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**CONVEX MULTI-CHOICE COOPERATIVE GAMES
AND THEIR MONOTONIC ALLOCATION SCHEMES**

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Convex multi-choice cooperative games and their monotonic allocation schemes

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Abstract

This paper focuses on new characterizations of convex multi-choice games using the notions of exactness and superadditivity. Furthermore, (level-increase) monotonic allocation schemes (limas) on the class of convex multi-choice games are introduced and studied. It turns out that each element of the Weber set of such a game is extendable to a limas, and the (total) Shapley value for multi-choice games generates a limas for each convex multi-choice game.

JEL Classification: C71.

Keywords: Multi-choice games, Convex games, Marginal games, Weber set, Monotonic allocation schemes.

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1 Introduction

Multi-choice games were introduced by Hsiao and Raghavan (1993a,b) to allow players in a cooperative environment to exert any of a finite number of activity levels suitable to the situation at stake. An extension of this model of cooperative games was introduced by Nouweland et al. (1995) to cope with situations where different players might have different sets of activity levels to participate with when cooperating with other players. Results on multi-choice games can be also found in Calvo and Santos (2000), Calvo, Gutiérrez and Santos (2000), Grabisch and Xie (2007), Klijn, Slikker and Zarzuelo (1999), Nouweland (1993), Peters and Zank (2005). Additionally, the reader can look at the survey on multi-choice cooperative games in Branzei, Dimitrov, and Tijs (2005). Our work on convex multi-choice games in this paper is based on definitions and results from Nouweland et al. (1995) and Branzei, Dimitrov and Tijs (2005), that we briefly recall in Section 2. Then, in Section 3, we give new characterizations of convex multi-choice games using the notions of exactness and superadditivity. Inspired by Sprumont (1990), we introduce (level-increase) monotonic allocation schemes (limas) for convex multi-choice games in Section 4, and prove that each element of the Weber set of a convex multi-choice game is extendable to a limas. We also show there that the (total) Shapley value of a convex multi-choice game (cf. Nouweland et al., 1995) generates a limas of the game.

2 Preliminaries on multi-choice games

Let N be a set of players, usually of the form $\{1, 2, \dots, n\}$, that consider cooperation in a multi-choice environment, i.e. each player $i \in N$ has a finite number of feasible participation levels whose set we denote by $M_i = \{0, 1, \dots, m_i\}$, where $m_i \in \mathbb{N} = \{1, 2, \dots\}$. We consider the product $\mathcal{M}^N = \prod_{i \in N} M_i$. Each element $s = (s_1, s_2, \dots, s_n) \in \mathcal{M}^N$ specifies a participation profile for players and is referred to as a multi-choice coalition. So, a multi-choice coalition indicates the participation level of each player. Then, $m = (m_1, m_2, \dots, m_n)$ is the players' maximal participation level profile that plays the role of the "grand coalition", whereas $0 = (0, 0, \dots, 0)$ plays the role of the "empty coalition". We also use the notation $M_i^+ = M_i \setminus \{0\}$ and $\mathcal{M}_+^N = \mathcal{M}^N \setminus \{0\}$. A cooperative multi-choice game is a triple $\langle N, m, v \rangle$, where $v : \mathcal{M}^N \rightarrow \mathbb{R}$ is the characteristic function with $v(0) = 0$ that specifies the

players' potential worth, $v(s)$, when they join their efforts at any activity level profile $s = (s_1, \dots, s_n)$. For $s \in \mathcal{M}^N$ we denote by (s_{-i}, k) the participation profile where all players except player i play at levels defined by s while player i plays at level $k \in M_i$. A useful particular case is $(0_{-i}, k)$, when only player i is active. We define the carrier of s by $\text{car}(s) = \{i \in N \mid s_i > 0\}$. For $s, t \in \mathcal{M}^N$ we use the notation $s \leq t$ iff $s_i \leq t_i$ for each $i \in N$ and define $s \wedge t = (\min(s_1, t_1), \dots, \min(s_n, t_n))$ and $s \vee t = (\max(s_1, t_1), \dots, \max(s_n, t_n))$. We denote the set of all multi-choice games with player set N and maximal participation profile m by $MC^{N,m}$. Often, we identify a multi-choice game $\langle N, m, v \rangle$ with its characteristic function v . For a game $v \in MC^{N,m}$ the zero-normalization of v is the game v_0 that is obtained by subtracting from v the additive game a with $a(je^i) := v(je^i)$ for all $i \in N$ and $j \in M_i^+$, where e^i is the i -th standard vector in \mathbb{R}^N . Recall that a game $v \in MC^{N,m}$ is called *additive* if the worth of each coalition s equals the sum of the worths of the players when they all work alone at their level in s , i.e. $v(s) = \sum_{i \in N} v(s_i e^i)$ for all $s \in \mathcal{M}^N$. A game $v \in MC^{N,m}$ is *zero-monotonic* if its zero-normalization is monotonic, that is $v_0(s) \leq v_0(t)$ for all $s, t \in \mathcal{M}^N$ with $s \leq t$. A game $v \in MC^{N,m}$ is called *superadditive* if $v(s \vee t) \geq v(s) + v(t)$ for all $s, t \in \mathcal{M}^N$ with $s \wedge t = 0$. A game $v \in MC^{N,m}$ is called *convex* if

