n'(C_n) \quad v.s. \quad \theta_{n+1} h_n(F_n).$$

The h_n function is calculated by "summarizing" the expectations over the future. This property allows us to convert an n-period problem into an *induced* $(n-1)$-period one, by regarding the time t_{n-1} as the new end of the system and F_{n-1} as the new final buffer with utility h_{n-1}.

This perspective is essential for the proofs later. As a first application, it can help us link the settings of CEB and OEB to each other. First, as we have discussed, any OEB problem can be converted into a CEB problem by regarding F_N as an extra cash outflow C_{N+1} at $t_{N+1} = t_N$. The following result shows that in the sense of Pareto efficiency, the OEB and CEB are equivalent, thus we can work with the two environments interchangeably.

PROPOSITION 3.6 *(Equivalence between CEB and OEB problems.) The CEB and the OEB are equivalent in the sense that they can always be converted into the form of the other which can produce the identical PE risk sharing rule.*

PROOF We only need to consider the direction from CEB to OEB. Given a CEB case with PE risk sharing rule $(C_1, \cdots, C_N)$, utility functions $(u_1, \cdots, u_N)$ and weights $(\theta_1, \cdots, \theta_N)$, we can create a corresponding OEB problem that replicates the original setting for $n = 1, \cdots, N-1$ and truncate the system at time t_{N-1} by defining

$$h(x) := \mathbb{E}^{\mathbb{P}}_{N-1}\left[u'_N(X_N + xR_N)R_N\right]$$

as the marginal utility function for the new end buffer F_{N-1} together with weight θ_N. Then according to the IBE for the CEB problem we have

$$\begin{aligned} \theta_{N-1}u'_{N-1}(C_{N-1}) &= \theta_N \mathbb{E}^{\mathbb{P}}_{N-1}\left[u'_N(C_N)R_N\right] \\ &= \theta_N \mathbb{E}^{\mathbb{P}}_{N-1}\left[u'_N(X_N + F_{N-1}R_N)R_N\right] \\ &= \theta_N h(F_{N-1}) \end{aligned}$$

which matches the final-period IBE in Theorem 3.2. Thus according to the theorem the two settings should produce the same PE risk sharing rules. The only thing left is to verify that the function $h(x)$ defined in this way is indeed a (stereotype) marginal utility function; this has been done in the proof of Theorem 3.5. $\square$

There is one degree of freedom extra in determining θ, as for any $c \in \mathbb{R}_{++}$, θ and $c \cdot \theta$ will produce essentially the same optimization target. But if we choose a way of normalizing the θ's, e.g. restrict the θ's to the open unit simplex in $\mathbb{R}^{N+1}_{++}$, then we will have the following theorem which tells that every PE risk-sharing rule $\rho \in \mathcal{P}$ can be uniquely characterized by the weights θ, and the function Φ is a meaningful bijection between all the PE risk sharing rules ρ's and the weights θ's.

THEOREM 3.7 *Φ is a one-to-one mapping between the set of all the Pareto efficient risk sharing rules $\mathcal{P}$ and the open unit simplex in $\mathbb{R}^{N+1}_{++}$, i.e. the set $\mathcal{U} := \{c \in \mathbb{R}^{N+1}_{++} | c_1 + \cdots c_{N+1} = 1\}$.*

PROOF This can be seen as a corollary of Theorem 3.2. We discuss the two directions.

1. $\mathcal{U} \to \mathcal{P}$: the mapping Φ maps any $\theta \in \mathbb{R}_{++}^{N+1}$ into $\mathcal{P}$. This mapping is not injective. Consider some θ and θ' s.t. $\Phi(\theta) = \Phi(\theta')$. Then we show that there will exist some $c \in \mathbb{R}_{++}$ s.t. $\theta = c\theta'$ thus Φ is injective if restricted on $\mathcal{U}$.

   By the IBEs we know that

   $$\frac{\theta_n}{\theta_{n+1}} = \frac{\mathbb{E}_n u'_{n+1}(C_{n+1})R_{n+1}}{u'_n(C_n)} \quad \forall n = 1, \cdots N-1$$

   and

   $$\frac{\theta_N}{\theta_p} = \frac{u'_p(F_N)}{u'_N(C_N)}.$$

   This indicates

   $$\frac{\theta_n}{\theta_{n+1}} = \frac{\theta'_n}{\theta'_{n+1}} \quad \forall n = 1, \cdots N-1$$

   and

   $$\frac{\theta_N}{\theta_p} = \frac{\theta'_N}{\theta'_p}$$

   We then have

   $$\theta = \frac{\theta_1}{\theta'_1}\theta'.$$

   Φ will be an injective mapping if restricted on $\mathcal{U}$.

2. $\mathcal{P} \to \mathcal{U}$: Theorem 3.2 tells that for any element $\rho \in \mathcal{P}$, there exists some $\theta \in \mathbb{R}_{++}^{N+1}$ s.t. $\Phi(\theta) = \rho$.

We conclude from the above discussion that Φ is both injective and surjective. It must be bijective. $\square$

We conclude this section by some useful properties of the PE risk sharing system. First, we give the following result which seems quite intuitive: every agent will be better off when the realization of the risks is (strictly) better. We call this the monotonicity property of the system w.r.t. the risks.

Lemma 3.8 *(Monotonicity property of the system w.r.t. the risks.) For any $\theta \in \mathbb{R}_{++}^{N+1}$, consider two trajectories $J, J^* \in \mathcal{J}_N$ s.t. $(X^J, R^J) \gneqq (X^{J^*}, R^{J^*})$. Then we have $\rho^J \gneqq \rho^{J^*}$.*

Proof See appendix. $\square$

The following result illustrates the impact of the weight θ on the cash flows: if some weight increases while the others stay the same, then along any trajectory, the corresponding cash flow will increase while the other cash flows will decrease.

Lemma 3.9 *(Monotonicity property of the system w.r.t. the weights.) Consider two weights $\theta = (\theta_1, \cdots, \theta_N, \theta_p), \theta' = (\theta'_1, \cdots, \theta'_N, \theta'_p) \in \mathbb{R}_{++}^{N+1}$ s.t. there exists some $n = 1, \cdots, N, p$ that*

$$\theta_n > \theta'_n, \quad \theta_i = \theta'_i \quad \forall i \neq n.$$

Then we have that for any trajectory $J \in \mathcal{J}_N$, the corresponding PE risk sharing rules satisfy

$$C_n^J > C_n^{\prime J}, \quad C_i^J < C_i^{\prime J} \quad \forall i \neq n.$$

Here for convenience we let $C_p = F_N$.

Proof See appendix. □

4 Financial Fairness

As we have discussed, the PE risk sharing rules can be totally characterized by the points on the open unit simplex in $\mathbb{R}_{++}^{N+1}$ and thus there will be infinitely many such PE rules. We will see in the following that the concept of financial fairness will help us narrow down our scope – finally we will arrive at a unique risk sharing rule that is both PE and FF.

The concept of financial fairness means that when the system starts, for each agent involved, the market value of the risks he contributes into the system should be equal to that of the cash outflows he gets after risk sharing. FF will work via the concept of *value profile*, which is the vector of the values of cash outflows under the risk-neutral measure $\mathbb{Q}$, that is, for any $\rho = (C_1, \cdots, C_N, F_N) \in \mathcal{RS}$

$$v = (v_1, v_2, \cdots, v_N, v_p) := \mathbb{E}^{\mathbb{Q}}\rho = \left(\mathbb{E}^{\mathbb{Q}}C_1, \mathbb{E}^{\mathbb{Q}}C_2, \cdots, \mathbb{E}^{\mathbb{Q}}C_N, \mathbb{E}^{\mathbb{Q}}F_N\right) \in \mathbb{R}^{N+1}. \tag{4.1}$$

As before we consider no discounting and we simply use the $\mathbb{Q}$–expectation as market values. These market values are totally determined by the market values of the risk positions of each agent before risk sharing.

The set of all the possible value profiles $\mathcal{V}$ can only be a restricted subset of $\mathbb{R}^{N+1}$. First note it is trivial that

$$v_n > b_n \quad \forall n = 1, \cdots, N; \quad v_p > b_p$$

according to the domain requirements of the utility functions. Next, according to the global budget constraint (2.3) we shall have, by taking the expectation under $\mathbb{Q}$ to both sides

$$\sum_{n=1}^{N-1}\left[v_n\left(\prod_{i=n+1}^{N}(1+r_i)\right)\right] + v_N + v_p = \sum_{n=1}^{N-1}\left[x_n\left(\prod_{i=n+1}^{N}(1+r_i)\right)\right] + x_N. \tag{4.2}$$

We can then write

$$\mathcal{V} = \left\{v \in \mathbb{R}^N \middle| \text{Eq (4.2) holds;} \quad v_n > b_n \ \forall\ n = 1, \cdots, N; \quad v_{N+1} > b_p\right\} \tag{4.3}$$

as the set of all possible value profiles. Note that $\mathcal{V}$ is totally determined by the risks and the utility functions.

Remark 4.1 The global budget constraint suggests that for any given value profile vector $v := (v_1, \cdots, v_N, v_p)$, we only have to consider any N coefficients. For instance, if the following hold

$$\mathbb{E}^{\mathbb{Q}} C_n = v_n \quad n = 1, \cdots, N$$

then

$$\mathbb{E}^{\mathbb{Q}} F_N = v_p$$

will automatically be satisfied.

5 Existence and Uniqueness of the PEFF Risk Sharing Rule

The theorems in this section will show that the solution exists and is actually unique if we combine the Pareto efficiency with financial fairness. We continue to work with the general situation when there are N cash outflows alongside the buffer, $N \geq 1$. For any given value profile $v := (v_1, \cdots, v_N, v_p) \in \mathcal{V}$, the corresponding PEFF risk-sharing rule is the solution to the following equation system:

1. budget constraints (BCs):

$$F_n + C_n = X_n + F_{n-1} R_n \quad n = 1, \cdots, N; \tag{5.1}$$

2. inter-temporal balance equations (IBEs):

$$\begin{aligned} \theta_n u_n'(C_n) &= \theta_{n+1} \mathbb{E}_n^{\mathbb{P}} \left[u_{n+1}'(C_{n+1}) R_{n+1} \right] \quad \forall n = 1, \cdots N-1, \\ \theta_N u_N'(C_N) &= \theta_p u_p'(F_N); \end{aligned} \tag{5.2}$$

3. financial fairness constraints (FFs):

$$\mathbb{E}^{\mathbb{Q}} C_n = v_n \quad \forall n = 1, \cdots, N. \tag{5.3}$$

Please note that the C's and F's are actually functions on a finite discrete domain. Each of the BC and IBE equations above then actually stands for a family of trajectory-indexed equations, i.e. the equation holds true for all possible trajectories.

The following theorem is one of the key results of this paper. It tells that for the equation system above, the solution always exists and is unique, thus it establishes the existence and uniqueness of the PEFF risk sharing rule.

Theorem 5.1 *(The existence and uniqueness of the PEFF risk sharing rule.) For any given value profile vector $v \in \mathcal{V}$, the PEFF risk-sharing rule exists and is unique. The corresponding θ is unique up to normalization.*

Proof See appendix. $\square$

Theorem 3.2 tells that function sets BC and IBE characterize all the possible PE risk-sharing rules by way of weights $\theta \in \mathbb{R}^{N+1}_{++}$. The theorem above then shows that the value profile determines a unique θ.

Recall that in Theorem 3.5 Φ defines a bijective mapping from $\mathcal{U}$ to the set of all PE risk sharing rules $\mathcal{P}$. The mapping Φ then induces a natural mapping Ψ from $\mathcal{U}$ to $\mathcal{V}$: $\Psi(\theta) = \mathbb{E}^{\mathbb{Q}}\Phi(\theta)$. This Ψ links the set of all the possible weights θ and the set of all the possible value profiles.

Theorem 5.2 *Ψ is a one-to-one mapping between the set of all possible value profiles $\mathcal{V}$ and the open unit simplex $\mathcal{U}$ in $\mathbb{R}^{N+1}_{++}$.*

Proof Theorem 5.1 tells that Ψ is surjective: for any given $v \in \mathcal{V}$ there exists a $\theta \in \mathbb{R}^{N+1}_{++}$ s.t. $\Psi(\theta) = \mathbb{E}^{\mathbb{Q}}\Phi(\theta) = v$.

This Ψ is also injective restricted on the open unit simplex $\mathcal{U}$ because of the uniqueness of θ up to normalization. Suppose there are $\theta_1, \theta_2 \in \mathcal{U}$ such that $\Psi(\theta_1) = \Psi(\theta_2)$. Theorem 5.1 indicates that $\Phi(\theta_1) = \Phi(\theta_2)$, as for each value profile, there will exist exactly one PE risk sharing rule s.t. the FF condition is satisfied. According to Theorem 3.7, it must be that $\theta_1 = \theta_2$ as they both belong to the open unit simplex $\mathcal{U}$. $\square$

We can then say that the θ uniquely determines the value profile of any PE risk sharing rule, and also *vice versa*. Instead of talking about the weights θ we can now talk about the value profiles which seem more tangible. However, we cannot say more of the mapping Ψ; the structure of it can be very complicated depending on the utility functions one uses.

6 A General Algorithm For Finding PEFF Solution

Looking for the PEFF risk sharing rule will come down to solving a system of both linear and non-linear equations. In most cases there's no hope for explicit solutions; fortunately, we have a numerical algorithm that helps to find the PEFF solution under all circumstances.

Recall that

$$L_n = \theta_n u_n'(C_n) \quad n = 1, \cdots, N$$

are the weighted marginal utilities of the cash outflows as determined by the risk sharing rule at time t_n. According to the IBEs

$$L_n = \mathbb{E}^{\mathbb{P}}[L_{n+1}R_{n+1}] \quad n = 1, \cdots, N-1,$$

thus the whole sequence $\{L_n\}$ is known once L_N is known.

In Theorem 3.5 we constructed a mapping $\Phi : \mathbb{R}^{N+1}_{++} \to \mathcal{P}$ from the sets of functions BC and IBE. Given the mapping Φ, we can deduce another mapping φ_1 by

$$\varphi_1(\theta) = L_N = \theta_N u_N'(C_N) = \theta_N u_N'(\Phi_{(N)}(\theta)),$$

where $\Phi_{(N)}(\cdot)$ stands for the N-th coordinate of this vector-valued function. φ_1 maps any $\theta \in \mathbb{R}_{++}^{N+1}$ into some L_N. For any L_N, another mapping $\varphi_2 : L_N \mapsto \theta$ can be constructed based on the FF constraints: note that according to the mapping Φ we have

$$C_n = I_n\left(\frac{L_n}{\theta_n}\right) \quad \forall n = 1, \cdots, N;$$
$$F_N = I_p\left(\frac{L_N}{\theta_p}\right),$$

and

$$L_n = \mathbb{E}_n[L_{n+1}R_{n+1}].$$

This allows us to find a θ' s.t. the FF conditions are satisfied for the given L_N:

$$\mathbb{E}^{\mathbb{Q}}C_n = \mathbb{E}^{\mathbb{Q}}I_n\left(\frac{L_n}{\theta'_n}\right) = v_n \quad \forall n = 1, \cdots, N; \tag{6.1}$$
$$\mathbb{E}^{\mathbb{Q}}F_N = \mathbb{E}^{\mathbb{Q}}I_p\left(\frac{L_N}{\theta'_p}\right) = v_p. \tag{6.2}$$

The function φ_2 is well defined since

$$\mathbb{E}^{\mathbb{Q}}C_n = \mathbb{E}^{\mathbb{Q}}I_n\left(\frac{L_n}{\theta_n}\right) = \sum_{J\in\mathcal{J}_n} \mathbb{Q}(J) I_n\left(\frac{L_n^J}{\theta_n}\right)$$

is a strictly increasing and continuous function in θ_n with $\theta_n \in \mathbb{R}_{++}$. Thus $\varphi_{2(n)}(L_N) := \left[\mathbb{E}^{\mathbb{Q}}I_n\left(\frac{L_n}{\cdot}\right)\right]^{-1}(v_n)$ is well defined. This holds for all $n = 1, \cdots, N$ and also for F_N, thus φ_2 is well-defined. Please note that one and only one coordinate of the weight vector θ is solved in every single equation (6.1) and (6.2).

Consider the composition of the two functions $\varphi = \varphi_2 \circ \varphi_1$: it is a mapping from $\mathbb{R}_{++}^{N+1}$ into itself. Theorem 5.1 tells that there always exists a unique fixed point of this mapping φ, which corresponds to the PEFF risk sharing rule. The next theorem shows that φ suggests an iterative algorithm for finding the PEFF solution.

Theorem 6.1 *(Feasibility of an iterative algorithm by φ.) For any given starting point $\theta \in \mathbb{R}_{++}^{N+1}$ with any proper normalization, the sequence of iterates $\{\varphi^{(n)}(\theta) | n \in \mathbb{N}_+\}$ will converge to the fixed point of φ.*

Proof See Appendix. □

Theorem 6.1 suggests that starting with any given θ, one first finds the corresponding L_N by φ_1 and then updates the value of θ by φ_2. It is more convenient, in fact, to use function Φ instead of φ_1, i.e. we map θ to ρ directly and in the second step we update the θ accordingly. In the first step, we need to calculate numerically the functions g's and h's backwards in time, and once all the functions are ready, we then go forwards in time and calculate all the C's and F's from the starting distribution X_1.

Algorithm 1 *(Numerical algorithm for finding the PEFF solution.)* The following gives a description of the numerical algorithm for finding the PEFF solution.

1. Start with some initial $\theta^{(0)} \in \mathbb{R}^{N+1}_{++}$.
2. For any given $\theta^{(m)}$ with $m \in \mathbb{N}$, calculate backwards in time that

$$\begin{aligned} G_N^{(m)}(x) &:= I_N\left(\frac{x}{\theta_N^{(m)}}\right) + I_p\left(\frac{x}{\theta_p^{(m)}}\right), \\ g_N^{(m)}(x) &:= \left(G_N^{(m)}\right)^{-1}(x), \end{aligned}$$

   and for $n = 1, \cdots, N-1$

$$\begin{aligned} h_n^{(m)}(x) &= \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\theta_{n+1}^{(m)}} g_{n+1}^{(m)}(X_{n+1} + xR_{n+1})R_{n+1}\right], \\ H_n^{(m)} &= \left(h_n^{(m)}\right)^{-1}, \\ G_n^{(m)}(x) &:= I_n\left(\frac{x}{\theta_n^{(m)}}\right) + H_n^{(m)}\left(\frac{x}{\theta_{n+1}^{(m)}}\right), \\ g_n^{(m)}(x) &:= \left(G_n^{(m)}\right)^{-1}. \end{aligned}$$

3. Calculate the decision variables forwards in time by

$$\begin{aligned} A_n^{(m)} &= X_n + F_{n-1}^{(m)} R_n \quad n = 1, \cdots, N, \\ C_n^{(m)} &= I_n\left(\frac{g_n^{(m)}(A_n^{(m)})}{\theta_n^{(m)}}\right) \quad n = 1, \cdots, N, \\ F_n^{(m)} &= H_n^{(m)}\left(\frac{g_n^{(m)}(A_n^{(m)})}{\theta_{n+1}^{(m)}}\right) \quad n = 1, \cdots, N-1, \\ F_N^{(m)} &= I_p\left(\frac{g_N^{(m)}(A_N^{(m)})}{\theta_p^{(m)}}\right). \end{aligned}$$

4. Update the θ from $\theta^{(m)}$ to $\theta^{(m+1)}$ by solving that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} C_n^{(m)} &= \mathbb{E}^{\mathbb{Q}} I_n\left(\frac{g_n^{(m)}(A_n^{(m)})}{\theta_n^{(m+1)}}\right) = v_n \quad n = 1, \cdots, N; \\ \mathbb{E}^{\mathbb{Q}} F_N^{(m)} &= \mathbb{E}^{\mathbb{Q}} I_p\left(\frac{g_N^{(m)}(A_N^{(m)})}{\theta_p^{(m+1)}}\right) = v_p. \end{aligned}$$

5. Normalize $\theta^{(m+1)}$.
6. If, for some pre-specified error tolerance ε

$$\left\|\theta^{(m)} - \theta^{(m+1)}\right\| < \varepsilon$$

we conclude that $\rho^{(m)}$ is the PEFF risk sharing rule we are looking for. Otherwise, go to step 2 with $\theta^{(m+1)}$.