$$v(s \wedge t) + v(s \vee t) \geq v(s) + v(t) \text{ for all } s, t \in \mathcal{M}^N. \quad (2.1)$$

Relation (2.1) is equivalent with

$$v(s + t) - v(s) \geq v(\bar{s} + t) - v(\bar{s}) \quad (2.2)$$

for all $s, \bar{s}, t \in \mathcal{M}^N$ satisfying $\bar{s} \leq s$, $\bar{s}_i = s_i$ for all $i \in \text{car}(t)$ and $s + t \in \mathcal{M}^N$. Clearly, a convex multi-choice game is superadditive. In the sequel, we denote the class of convex multi-choice games with player set N and maximal participation profile m by $CMC^{N,m}$. Let $v \in MC^{N,m}$. We define $M := \{(i, j) \mid i \in N, j \in M_i\}$ and $M^+ := \{(i, j) \mid i \in N, j \in M_i^+\}$. A (level) payoff vector for the game v is a function $x : M \rightarrow \mathbb{R}$, where for $i \in N$ and $j \in M_i^+$, x_{ij} denotes the payoff to player i corresponding to a change of activity level of i from $j - 1$ to j , and $x_{i0} = 0$ for all $i \in N$. One can represent a payoff vector for a game v as a $\sum_{i \in N} m_i$ -dimensional vector whose coordinates are numbered by corresponding elements of M^+ , where the first m_1 coordinates represent payoffs for successive levels of player 1, the next m_2 coordinates are payoffs for successive levels of player 2, and so on. Let x

and y be two payoff vectors for the game v . We say that x is *weakly smaller than* y if for each $s \in \mathcal{M}^N$,

$$X(s) = \sum_{i \in N} \sum_{j=0}^{s_i} x_{ij} \leq \sum_{i \in N} \sum_{j=0}^{s_i} y_{ij} =: Y(s).$$

A level payoff vector $x : M \rightarrow \mathbb{R}$ is called *efficient* if $X(m) = \sum_{i \in N} \sum_{j=1}^{m_i} x_{ij} = v(m)$, and is called *level-increase rational* if, for all $i \in N$ and $j \in M_i^+$, x_{ij} is at least the increase in worth that player i can obtain on his own (i.e. working alone) when he changes his activity level from $j - 1$ to j , that is $x_{ij} \geq v(je^i) - v((j - 1)e^i)$, or, equivalently, $x_{ij} \geq v(0_{-i}, j) - v(0_{-i}, j - 1)$. A payoff vector which is both efficient and level-increase rational is called an *imputation*. We denote by $I(v)$ the set of imputations of $v \in MC^{N,m}$. The core $C(v)$ of a game $v \in MC^{N,m}$ consists of all $x \in I(v)$ that satisfy $X(s) \geq v(s)$ for all $s \in \mathcal{M}^N$, i.e.

$$C(v) = \{x \in I(v) \mid X(s) \geq v(s) \text{ for each } s \in \mathcal{M}^N\}.$$

A game whose core is nonempty is called a *balanced game*. The set $C_{\min}(v)$ of minimal core elements of v is defined as

$$\{x \in C(v) \mid \nexists y \in C(v) \text{ s.t. } y \neq x \text{ and } y \text{ is weakly smaller than } x\}.$$