Remark 6.2 *(Comparison to the algorithm proposed by Pazdera et al. [18].)* As has been mentioned, the framework introduced here can also deal with the single-period situation, which has been investigated by Pazdera et al. [18]. There is a significant difference between the two numerical algorithms, though. The algorithm here makes use of the induction technique that the number of cash outflows is reduced by one recursively, thus in each iteration the algorithm always calculate the functions backwards and then the distributions of the decision variables forwards. In contrast, the algorithm in [18] need not use such an induction technique; functions and decision variables can be calculated simultaneously in each iteration. The algorithm in [18] offers more efficiency for the single-period problem, while the algorithm here is more versatile and can deal with multi-period problems.

7 Explicit PEFF Solution: Example

This section discusses a special case when we assume the R_n's are constants (thus only the risks X are stochastic) and exponential utility functions (the constant-absolute-risk-aversion (CARA) utility) are used for all the cash outflows

$$u_n(x) = 1 - e^{-\alpha_n x}, \quad \forall\, n = 1, \cdots, N,$$

and also for the buffer provider

$$u_p(x) = 1 - e^{-\alpha_p x}.$$

Then we will have explicit PEFF solutions: the benefits are actually linear functions of the risks.

Theorem 7.1 *(PEFF solution under CARA utility.) The PEFF solution to an N-period model with exponential utility functions is of the form*

$$\begin{aligned} C_n &= a_n\left[(X_n + F_{n-1}R_n) - w_n\right] + v_n = a_n(A_n - w_n) + v_n, \\ F_n &= A_n - C_n = (1 - a_n)A_n - (v_n - a_n w_n), \end{aligned}$$

where

$$w_n := \mathbb{E}^{\mathbb{Q}} A_n$$

which can be calculated recursively from the budget constraints and the a_n's are defined recursively by

$$a_N = \frac{\alpha_p}{\alpha_p + \alpha_N}, \tag{7.1}$$

$$a_n = \frac{a_{n+1}\alpha_{n+1}R_{n+1}}{\alpha_n + a_{n+1}\alpha_{n+1}R_{n+1}} \quad n = 1, \cdots, N-1. \tag{7.2}$$

PROOF See appendix. □

Theorem 7.1 shows that under CARA utility, each cash outflow only takes a proportion a_n of $A_n - w_n$ which is the excess return from total available asset, thus only takes part of the risk. The remaining part $(1 - a_n)$ is shifted into the future. Under the CARA utility assumption, the risk-sharing rules don't depend on the distribution of the random variables.

REMARK 7.2 Suppose $R_n \equiv R = 1 + r$ for $n = 1, \cdots, N$. Also, let $\alpha_n \equiv \alpha$ for $n = 1, \cdots, N$, that is, we assume the same risk aversion level for all the agents except the buffer. The equations (7.2) become

$$a_n = \frac{a_{n+1}R}{1 + a_{n+1}R}.$$

If we let $N \to \infty$, then we shall have

$$a_n \to \frac{R-1}{R} \approx r,$$

that is, given a sufficiently long horizon, the proportion that each agent takes from the total excess return is approximately equal to the risk-free rate.

8 Concluding Remarks

In this paper we explored solving a multi-period risk sharing problem under the concept of Pareto efficiency and financial fairness. The important results are:

1. Theorem 3.2 characterizes the Pareto efficient risk sharing rules: every PE risk sharing rule can be associated uniquely to an optimization problem with the objective function being the weighted aggregate expected utility of the cash outflows, which can be further translated into the inter-temporal balance equations. Theorem 3.5 tells how to compute the risk sharing rule given the weights.
2. Theorem 5.1 establishes the existence and uniqueness of a PEFF risk sharing rule. Furthermore, Theorem 5.2 tells that the value profile will uniquely determine the weights.
3. Theorem 6.1 guarantees the possibility to find unique the PEFF rule numerically by a universal algorithm.

We conclude this paper with some comments on further possibilities. First, this paper assumes that each agent can have only one cash outflow as a way of simplification. As a result, the optimization target (3.1) is time-additive and the value profile is straightforward to determine. If we make the generalization that each agent can have multiple cash outflows in different periods, two issues need to be solved. Utility-wise, one needs to choose a preference functional for evaluating the welfare; value-wise, the value profile needs to be determined following extra principles. Some cases are essentially different from the setting in this paper, and the existence and uniqueness of the PEFF solution may have to be re-established.

In this paper the financial fairness is defined in an *ex ante* sense, i.e. the market values of the cash flows will match the given value profile only at the time when the system starts. The FF will generally not hold *ex interim*, as the cash outflows are by nature contingent claims and their market values will change after the system starts. This is not a problem when, like in a multi-period reinsurance arrangement, all the agents are already physically present when the system starts; however, for a CDC pension system which may include already the unborn cohorts at start, this issue may result in the so-called *discontinuity problem*: some future cohort may find themselves in a very disadvantageous position when they have to face a large deficit in the buffer left by the previous generations because of some preceding bad financial performance. The later cohort may argue that they didn't have a say when the system was initiated, thus they may choose not to step into the system.

Strict ex-interim FF is meaningful, but essentially excludes any possibility of intertemporal capital transfer, thus there is no space for intergenerational risk sharing. One may then adopt some fairness condition that lies between the two extremes as a compromise. We may also introduce a second-best solution by imposing extra constraints on the size of the buffer such that the deviation from ex-interim FF still remains acceptable. These possibilities are beyond the scope of this paper and may be future topics of interest.

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A Proofs for Section 3

For any risk sharing rule $\rho = (C_1, \cdots, C_N, F_N) \in \mathcal{RS}$, let

$$u(\rho) := (u_1(C_1), \cdots, u_N(C_N), u_p(F_N))$$

and

$$\phi := \mathbb{E}^{\mathbb{P}} u(\rho) = (\mathbb{E}^{\mathbb{P}} u_1(C_1), \cdots, \mathbb{E}^{\mathbb{P}} u_N(C_N), \mathbb{E}^{\mathbb{P}} u_p(F_N)) \in \mathbb{R}^{N+1}.$$

First note that ϕ is a strictly concave and increasing function of ρ with co-domain $\mathbb{R}^{N+1}$. The PE optimization target then becomes

$$\mathbb{E}^{\mathbb{P}} \left[\sum_{n=1}^{N} \theta_n u_n(C_n) + \theta_p u_p(F_N) \right] = \langle \theta, \phi \rangle$$

where $\theta = (\theta_1, \cdots, \theta_N, \theta_p) \in \mathbb{R}^{N+1}_{++}$.

We need the following definitions and results in preparation for the proof of Theorem 3.2.

Lemma A.1 *Consider n concave functions $\{f_i | i = 1, \cdots, n\}$ from a common domain K to $\mathbb{R} \cup \{-\infty\}$. Then $F(K) - \mathbb{R}^n_+ := \{x - y | \forall x \in F(K), y \in \mathbb{R}^n_+\}$ is convex where $F := (f_1, f_2, \cdots, f_n)$.*

Proof cf. the proof of Proposition 2.6 from Aubin [1]. □

We will use a separation theorem in the proof of Theorem 3.2. We then need to introduce the following definitions. [1]

Definition A.2 *(Affine sets in $\mathbb{R}^n$.)* A subset $M \in \mathbb{R}^n$ is called an affine set if $(1-\lambda)x + \lambda y \in M$ for any $x, y \in M$ and $\lambda \in \mathbb{R}$.

Definition A.3 *(Affine hull.)* The affine hull of any subset $M \in \mathbb{R}^n$, which is denoted as $\mathit{aff}(M)$, is the smallest affine set that contains M.

Definition A.4 *(Relative interior and boundary.)* The relative interior of a convex set $C \subset \mathbb{R}$, which is denoted as $ri(C)$, is defined as the interior of C when it is regarded as a subset of $\mathit{aff}(C)$. The relative boundary of C is the difference of the closure of C and the relative interior of C.

The following lemma is crucial in proving Theorem 3.2.

Lemma A.5 *Let C be a convex set. A point $x \in C$ is a relative boundary point of C if and only if there exists a linear function not constant on C s.t. it achieves its maximum over C at x.*

[1] Interested readers are referred to Rockafeller [20] for more details.

Proof cf. Corollary 11.6.2 by Rockafellar [20]. □

Proof of Theorem 3.2.

$1 \Rightarrow 2$: Let $\rho = (C_1, C_2, \cdots C_N, F_N)$ be PE. Then we have that $\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}$ is convex by Lemma A.1. Note that an element ρ^* is PE if and only if

$$\{\phi(\rho^*)\} \cap \left(\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}\right) = \{\phi(\rho^*)\}$$

and

$$\{\phi(\rho^*)\} \cap \left(\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}\right)^\circ = \varnothing.$$

Otherwise, if $\{\phi(\rho^*)\} \in \left(\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}\right)^\circ$, then there exist $\rho' \in \mathcal{RS}$ and $c \in \mathbb{R}_+^{N+1}$ with $c \neq 0$ s.t. $\phi(\rho^*) = \phi(\rho') - c$, which means ρ' results in a Pareto improvement. This is in contradiction with the assumption that ρ^* is PE.

$\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}$ is a full-dimensional set thus its relative interior is the same as its interior. Write $\phi^* = \phi(\rho^*)$. Then ϕ^* is a relative boundary point of $\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}$, as it belongs to $\phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}$, thus to its closure, but not its relative interior. According to Lemma A.5, for this ϕ^*, there exists a $\theta^* \neq 0$ s.t.

$$\sup_{\phi \in \phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}} \langle \theta^*, \phi \rangle \leq \langle \theta^*, \phi^* \rangle.$$

First note that any coordinates of θ^* cannot be negative as then

$$\sup_{\phi \in \phi(\mathcal{RS}) - \mathbb{R}_+^{N+1}} \langle \theta^*, \phi \rangle = +\infty.$$

No coordinates of θ^* can be 0. If this would be the case, suppose $\theta_1^* = 0$ while $\theta_2^* > 0$ without loss of generality. Then any $\rho = (C_1, C_2, \cdots C_N, F_N)$ cannot be optimal since for any small $\epsilon > 0$ such that $C_1^{j_1} - \epsilon > b_1$ for all $j_1 \in \mathcal{J}_1$, $\rho_\epsilon = (C_1 - \epsilon, C_2 + R_2\epsilon, \cdots C_N, F_N)$ will result in a larger optimization target because u_2 is strictly increasing.

$2 \Rightarrow 1$: consider a risk-sharing rule ρ that maximizes $\langle \theta, \phi \rangle$ for some $\theta \in \mathbb{R}_+^{N+1}$. If ρ is not PE, then there exists another $\tilde{\rho}$ s.t. $\tilde{\phi} \gneqq \phi$ and hence

$$\langle \theta, \tilde{\phi} \rangle > \langle \theta, \phi \rangle$$

which results in a contradiction.

$2 \Leftrightarrow 3$: as we are working with a finite probability space, we may use the Lagrangian multiplier method to solve the maximization problem.

For $n = 1, \cdots, N$, reorganize the budget constraint and we have

$$F_n^{J_{n-1}j_n} + C_n^{J_{n-1}j_n} - X_n^{j_n} - F_{n-1}^{J_{n-1}} R_n^{j_n} = 0.$$

Define

$$F_n^{J_{n-1}j_n} + C_n^{J_{n-1}j_n} - X_n^{j_n} - F_{n-1}^{J_{n-1}} R_n^{j_n}$$

as $BC^{J_{n-1}j_n}$ or BC^{J_n}.

We then maximize

$$\sum_{n=1}^{N}\left\{\theta_n \sum_{J_n\in\mathcal{J}_n} \mathbb{P}(J_n)u_n(C_n^{J_n}) + \sum_{J_n\in\mathcal{J}_n} \lambda^{J_n} BC^{J_n}\right\} + \theta_p \sum_{J_N\in\mathcal{J}_N} \mathbb{P}(J_N)u_p(F_N^{J_N})$$

where the λ's are the Lagrangian multipliers.

For any $n < N$, setting the first-order partial derivative w.r.t. $C_n^{J_n}$ to 0 will help us find a stationary point of the optimization problem. It gives

$$\mathbb{P}(J_n)\theta_n u_n'(C_n^{J_n}) + \lambda^{J_n} = 0 \quad \forall J_n \in \mathcal{J}_n.$$

For $n+1$ similarly we have, along the trajectory J_n

$$\mathbb{P}(J_n j_{n+1})\theta_{n+1} u_{n+1}'(C_{n+1}^{J_n j_{n+1}}) + \lambda^{J_n j_{n+1}} = 0 \quad \forall j_{n+1} \in \mathcal{J}_n^{n+1}.$$

Now take the partial derivative w.r.t. F_n and set to 0

$$\lambda^{J_n} = \sum_{j_{n+1}\in\mathcal{J}_n^{n+1}} \lambda^{J_n j_{n+1}} R_{n+1}^{j_{n+1}}.$$

This will lead to

$$\theta_n u_n'(C_n^{J_n}) = \theta_n \sum_{j_{n+1}\in\mathcal{J}_n^{n+1}} u_{n+1}'(C_{n+1}^{J_n j_{n+1}}) R_{n+1}^{j_{n+1}} \frac{\mathbb{P}(J_n j_{n+1})}{\mathbb{P}(J_n)}.$$

By the assumption of sequential independence we have

$$\frac{\mathbb{P}(J_n j_{n+1})}{\mathbb{P}(J_n)} = \mathbb{P}(j_{n+1}).$$

Then the equation can be further rewritten as

$$\theta_n u_n'(C_n) = \theta_{n+1}\mathbb{E}_n^{\mathbb{P}}\left[u_{n+1}'(C_{n+1})R_{n+1}\right] \quad \forall n = 1, \cdots N-1.$$

C_N and F_N are both $\mathcal{F}_N$-measurable and we have

$$\theta_N u_N'(C_N^{J_N}) = \theta_p u_p'(F_N^{J_N}) = -\lambda^{J_N}$$

by taking partial derivatives w.r.t. C_N and F_N and setting them to be 0.

We have arrived at a stationary point thanks to the Lagrangian multiplier method; this stationary point is the unique global optimum once we note that the optimization target is a concave function w.r.t. the decision variables and the feasible set is convex. □

Proof of Theorem 3.5. The optimization target 3.1 is a parameterized optimization problem of time-additive utility functions:

$$\begin{aligned} &\max_{C_1,\cdots,C_N} \mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N}\theta_n u_n(C_n) + \theta_p u_p(F_N)\right] \\ s.t.\quad & F_n + C_n = X_n + F_{n-1}R_n \quad n = 1, \cdots, N, \\ & F_0 = 0. \end{aligned}$$

This optimization problem can be solved by dynamic programming. Add in a new time point $t_{N+1} = t_N$, and

$$X_{N+1} \equiv 0, \quad R_{N+1} \equiv 1.$$

Define

$$A_n := X_n + F_{n-1} R_n \quad n = 1, \cdots, N+1,$$

which has the interpretation as the total available asset at time t_n to be divided into the current cash flow and the buffer for later use. Note that by definition $A_{N+1} = F_N$. The A's are the state variables, the C's are the decision variables and the X's and R's are the risks. Then we shall have the optimization problem formulated as, in line with the routine by Bertsekas [5]

$$\begin{aligned} &\max_{C_1, \cdots, C_N} \mathbb{E}^{\mathbb{P}} \left[\sum_{n=1}^{N} \theta_n u_n(C_n) + \theta_p u_p(A_{N+1}) \right] \\ s.t. \quad & A_{n+1} = X_{n+1} + (A_n - C_n) R_{n+1}, \quad n = 1, \cdots, N, \\ & A_1 = X_1. \end{aligned}$$

Proposition 1.3.1 in [5] tells that in order to solve the problem one needs to define first

$$V_{N+1}(A_{N+1}) = \theta_p u_p(A_{N+1}),$$

and then define backwards, for $n = 1, \cdots, N$

$$V_n(A_n) = \max_{C_n} \mathbb{E}_n^{\mathbb{P}} \left[\theta_n u_n(C_n) + V_{n+1}(X_{n+1} + (A_n - C_n) R_{n+1}) \right]. \tag{A.1}$$

This can be solved by taking the derivative of

$$\mathbb{E}_n^{\mathbb{P}} \left[\theta_n u_n(C_n) + V_{n+1}(X_{n+1} + (A_n - C_n) R_{n+1}) \right] \tag{A.2}$$

w.r.t. C_n and setting it to 0. We will start from period N and go backwards in time in order to verify the differentiability of the V_n's. For period N, note that the target A.2 becomes

$$\theta_N u_N(C_N) + \theta_p u_p(F_N) = \theta_N u_N(C_N) + \theta_p u_p(A_N - C_N).$$

The conditional expectation vanishes because of the measurability of C_N and F_N. It is continuous and differentiable w.r.t. C_N. Take the derivative and set it to 0; we get

$$\theta_N u_N'(C_N^*) = \theta_p u_p'(A_N^* - C_N^*) := L_N^*.$$

Here the star indicates that it is the optimal solution. Next, define

$$\begin{aligned} G_N(x) &:= I_N\left(\frac{x}{\theta_N}\right) + I_p\left(\frac{x}{\theta_p}\right), \\ g_N(x) &:= G_N^{-1}. \end{aligned}$$

Both G_N and g_N are well-defined. G_N is the sum of two strictly decreasing bijective functions thus it is strictly decreasing and bijective from $\mathbb{R}_{++}$ to $(\max\{b_N, b_p\}, +\infty)$, and it follows that g_N is also strictly decreasing and bijective from $(\max\{b_N, b_p\}, +\infty)$ to $\mathbb{R}_{++}$. The Inada conditions tell

$$\lim_{x\to 0} G_N(x) = +\infty, \quad \lim_{x\to+\infty} G_N(x) = \max\{b_N, b_p\}$$

and thus

$$\lim_{x\to\max\{b_N,b_p\}} g_N(x) = +\infty, \quad \lim_{x\to+\infty} g_N(x) = 0.$$

L_N^* can then be calculated as

$$L_N^* = g_N(A_N^*)$$

and

$$C_N^* = I_N\left(\frac{L_N^*}{\theta_N}\right), \quad F_N^* = I_p\left(\frac{L_N^*}{\theta_p}\right).$$

The value function is

$$V_N(A_N^*) = \theta_N u_N(C_N^*) + \theta_p u_p(A_N^* - C_N^*)$$

which is a differentiable function w.r.t. A_N. The envelope theorem tells that

$$\frac{\mathrm{d}V_N}{\mathrm{d}A_N^*} = \theta_p u_p'(A_N^* - C_N^*) = \theta_p u_p'(F_N^*) = g_N(A_N^*).$$