Two important solution concepts for multi-choice games, namely the Shapley value (cf. Nouweland et al. (1995)) and the Weber set (cf. Nouweland et al. (1995)), are based on marginal payoff vectors which are defined by using admissible orderings. Let $v \in MC^{N,m}$. An *admissible ordering (for v)* is a

bijection $\sigma : M^+ \rightarrow \left\{1, \dots, \sum_{i \in N} m_i\right\}$ satisfying $\sigma((i, j)) < \sigma((i, j + 1))$ for all $i \in N$ and $j \in \{1, \dots, m_i - 1\}$. The number of admissible orderings for

v is $\left(\sum_{i \in N} m_i\right)! / \prod_{i \in N} (m_i!)$; we denote the set of all admissible orderings for a

game v by $\Xi(v)$. Now, let $\sigma \in \Xi(v)$ and $k \in \left\{1, \dots, \sum_{i \in N} m_i\right\}$. Denote by $s^{\sigma, k}$ the coalition defined by

$$s_i^{\sigma, k} := \max(\{j \in M_i \mid \sigma((i, j)) \leq k\} \cup \{0\})$$

for all $i \in N$. The coalition $s^{\sigma,k}$ is the participation profile reached after k steps according to the ordering σ . The *marginal vector* $w^{\sigma,v} : M \rightarrow \mathbb{R}$ of v corresponding to σ is defined by

$$w_{ij}^{\sigma,v} := v(s^{\sigma,\sigma((i,j))}) - v(s^{\sigma,\sigma((i,j))^{-1}}),$$

for all $i \in N$ and $j \in M_i^+$. In general, the marginal vectors $w^{\sigma,v}$, $\sigma \in \Xi(v)$, of a multi-choice game v are not necessarily imputations, but for zero-monotonic games they are. For multi-choice games, several different Shapley-like values are known. The Shapley value $\Phi(v)$ of $v \in MC^{N,m}$ is (cf. Nouweland et al. (1995)) the average of all marginal vectors of $w^{\sigma,v}$, in formula

$$\Phi(v) = (\Phi_{ij}(v))_{i \in N, j \in M_i^+}, \quad \Phi_{ij}(v) := \frac{\prod_{i \in N} (m_i!)}{\left(\sum_{i \in N} m_i\right)!} \sum_{\sigma \in \Xi(v)} w_{ij}^{\sigma,v}.$$

The Weber set, $W(v)$, of a multi-choice game v is the convex hull of the marginal vectors of v , i.e. $W(v) = \text{co}\{w^{\sigma,v} \mid \sigma \in \Xi(v)\}$. Basic results for convex multi-choice games that are used in this paper are: $v \in CMC^{N,m}$ iff $W(v) = \text{co}(C_{\min}(v))$ (Theorem 11.12 in Branzei, Dimitrov and Tijs (2005)), and if $v \in CMC^{N,m}$ then $W(v) \subset C(v)$ (Theorem 11.9 in Branzei, Dimitrov and Tijs¹ (2005)).

3 New characterizations of convex multi-choice games

Our aim is to extend some characterizations of traditional convex games for convex multi-choice games. Recall that a traditional cooperative game is a pair $\langle N, v \rangle$, where N is a set of players and v is a characteristic function $v : 2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$. A game $\langle N, v \rangle$ is called convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$.

We start this section by introducing the notions of exact multi-choice game, subgame of a multi-choice game, and marginal game of a multi-choice game with respect to a multi-choice coalition.

¹Grabisch and Xie (2007) proposed notions related to the core and the Weber set of a multi-choice game, and showed that in case of convexity there is still equality between that core and that Weber set of the game.

We call a balanced multi-choice game $\langle N, m, v \rangle$ an *exact game* if for each $s \in \mathcal{M}^N$ there is an $x \in C(v)$ such that $x(s) = v(s)$. Let $v \in MC^{N,m}$ and let $u \in \mathcal{M}^N$. We denote by \mathcal{M}_u^N the subset of \mathcal{M}^N consisting of multi-choice coalitions $s \leq u$. The *subgame* of v with respect to u , $\langle N, u, v_u \rangle$, is defined by $v_u(s) := v(s)$ for each $s \in \mathcal{M}_u^N$. We define the *marginal game* of v based on u (or the *u-marginal game* of v), $\langle N, m - u, v^{-u} \rangle$, by $v^{-u}(s) := v(s + u) - v(u)$ for each $s \in \mathcal{M}_{m-u}^N$.