Going one period backwards, we have the value function

$$V_{N-1}(A_{N-1}) = \max_{C_{N-1}} \mathbb{E}_{N-1}^{\mathbb{P}}\left[\theta_{N-1}u_{N-1}(C_{N-1}) + V_N(X_N + (A_{N-1} - C_{N-1})R_N)\right].$$

For the part

$$\mathbb{E}_{N-1}^{\mathbb{P}}\left[\theta_{N-1}u_{N-1}(C_{N-1}) + V_N(X_N + (A_{N-1} - C_{N-1})R_N)\right]$$

differentiation and conditional expectation can be interchanged since we are working on a finite probability space and we get

$$\theta_{N-1}u_{N-1}'(C_{N-1}) + \mathbb{E}_{N-1}^{\mathbb{P}}\left[\frac{\mathrm{d}V_N(A_N)}{\mathrm{d}C_{N-1}}\right] = \theta_{N-1}u_{N-1}'(C_{N-1}) + \mathbb{E}_{N-1}^{\mathbb{P}}\left[\frac{\mathrm{d}V_N(A_N)}{\mathrm{d}A_N}\frac{\mathrm{d}A_N}{\mathrm{d}C_{N-1}}\right] = 0,$$

which leads us to

$$L_{N-1}^* = \theta_{N-1}u_{N-1}'(C_{N-1}^*) = \mathbb{E}_{N-1}^{\mathbb{P}}\left[g_N(X_N + F_{N-1}^*R_N)R_N\right] = \mathbb{E}_{N-1}^{\mathbb{P}}[L_N^*R_N].$$

We then define

$$h_{N-1}(x) := \frac{1}{\theta_N}\mathbb{E}_{N-1}^{\mathbb{P}}\left[g_N(X_N + xR_N)R_N\right].$$

Due to the assumption of sequential independence, $h_{N-1}(x)$ can further be written in the form of an unconditional expectation

$$h_{N-1}(x) = \frac{1}{\theta_N}\mathbb{E}^{\mathbb{P}}\left[g_N(X_N + xR_N)R_N\right]$$

since both X_N and R_N are independent from $\mathcal{F}_{N-1}$. Note that h_{N-1} is invertible since by definition it is a weighted sum of strictly decreasing functions; thus h_{N-1} is also a strictly decreasing function with domain $(d_{N-1}, +\infty)$, where d_{N-1} is defined as

$$d_{N-1} = \inf \left\{ d \in \mathbb{R} \middle| X_N^j + dR_N^j \geq \max\{b_N, b_p\} \quad \forall j \in \mathcal{J}_{N-1}^N \right\}.$$

Furthermore, h_{N-1} can be viewed as the marginal utility of a stereotype utility function since

- it is continuous and strictly decreasing,
- it satisfies

$$\lim_{x \to d_{N-1}} h_{N-1}(x) = +\infty, \quad \lim_{x \to +\infty} h_{N-1}(x) = 0.$$

Write

$$H_{N-1} := h_{N-1}^{-1}.$$

Then once we combine

$$C_{N-1}^* + F_{N-1}^* = X_{N-1} + F_{N-2}^* R_{N-1} = A_{N-1}^*$$

with

$$L_{N-1}^* = \theta_{N-1} u_{N-1}'(C_{N-1}^*) = \theta_N h_{N-1}(F_{N-1}^*)$$

we have

$$I_{N-1}\left(\frac{L_{N-1}^*}{\theta_{N-1}}\right) + H_{N-1}\left(\frac{L_{N-1}^*}{\theta_N}\right) = A_{N-1}^*.$$

Next, define

$$G_{N-1}(x) := I_{N-1}\left(\frac{x}{\theta_{N-1}}\right) + H_{N-1}\left(\frac{x}{\theta_N}\right),$$
$$g_{N-1}(x) := G_{N-1}^{-1}.$$

G_{N-1} and g_{N-1} are well-defined just as G_N and g_N. L_{N-1} can then be calculated as

$$L_{N-1}^* = g_{N-1}(A_{N-1}^*)$$

and

$$C_{N-1}^* = I_{N-1}\left(\frac{L_{N-1}^*}{\theta_{N-1}}\right), \quad F_{N-1}^* = H_{N-1}\left(\frac{L_{N-1}^*}{\theta_N}\right).$$

For the value function

$$V_{N-1}(A_{N-1}^*) = \theta_{N-1} u_{N-1}(C_{N-1}^*) + \mathbb{E}_{N-1}^{\mathbb{P}} V_N\left[X_N + (A_{N-1}^* - C_{N-1}^*) R_N\right],$$

it follows that V_{N-1} is differentiable and one can calculate by the envelope theorem that

$$\frac{\mathrm{d}V_{N-1}}{\mathrm{d}A_{N-1}^*} = \mathbb{E}_{N-1}^{\mathbb{P}}[V_N' \cdot R_N] = \mathbb{E}_{N-1}^{\mathbb{P}}[L_N^* R_N] = L_{N-1}^* = g_{N-1}(A_{N-1}^*).$$

Proceeding one period backwards, solving Equation (A.1) gives

$$\theta_{N-2}u'_{N-2}(C_{N-2}) + \mathbb{E}^{\mathbb{P}}_{N-2}\left[\frac{\mathrm{d}V_{N-1}(A_{N-1})}{\mathrm{d}C_{N-2}}\right] = 0,$$

and

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}_{N-2}\left[\frac{\mathrm{d}V_{N-1}(A_{N-1})}{\mathrm{d}C_{N-2}}\right] &= \mathbb{E}^{\mathbb{P}}_{N-2}\left[\frac{\mathrm{d}V_{N-1}(A_{N-1})}{\mathrm{d}A_{N-1}}\cdot\frac{\mathrm{d}A_{N-1}}{\mathrm{d}C_{N-2}}\right] \\
&= \mathbb{E}^{\mathbb{P}}_{N-2}\left[g_{N-1}(A_{N-1})\cdot(-R_{N-1})\right] \\
&= -\mathbb{E}^{\mathbb{P}}_{N-2}\left[g_{N-1}(X_{N-1}+F_{N-2}R_{N-1})R_{N-1}\right].
\end{aligned}$$

We can then repeat what has been done in period $N-1$. This recursive procedure can be continued backwards in time until we arrive at the first period. That is, we can always define recursively for $n = 1, \cdots, N-2$

$$\begin{aligned}
h_n(x) &= \mathbb{E}^{\mathbb{P}}_{n}\left[\frac{1}{\theta_{n+1}}g_{n+1}(X_{n+1}+xR_{n+1})R_{n+1}\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\theta_{n+1}}g_{n+1}(X_{n+1}+xR_{n+1})R_{n+1}\right], \\
H_n &= h_n^{-1}, \\
G_n(x) &:= I_n\left(\frac{x}{\theta_n}\right) + H_n\left(\frac{x}{\theta_{n+1}}\right), \\
g_n(x) &:= G_n^{-1},
\end{aligned}$$

and the decision variables are given by

$$C_n^* = I_n\left(\frac{g_n(A_n^*)}{\theta_n}\right), \quad F_n^* = H_n\left(\frac{g_n(A_n^*)}{\theta_{n+1}}\right).$$

This will be the unique solution of the optimization problem, as the optimization target is concave w.r.t. the decision variables and the feasible set is convex. □

Proof of Lemma 3.8. By definition the function g's are all strictly decreasing. We have

$$\begin{aligned}
C_n &= I_n\left(\frac{g_n(X_n+F_{n-1}R_n)}{\theta_n}\right) \quad \forall n = 1,\cdots,N, \\
F_n &= H_n\left(\frac{g_n(X_n+F_{n-1}R_n)}{\theta_{n+1}}\right) \quad \forall n = 1,\cdots,N-1, \\
F_N &= I_p\left(\frac{g_N(X_N+F_{N-1}R_N)}{\theta_p}\right),
\end{aligned}$$

thus both C_n and F_n are increasing functions of $A_n = X_n + F_{n-1}R_n$.

We only have to consider the case when only one coordinate of $(X, R) = (X_1, \cdots, X_N, R_2, \cdots, R_N)$ increases. First consider two trajectories J, J^* such that there is a time point $\tau = 1, \cdots, N$ s.t. $X^J_\tau > X^{J^*}_\tau$ and other random variables from (X, R) are equal. Since

$$F_n = H_n\left(\frac{g_n(X_n+F_{n-1}R_n)}{\theta_{n+1}}\right) \quad \forall n = 1,\cdots,N-1$$

then $F_1^J = F_1^{J^*}$, and this will lead to $F_2^J = F_2^{J^*}$. Doing this recursively we conclude that $F_n^J = F_n^{J^*}$ for any $n < \tau$. Then as

$$X_\tau^J + F_{\tau-1}^J R_\tau^J > X_\tau^{J^*} + F_{\tau-1}^{J^*} R_\tau^{J^*}$$

we have

$$C_\tau^J > C_\tau^{J^*}, \quad F_\tau^J > F_\tau^{J^*},$$

and the latter will tell that $C_n^J > C_n^{J^*}$ for all $n > \tau$. Also $F_N^J > F_N^{J^*}$. Then $\rho^J \gneqq \rho^{J^*}$.

The cases when only $R_\tau^J > R_\tau^{J^*}$ follows analogously. □

It is convenient to have the following definition before we continue to the proof of Lemma 3.9.

Definition A.6 *(N-PE Problem.)* An N-PE problem refers to the 4-tuple $((X,R),\rho,u',\theta)$ and the corresponding equation systems BC (5.1) and IBE (5.2), where (X,R) is a vector of random variables, ρ a vector of decision variables, u' an $N+1$ tuple of stereotype marginal utility functions and θ a constant vector, i.e.

$$\begin{aligned}
(X,R) &= (X_1,\cdots,X_N,R_2,\cdots,R_N) \in \mathcal{L}^{2N+1},\\
\rho &= (C_1,\cdots,C_N,F_N) \in \mathcal{L}^{N+1},\\
u' &= (u_1',\cdots,u_N',u_p'),\\
\theta &= (\theta_1,\cdots,\theta_N,\theta_p) \in \mathbb{R}_{++}^{N+1},
\end{aligned}$$

where $\mathcal{L} := \mathbb{R}^\Omega$ is the space of random variables over the underlying probability space.

Proof of Lemma 3.9. The key point of the proof is that otherwise, the IBE and the BC cannot hold simultaneously.

We use mathematical induction to show this. First consider $N = 1$. For a 1-PE problem this is true; we only have two agents including the buffer and there will be only one family of IBE:

$$\theta_1 u_1'(C_1) = \theta_p u_p'(F_1),$$

and the budget constraints are

$$C_1 + F_1 = X_1.$$

For any trajectory $J \in \mathcal{J}_1$, if θ_1 increases, then we argue that C_1^J cannot decrease. Otherwise (i.e. C_1^J decreases), by the budget constaint F_1^J will increase, but according to the IBE it will decrease, which is a contradiction. For the same reason C_1^J cannot stay the same. Thus C_1^J will increase and F_1^J has to decrease. As there is a symmetry between C_1 and F_1, we conclude that the argument is true for single-period problems.

Assume the statement holds true for an N-PE problem, $N > 1$. Then consider the case of an $(N+1)$-PE problem with the the conventional notations

$$\begin{aligned}(X, R) &= (X_1, \cdots, X_{N+1}, R_2, \cdots, R_{N+1}),\\ \rho &= (C_1, \cdots, C_{N+1}, F_{N+1}),\\ u' &= (u'_1, \cdots, u'_{N+1}, u'_p),\\ \theta &= (\theta_1, \cdots, \theta_{N+1}, \theta_p).\end{aligned}$$

First consider if some θ_n increases, $n < N+1$. Then as we have discussed, this $(N+1)$-PE problem can be converted into an induced N-PE problem by truncation at time point t_N and define h_N as has been defined in Theorem 3.5, that is, the 4-tuple

$$\begin{aligned}(X, R)_{[N]} &= (X_1, \cdots, X_N, R_2, \cdots, R_N),\\ \rho_{[N]} &= (C_1, \cdots, C_N, F_N),\\ u'_{[N]} &= (u'_1, \cdots, u'_N, h_N),\\ \theta_{[N]} &= (\theta_1, \cdots, \theta_N, \theta_{N+1}).\end{aligned}$$

Consider this N-PE problem. According to the induction assumption, we will have that for any $J \in \mathcal{J}_{N+1}$, C_n^J will increase if θ_n increases, while other cash outflows will decrease. So F_N^J will decrease and so is A_{N+1}^J. Note that by definition the function g_{N+1} will stay the same if θ_n increases. Thus C_{N+1}^J and F_{N+1}^J will both decrease as they are increasing functions of A_{N+1}^J.

Now consider the situation if θ_{N+1} increases. We will show that $F_{N+1}^{J_{N+1}}$ will decrease. Otherwise (i.e. $F_{N+1}^{J_{N+1}}$ either increases or stays the same), by the final period IBE

$$\theta_{N+1} u'_{N+1}(C_{N+1}^{J_{N+1}}) = \theta_p u'_p(F_{N+1}^{J_{N+1}}) = L_{N+1}^{J_{N+1}}$$

we have that $C_{N+1}^{J_{N+1}}$ has to increase because of the monotonicity of u'_{N+1} and u'_p. Then by the budget constraint for that period

$$C_{N+1}^{J_{N+1}} + F_{N+1}^{J_{N+1}} = X_{N+1}^{j_{N+1}} + F_N^{J_N} R_{N+1}^{j_{N+1}}$$

$F_N^{J_N}$ also has to increase. This will lead to the fact that $L_{N+1}^{J_N j^*_{N+1}}$ will not increase for any $j^*_{N+1} \in \mathcal{J}_N^{N+1}$. This is because we have

$$L_{N+1}^{J_N j_{N+1}} = g_{N+1}(X_{N+1}^{j_{N+1}} + F_N^{J_N} R_{N+1}^{j_{N+1}})$$

and

$$L_{N+1}^{J_N j^*_{N+1}} = g_{N+1}(X_{N+1}^{j^*_{N+1}} + F_N^{J_N} R_{N+1}^{j^*_{N+1}})$$

which shows that $L_{N+1}^{J_N j_{N+1}}$ and $L_{N+1}^{J_N j^*_{N+1}}$ should have the same monotonicity property w.r.t. θ_{N+1}. The result is that $\mathbb{E}_N(L_{N+1}^{J_N} R_{N+1})$ will not increase.

According to the global budget constraint along that trajectory, there has to be at least one n such that $C_n^{J_n}$ will decrease. Let the set of such n's be denoted by $\mathcal{T}$. Consider first the situation that $\max\{\mathcal{T}\} = N$. Then $L_N^{J_N} = \theta_N u_N'(C_N^{J_N})$ will increase. On the other hand, $\mathbb{E}_N(L_{N+1}^{J_N} R_{N+1})$ will not increase. We then arrive at a contradiction by noting that by IBE we should have

$$L_N^{J_N} = \mathbb{E}_N(L_{N+1}^{J_N} R_{N+1}).$$

Then consider more generally that $\tau = \max\{\mathcal{T}\} < N$. Then as $F_N^{J_N}$ will increase and $C_N^{J_N}$ will not decrease, by budget constraint we know $F_{N-1}^{J_{N-1}}$ will increase. Repeat this reasoning until we get that $F_\tau^{J_\tau}$ will have to increase. Then by analogy as above we will have that $\mathbb{E}_\tau(L_{\tau+1}^{J_\tau} R_{\tau+1})$ will not increase. However, $L_\tau = \theta_\tau u_\tau'(C_\tau^{J_\tau})$ will increase as $C_\tau^{J_\tau}$ decreases. The IBE will then not hold. We conclude that $F_{N+1}^{J_{N+1}}$ will decrease and $L_{N+1}^{J_{N+1}}$ will increase. According to

$$L_n = \mathbb{E}^{\mathbb{P}}[L_{n+1} R_{n+1}]$$

we know that for any $n < N+1$, along the trajectory J_n which is the up-to-time-t_n part of J_{N+1}, $L_n^{J_n}$ will increase. Then $C_n^{J_n}$ will decrease since

$$L_n^{J_n} = \theta_n u_n'(C_n^{J_n}).$$

Finally, consider the global budget constraint (2.3) along the trajectory J_{N+1}. It must be that $C_{N+1}^{J_{N+1}}$ will have to increase since all the other C's and F_{N+1} will decrease.

The case when only θ_p increases follows analogously as there is symmetry between C_{N+1} and F_{N+1}. This completes the proof. $\square$

B Proofs for Section 5

Please note that some of the proofs in this section make use of the mapping φ defined in Section 6.

Definition B.1 *(N-PEFF Problem.)* An N-PEFF problem refers to the 4-tuple $((X, R), \rho, u', v)$ and the corresponding equation systems (5.1), (5.2) and (5.3), where (X, R) is a vector of random variables, ρ a vector of decision variables, u' an $N+1$ tuple of stereotype marginal utility functions and v a value profile vector, i.e.

$$\begin{aligned}
(X, R) &= (X_1, \cdots, X_N, R_2, \cdots, R_N) \in \mathcal{L}^{2N+1}, \\
\rho &= (C_1, \cdots, C_N, F_N) \in \mathcal{L}^{N+1}, \\
u' &= (u_1', \cdots, u_N', u_p'), \\
v &= (v_1, \cdots, v_N, v_p) \in \mathcal{V}.
\end{aligned}$$

The set $\mathcal{V}$ is totally determined by (X, R) and u' according to Expression (4.3).

Definition B.2 *(Hilbert metric on $\mathbb{R}^n_{++}$.)* The Hilbert metric defines a distance as

$$d(x, y) = \log \frac{\max_i\{x_i/y_i\}}{\min_i\{x_i/y_i\}}$$

for any $x, y \in \mathbb{R}^n_{++}$. It is not a real metric as

$$d(x, y) = 0 \quad \Leftrightarrow \quad \exists c \in \mathbb{R}^+ \quad s.t. \quad y = cx.$$

It will become a true metric if restricted on e.g. the open unit simplex in $\mathbb{R}^n_{++}$.

Lemma B.3 *If $\phi : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is homogeneous and strongly monotone, then ϕ is contractive w.r.t. the Hilbert metric.*

Proof See for instance Lemma 4.5 in Pazdera et al [18]. □

Any contractive mapping ϕ can only have one fixed point. Suppose there are two, namely x and y with $d(x, y) > 0$. Then by contractiveness we have

$$d(x, y) = d(\phi(x), \phi(y)) < d(x, y)$$

which is contradictory. Then $d(x, y) = 0$. Note that the uniqueness is in the sense of Hilbert metric.

The following lemma is the key part of proving the uniqueness of the PEFF solution.

Lemma B.4 *The mapping φ_1 defined in Section 6 is strictly increasing, i.e. for any trajectory $J \in \mathcal{J}_N$, we have that*

$$L^J_N(\theta') \gneqq L^J_N(\theta'') \quad \forall \theta' \gneqq \theta''.$$

Proof To show this we only need to show that L^J_N is strictly increasing w.r.t. any one of the coordinates of θ. We can utilize Lemma 3.9.