Lemma 3.1 *Let $v \in CMC^{N,m}$ and let $u \in \mathcal{M}_+^N$. Then, $v^{-u} \in CMC^{N,m-u}$.*

Proof. Note that for $s, t \in \mathcal{M}_{m-u}^N$ we have

$$\begin{aligned} v^{-u}(s \vee t) + v^{-u}(s \wedge t) &= v((s \vee t) + u) + v((s \wedge t) + u) - 2v(u) \\ &= v((s + u) \vee (t + u)) + v((s + u) \wedge (t + u)) - 2v(u) \\ &\geq v(s + u) + v(t + u) - 2v(u) = v^{-u}(s) + v^{-u}(t), \end{aligned}$$

where the inequality follows from the convexity of $\langle N, m, v \rangle$. ▪

Remark 3.1 Since each convex game is also superadditive, we conclude from Lemma 3.1 that if $v \in CMC^{N,m}$ then all its marginal games are superadditive. The converse also holds true. This result has been independently obtained for traditional cooperative games $\langle N, v \rangle$ by Branzei, Dimitrov and Tijs (2004) and Martinez-Legaz (2006).

Theorem 3.1 *Let $v \in MC^{N,m}$ and let $u \in \mathcal{M}_+^N$. Then the following assertions are equivalent:*

- (i) *Each u-marginal game of v , v^{-u} , is superadditive;*
- (ii) *v is a convex game.*

Proof. We need still to prove that (i) \implies (ii). Suppose that v^{-u} is superadditive. Then (2.1) holds true for all $s, t \in \mathcal{M}^N$ with $s \wedge t = 0$ because $v = v^{-0}$ is superadditive.

For $s \wedge t = f \neq 0$, take $p = s - f$ and $q = t - f$. Since $\langle N, m - s \wedge t, v^{-f} \rangle$ is superadditive, we obtain

$$\begin{aligned} 0 &\leq v^{-f}(p \vee q) - v^{-f}(p) - v^{-f}(q) = \\ &= v(p \vee q + f) - v(p + f) - v(q + f) + v(f) = \\ &= v(s \vee t) - v(s) - v(t) + v(s \wedge t), \end{aligned}$$

i.e. v is convex. ▪

For a traditional cooperative game $\langle N, v \rangle$, Biswas et al. (1999) (see also Azrieli and Lehrer (2005)) proved that the game is convex if and only if each subgame $\langle S, v \rangle$, with $S \subset N$, is an exact game. In the sequel, we prove that a similar characterization holds true for multi-choice games.

Proposition 3.1 *Each convex multi-choice game v is an exact game.*

Proof. According to Theorem 11.12 in Branzei, Dimitrov and Tijs (2005), for $v \in CMC^{N,m}$, $W(v) = \text{co}(C_{\min}(v))$, implying that all marginal vectors $w^{\sigma,v}$ are core elements. Take σ such that s is one of the "intermediate coalitions". Then, $x(s) = w^{\sigma,v}(s) = v(s)$. ▪

Theorem 3.2 *Let $v \in MC^{N,m}$. Then the following assertions are equivalent:*

- (i) $\langle N, m, v \rangle$ is convex;
- (ii) $\langle N, u, v_u \rangle$ is exact for each $u \in \mathcal{M}_+^N$.

Proof. (i) \rightarrow (ii) follows from Proposition 3.1 because each subgame of a convex game is convex, and hence exact.

(ii) \rightarrow (i): Take $s, t \in \mathcal{M}^N$. Since the subgame $v_{s \vee t}$ is exact, there is $x \in C(v_{s \vee t})$ such that $x(s \wedge t) = v_{s \vee t}(s \wedge t) = v(s \wedge t)$.

Now, using $x(s \vee t) = v_{s \vee t}(s \vee t) = v(s \vee t)$, we obtain

$$v(s \vee t) + v(s \wedge t) = x(s \vee t) + x(s \wedge t) = x(s) + x(t) \geq v(s) + v(t). \quad \blacksquare$$

4 Monotonic allocation schemes for multi-choice games

Inspired by Sprumont (1990) (see also Hokari (2000), Thomson (1983,1995)) who introduced and studied the interesting notion of population monotonic allocation scheme (pmas) for traditional cooperative games, we introduce here for multi-choice games the notion of level-increase monotonic allocation scheme (limas). Recall that a pmas for a (traditional) cooperative game $\langle N, v \rangle$ is an allocation scheme $[a_{S,i}]_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ such that:

- (i) $(a_{S,i})_{i \in S} \in C(v_S)$ for each $S \in 2^N \setminus \{\emptyset\}$, where v_S is the subgame corresponding to S , i.e. $v_S : 2^S \rightarrow \mathbb{R}$ is the restriction of $v : 2^N \rightarrow \mathbb{R}$ to 2^S ;
- (ii) $a_{S,i} \leq a_{T,i}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and $i \in S$.