Consider first that only θ_n increases while the other θ's stay the same, $n = 1, \cdots, N$. Then according to Lemma 3.9, F^J_N will decrease thus

$$L^J_N = \theta_p u'_p(F^J_N)$$

will increase. The case when only θ_p increases follows analogously as there is symmetry between C_N and F_N. □

Lemma B.5 *(The uniqueness of the PEFF rule.) For any given value profile $v = (v_1, \cdots, v_N, v_p) \in \mathcal{V}$, the corresponding PEFF risk-sharing rule will be unique if it exists.*

Proof The main point of this proof is to show that φ defined in Section 6 is homogeneous and strictly monotone thus by Lemma B.3 it can only have one fixed point (up to normalization) if it has.

The mapping is homogeneous by definition thus we only have to consider monotonicity. First, according to Lemma B.4 φ_1 is strictly increasing w.r.t. θ along all possible trajectories. Then L_n^J is also increasing since

$$L_n = \mathbb{E}_n^{\mathbb{P}}[L_{n+1}R_{n+1}].$$

Now consider $\theta' \gneqq \theta''$. Then for any $J \in \mathcal{J}_N$ we have that $L_N^J(\theta') > L_N^J(\theta'')$. For any possible n, the n-th coordinate of φ_2: $\varphi_{2(n)}(L_N) = \left[\mathbb{E}^{\mathbb{Q}} I_n\left(\frac{L_n}{\cdot}\right)\right]^{-1}(v_n)$ will lead to that $\varphi_{2(n)}(L_N(\theta')) > \varphi_{2,n}(L_N(\theta''))$. This is because φ_2 will always require that

$$\mathbb{E}^{\mathbb{Q}} C_n = \mathbb{E}^{\mathbb{Q}} I_n\left(\frac{L_n}{\theta_n}\right) = \sum_{J \in \mathcal{J}_n} \mathbb{Q}(J) I_n\left(\frac{L_n^J}{\theta_n}\right) = v_n.$$

If L_n^J increases for all $J \in \mathcal{J}_n$, then θ_n also will increase according to this φ_2. The result is that

$$\varphi(\theta') = \varphi_2(L_N(\theta')) > \varphi_2(L_N(\theta'')) = \varphi(\theta''),$$

i.e. φ is strictly increasing w.r.t. θ. $\square$

Proof of Theorem 5.1. The proof uses mathematical induction. Note that we can always fix $\theta_p = 1$ as a normalization to the θ's unless specified otherwise.

For any 1-PEFF problem, there is only one random variable X_1 to be shared. One needs to solve

$$\begin{aligned} C_1 + F_1 &= X_1, \\ \theta_1 u_1'(C_1) &= \theta_p u_p'(F_1), \\ \mathbb{E}^{\mathbb{Q}} C_1 &= v_1. \end{aligned}$$

For any given θ_1, the equations of BC and IBE will jointly produce a certain risk sharing rule according to the mapping Φ in Theorem 3.5. However, the third FF equation may not hold. We need to show that there will exist some θ_1 such that the FF equation will hold. We define

$$w(\theta_1) = \mathbb{E}^{\mathbb{Q}} C_1 = \sum_{J \in \mathcal{J}_1} \mathbb{Q}(J) C_1^J.$$

It is a continuous function of θ_1 which follows as a property of the mapping Φ. Next we will show that the value of the function w can be both above and below v_1, so that there exists some θ_1^* s.t. $w(\theta_1^*) = v_1$ since w is continuous. This will be done by taking θ_1 to the limits.

First consider $\lim\limits_{\theta_1 \to 0} w(\theta_1)$. Then along any trajectory $J \in \mathcal{J}_1$ it must be that $C_1^J \to b_1 < v_1$. Otherwise, suppose there exists some sequence of θ_1, say $\{\hat{\theta}^{[m]}\}$ with $\hat{\theta}^{[m]} \to 0$ as $m \to \infty$, such that

$$\lim_{m \to \infty} C_1^J(\hat{\theta}^{[m]}) \geq b_1 + \varepsilon$$

for some trajectory J and some $\varepsilon \in \mathbb{R}_{++}$. If $b_1 = -\infty$ then this is interpreted as bounded from below. Then according to the IBE

$$\theta_1 u_1'(C_1^J) = \theta_p u_p'(F_1^J)$$

the left hand side will go to 0 as $u_1'(C_1^J)$ will be bounded. As a result, F_1^J will have to go to $+\infty$ which is not possible if we take into consideration the budget constraint. We conclude that $C_1^J \to b_1 < v_1$ along all the J's if we let $\theta_1 \to 0$.

Next consider $\lim\limits_{\theta_1 \to \infty} w(\theta_1)$. Now we drop the normalization constraint $\theta_p = 1$. Taking into consideration the freedom of choosing a way of normalization, it follows that the following two statements are equivalent:

- fix θ_p and let $\theta_1 \to +\infty$;
- fix θ_1 and let $\theta_p \to 0$.

Then following the analogy above we have $F_1^J \to b_p$ for all $J \in \mathcal{J}_1$ as $\theta_p \to 0$. Thus $\lim\limits_{\theta_1 \to \infty} \mathbb{E}^{\mathbb{Q}} F_1 = b_p$ and according to the budget constraint

$$\lim_{\theta_1 \to \infty} w(\theta_1) = \lim_{\theta_1 \to \infty} \mathbb{E}^{\mathbb{Q}} C_1 = v_1 + v_p - b_p.$$

Then since

$$v_p - b_p > 0$$

must hold, we have

$$v_1 < v_1 + v_p - b_p.$$

By a simple intermediate value theorem we know that there will exist some θ_1^* s.t. $w(\theta_1^*) = v_1$. Then we have found a weight vector θ (i.e. $(\theta_1^*, \theta_p = 1)$) that leads to a PEFF solution to the system. This indicates that the fixed points of the mapping φ will exist; the fixed point must be unique according to Lemma B.5, i.e. the vector θ is unique. The uniqueness is up to normalization.

Let's assume that there always exists a unique solution for an N-PEFF problem, $N > 1$. Consider an $(N+1)$-PEFF problem using our conventional notations

$$\begin{aligned}
(X, R) &= (X_1, \cdots, X_{N+1}, R_2, \cdots, R_{N+1}), \\
\rho &= (C_1, \cdots, C_{N+1}, F_{N+1}), \\
u' &= (u_1', \cdots, u_{N+1}', u_p'), \\
v &= (v_1, \cdots, v_N, v_{N+1}, v_p)
\end{aligned}$$

Consider the corresponding $(N+1)$-PE problem with some given weight θ. Use $\theta_p = 1$ as a normalization. As we have discussed, the whole system will degrade to an induced N-PEFF problem with F_N now being the "final" buffer whose risk aversion is characterized by h_N given by Theorem 3.5. That is,

$$\begin{aligned}(X, R)_{[N]} &= (X_1, \cdots, X_N, R_2, \cdots, R_N),\\ \rho_{[N]} &= (C_1, \cdots, C_N, F_N),\\ u'_{[N]} &= (u'_1, \cdots, u'_N, h_N),\\ v_{[N]} &= (v_1, \cdots, v_N, \mathbb{E}^{\mathbb{Q}} F_N)\end{aligned}$$

where $\mathbb{E}^{\mathbb{Q}} F_N$ can be calculated according to the global budget constraint of the induced N-PEFF problem.

For any given θ_{N+1}, according to the assumption, the degraded system has a unique PEFF solution with coefficients $(\theta_1, \cdots, \theta_N)$. This solution, together with the θ_{N+1} and $\theta_p = 1$, satisfies all the equations except the following one[2]

$$\mathbb{E}^{\mathbb{Q}} C_{N+1} = \sum_{J \in \mathcal{J}_{N+1}} \mathbb{Q}(J) C^J_{N+1} = v_{N+1}.$$

Next we will show that there exists θ_{N+1} such that the equation above will hold. Then by Theorem B.5 the solution θ_{N+1} will be unique.

Define

$$w(\theta_{N+1}) = \mathbb{E}^{\mathbb{Q}} C_{N+1} = \sum_{J_{N+1} \in \mathcal{J}_{N+1}} \mathbb{Q}(J_{N+1}) C^{J_{N+1}}_{N+1}.$$

Note that $C^{J_{N+1}}_{N+1}$ is a continuous function of θ_{N+1} for any J which follows from Theorem 3.5 and so is w itself. Next we will show that the value of the function w can be both above and below v_{N+1}, so that there exists θ^*_{N+1} s.t. $w(\theta^*_{N+1}) = v_{N+1}$ since w is continuous. This will be done by taking θ_{N+1} to the limits.

First consider $\lim\limits_{\theta_{N+1} \to 0} w(\theta_{N+1})$. We will distinguish between the following two cases.

A. The lower bounds of the utility functions b_n are all finite. We will then have

$$\lim_{\theta_{N+1} \to 0} C^{J_{N+1}}_{N+1} = b_{N+1} \quad \forall J_{N+1} \in \mathcal{J}_{N+1}.$$

Otherwise, suppose there exists a sequence of θ_{N+1}, say $\{\hat{\theta}^{[m]}\}$ with $\hat{\theta}^{[m]} \to 0$ as $m \to \infty$, such that

$$\lim_{m \to \infty} C^{J_{N+1}}_{N+1}(\hat{\theta}^{[m]}) \geq b_{N+1} + \varepsilon$$

for some trajectory J_{N+1} and some $\varepsilon \in \mathbb{R}_{++}$. Then according to the final period IBE

$$\theta_{N+1} u'_{N+1}(C^{J_{N+1}}_{N+1}) = \theta_p u'_p(F^{J_{N+1}}_{N+1})$$

the left hand side will go to 0 as $u'_{N+1}(C^{J_{N+1}}_{N+1})$ will be bounded. As a result, $F^{J_{N+1}}_{N+1}$ will have to go to $+\infty$ which is not possible when all the C's can only be finite.

[2]The equation $\mathbb{E}^{\mathbb{Q}} F_{N+1} = v_p$ will also not hold. However, as we have discussed, we don't have to consider this equation, since it will be automatically satisfied if other FF conditions hold.

B. Consider when $b_n = -\infty$ for some n. $\mathcal{T}$ denotes the set of all such n's. We will show that still

$$\lim_{\theta_{N+1}\to 0} C_{N+1}^{J_{N+1}} = b_{N+1} \quad \forall J_{N+1} \in \mathcal{J}_{N+1}.$$

We then only have to show this for a special J' which satisfies that for any n, $X_n^{J'} = \max_{J\in\mathcal{J}_{N+1}} X_n^J$ and $R_n^{J'} = \max_{J\in\mathcal{J}_{N+1}} R_n^J$, i.e. $(X^{J'}, R^{J'})$ is the attainable "upper bound" of all trajectories. This is possible because the number of trajectories is finite, the condition 2.1 holds and the risk stream is sequentially independent. Once we show that $\lim_{\theta_{N+1}\to 0} C_{N+1}^{J'} = b_{N+1}$, by Lemma 3.8, the limit of C_{N+1} of all other trajectories cannot be larger than b_{N+1}, and also cannot be smaller than b_{N+1}.

Otherwise, suppose there exist a sequence of θ_{N+1}, say $\{\hat{\theta}^{[m]}\}$ with $\hat{\theta}^{[m]} \to 0$ as $m \to \infty$, and $\varepsilon > 0$, such that

$$\lim_{m\to\infty} C_{N+1}^{J'}(\hat{\theta}^{[m]}) \geq b_{N+1} + \varepsilon.$$

If $b_{N+1} = -\infty$ then the equation above is interpreted as that the sequence $\{C_{N+1}^{J'}(\hat{\theta}^{[m]})\}$ is bounded from below.

Then by final period IBE

$$\theta_{N+1} u'_{N+1}(C_{N+1}^{J'}) = \theta_p u'_p(F_{N+1}^{J'})$$

we have that $F_{N+1}^{J'}$ will have to go to $+\infty$ since $u'_{N+1}(C_{N+1}^{J'})$ will be bounded. Consider the global budget constraint: now since $C_{N+1}^{J'} + F_{N+1}^{J'} \to +\infty$, there will exist $\tau \in \mathcal{T}$ s.t. $C_\tau^{J'} \to -\infty$. By the definition of J', we have that $C_\tau^J \to -\infty$ for any other possible $J \in \mathcal{J}_{N+1}$, thus the value profile condition for C_τ will not hold. This is a contradiction.

To conclude: we have shown that

$$\lim_{\theta_{N+1}\to 0} C_{N+1}^{J} = b_{N+1} \quad \forall J \in \mathcal{J}_{N+1}$$

whatever the value of b_{N+1} is. Thus

$$w(\theta_{N+1}) = \mathbb{E}^{\mathbb{Q}} C_{N+1} = \sum_{J\in\mathcal{J}_{N+1}} \mathbb{Q}(J) C_{N+1}^J \to b_{N+1}.$$

Next consider $\lim_{\theta_{N+1}\to\infty} w(\theta_{N+1})$. Now we drop the normalization constraint $\theta_p = 1$. Taking into consideration the freedom of choosing a way of normalization, we can conclude that the following two statements are equivalent:

- fix θ_p and let $\theta_{N+1} \to +\infty$;
- fix θ_{N+1} and let $\theta_p \to 0$.

Then following the analogy we have $F_{N+1}^J \to b_p$ for all $J \in \mathcal{J}_{N+1}$ as $\theta_p \to 0$. Thus according to the budget constraint for the last period, we conclude that

$$\lim_{\theta_{N+1}\to\infty} w(\theta_{N+1}) = \lim_{\theta_{N+1}\to\infty} \sum_{J\in\mathcal{J}_{N+1}} \mathbb{Q}(J) C_{N+1}^J = v_{N+1} + v_p - b_p.$$

Then since

$$v_p - b_p > 0$$

must hold, we have

$$v_{N+1} < v_{N+1} + v_p - b_p.$$

By a simple intermediate value theorem we know that there will exist some θ^*_{N+1} s.t. $w(\theta^*_{N+1}) = v_{N+1}$. Then we have found a weight vector θ that leads to a PEFF solution to the system. This indicates that the fixed points of the mapping φ will exist; the fixed point must be unique according to Theorem B.5, i.e. the solution θ is unique. The uniqueness is up to normalization. This finishes the proof. □

C Proofs for Section 6

LEMMA C.1 *When $(\mathcal{X}, d)$ is a locally compact and connected metric space, and $f : \mathcal{X} \to \mathcal{X}$ is a contractive mapping with fixed point $x^* \in \mathcal{X}$, then for every $x \in \mathcal{X}$ the sequence of iterates $\{f^{(n)}(x) | n \in \mathbb{N}_+\}$ converges to x^*.*

PROOF cf. Thm. 1 by Nadler [17]. □

PROOF OF THEOREM 6.1. Lemma B.5 has shown that the mapping φ is contractive w.r.t. the Hilbert metric. The theorem then is a direct result of Lemma C.1. □

D Proofs for Section 7

PROOF OF THEOREM 7.1. The proof of the theorem is actually a process of calculation. First we need the following preparations. By Theorem 3.5, for any given θ we can define $f_n(\cdot)$ such that

$$C_n = f_n(X_n + F_{n-1}R_n).$$

By the IBE for the last period we have

$$\theta_p u'_p(X_N + F_{N-1}R_N - C_N) = \theta_N u'_N(C_N),$$

which will translate into

$$\theta_p \alpha_p \exp[-\alpha_p(X_N + F_{N-1}R_N - C_N)] = \theta_N \alpha_N \exp[-\alpha_N C_N].$$

Take the logarithm on both sides and after rearranging the items we get

$$C_N = \frac{\alpha_p}{\alpha_p + \alpha_N}(X_N + F_{N-1}R_N) + \frac{1}{\alpha_p + \alpha_N} \ln \frac{\theta_N \alpha_N}{\theta_p \alpha_p}.$$

Take the $\mathbb{Q}$-expectation and we shall have

$$\mathbb{E}^{\mathbb{Q}} C_N = v_N$$

which gives us

$$C_N = \frac{\alpha_p}{\alpha_p + \alpha_N}\left[X_N + F_{N-1}R_N\right] + (v_N - \frac{\alpha_p}{\alpha_p + \alpha_N} w_N),$$

where

$$w_N := \mathbb{E}^{\mathbb{Q}} A_N = \mathbb{E}^{\mathbb{Q}}(B_N + F_N) = v_N + v_p.$$

Next we show that for any possible n, f_n should be linear. We will show this by first showing that if f_{n+1} is linear with positive slope, then so is f_n.

By IBE

$$\theta_n u_n'(C_n) = \theta_{n+1}\mathbb{E}_n^{\mathbb{P}}\left[u_{n+1}'(C_{n+1})R_{n+1}\right]$$

we have

$$\frac{\theta_n}{R_{n+1}\theta_{n+1}} u_n'(f_n(x)) = \sum_{j\in\mathcal{J}_n^{n+1}} \mathbb{P}(j) u_{n+1}'\left[f_{n+1}(X_{n+1}^j + (x - f_n(x))R_{n+1})\right]$$

where x is the variable standing for the available assets. Assume that $f_{n+1}(x) = a_{n+1}x + e_{n+1}$ with $a_{n+1} > 0$. We have

$$\frac{\theta_n}{R_{n+1}\theta_{n+1}} \exp(-\alpha_n f_n(x)) = \sum_{j\in\mathcal{J}_n^{n+1}} \mathbb{P}(j)\exp\left\{-\alpha_{n+1}[a_{n+1}(X_{n+1}^j + (x - f_n(x))R_{n+1}) + e_{n+1}]\right\}$$

$$= \exp\{-\alpha_{n+1}a_{n+1}R_{n+1}(x - f_n)\} \cdot \left\{\sum_{j\in\mathcal{J}_n^{n+1}} \mathbb{P}(j)\exp\left[-\alpha_{n+1}(a_{n+1}X_{n+1}^j + e_{n+1})\right]\right\}$$

Take the logarithm on both sides:

$$\ln\left(\frac{\theta_n}{R_{n+1}\theta_{n+1}}\right) - \alpha_n \cdot f_n = \ln\kappa_{n+1} - \alpha_{n+1}a_{n+1}R_{n+1}(x - f_n),$$

finally

$$f_n(x) = \frac{a_{n+1}\alpha_{n+1}R_{n+1}}{\alpha_n + a_{n+1}\alpha_{n+1}R_{n+1}} x + \frac{1}{\alpha_n + a_{n+1}\alpha_{n+1}R_{n+1}} \ln\left(\frac{\theta_n}{R_{n+1}\theta_{n+1}}\frac{1}{\kappa_{n+1}}\right)$$

where

$$\kappa_{n+1} = \frac{\mathbb{E}u_{n+1}'(f_{n+1}(X_{n+1}))}{\alpha_{n+1}}.$$

It follows that all f_n should be linear with positive slope since f_N is. The slope satisfies

$$a_n = \frac{a_{n+1}\alpha_{n+1}R_{n+1}}{\alpha_n + \alpha_{n+1}a_{n+1}R_{n+1}}.$$

By recursion we know that if we start with $a_N = \frac{\alpha_p}{\alpha_p+\alpha_N}$, then all the a_n's can be calculated. Hence

$$C_n = f_n(X_n + F_{n-1}R_n) = a_n(X_n + F_{n-1}R_n) + \textit{constant}.$$

Taking the expectation under $\mathbb{Q}$ immediately gives the constant part and finally we have

$$C_n = a_n(X_n + F_{n-1}R_n) + (v_n - a_n w_n),$$

where $w_n = \mathbb{E}^{\mathbb{Q}} A_n$ can be recursively calculated according to the relationship

$$A_{n+1} = X_{n+1} + (A_n - C_n)R_{n+1}.$$

$\square$