Let $v \in MC^{N,m}$ and let $t \in \mathcal{M}_+^N$. For $i \in N$, denote the set $\{1, 2, \dots, t_i\}$ by M_i^t . A scheme $a = [a_{ij}^t]_{i \in N, j \in M_i^t}^{t \in \mathcal{M}_+^N}$ is called a *level-increase monotonic allocation scheme (limas)* if:

- (i) $a^t \in C(v_t)$ for all $t \in \mathcal{M}_+^N$ (*stability condition*);
- (ii) $a_{ij}^s \leq a_{ij}^t$ for all $s, t \in \mathcal{M}_+^N$ with $s \leq t$, for all $i \in \text{car}(s)$ and for all $j \in M_i^s$ (*level-increase monotonicity condition*).

Remark 4.1 Note that such a limas is a defective $|\mathcal{M}_+^N| \times |M^+|$ -matrix, whose rows correspond to multi-choice coalitions and whose columns correspond to elements of M^+ arranged according with the increasing ordering for players and for each player with respect to his participation levels. In each row t there is a core element of the multi-choice subgame v_t , with "*" for all components x_{ij} , with $i \in N$ and $j \in \{t_i + 1, \dots, m_i\}$. The level-increase monotonicity condition implies that, if the scheme is used as regulator for the (level) payoff distributions in the multi-choice subgames players are paid for each one-unit level increase (weakly) more in larger coalitions than in smaller coalitions.

Remark 4.2 A necessary condition for the existence of a limas for a multi-choice game v is the existence of core elements for v_t for each $t \in \mathcal{M}_+^N$. But this is not sufficient, as in the case of traditional cooperative games which can be seen as multi-choice games where each player has exactly two participation levels. A sufficient condition is the convexity of the game as we see in Theorem 4.1.

Let $v \in MC^{N,m}$ and $x \in W(v)$. Then we call x *limas extendable* if there exists a limas $[a_{ij}^t]_{i \in N, j \in M_i^t}^{t \in \mathcal{M}_+^N}$ such that $a_{ij}^m = x_{ij}$ for each $i \in N$ and $j \in M_i^+$.

In the next theorem we show that convex multi-choice games have a limas. Specifically, we prove that each Weber set element is limas extendable. In the proof, restrictions of $\sigma \in \Xi(v)$ to subgames v_t of v will play a role. It

will be useful to look at such σ as being a sequence of flags f^i , $i \in N$, signaling the players' turns to one-unit level increase according with their sets of participation levels. Then, for each $t \in \mathcal{M}_+^N$, the restriction of σ to t , denoted here by σ_t , can be obtained from the sequence of flags of σ by "removing" (notation "*"") for each player $i \in N$ exactly $m_i - t_i$ flags f^i starting from the back of that sequence. We illustrate this procedure in Example 4.1.

Example 4.1 Consider the multi-choice game $\langle N, m, v \rangle$ with $N = \{1, 2, 3\}$, $m = (2, 1, 2)$ and v a supermodular function. Consider $\sigma^1 \in \Xi(v)$ expressed in terms of flags as $\sigma^1 = (f^3, f^1, f^3, f^2, f^1)$. Note that this ordering generates the following maximal chain of multi-choice coalitions in \mathcal{M}^N :

$$(0, 0, 0) \xrightarrow{f^3} (0, 0, 1) \xrightarrow{f^1} (1, 0, 1) \xrightarrow{f^3} (1, 0, 2) \xrightarrow{f^2} (1, 1, 2) \xrightarrow{f^1} (2, 1, 2).$$

Now, consider the multi-choice coalition $t = (1, 1, 1)$ and the corresponding subgame $\langle N, t, v_t \rangle$. Then, the restriction of σ^1 to t is the ordering σ_t^1 which can be expressed in terms of flags as $(f^3, f^1, *, f^2, *)$; it generates the following maximal chain of multi-choice coalitions in \mathcal{M}_t^N :

$$(0, 0, 0) \xrightarrow{f^3} (0, 0, 1) \xrightarrow{f^1} (1, 0, 1) \xrightarrow{f^2} (1, 1, 1).$$

Theorem 4.1 *Let $v \in CMC^{N,m}$ and let $x \in W(v)$. Then x is limas extendable.*

Proof. Since x is in the convex hull of the marginal vectors $w^{\sigma,v}$ of v , it suffices to prove that each marginal vector $w^{\sigma,v}$ is limas extendable, because then the right convex combination of these limas extensions gives a limas extension of x .

Take $\sigma \in \Xi(v)$ and define $[a_{ij}^t]_{i \in N, j \in M_i^t}^{t \in \mathcal{M}_+^N}$ by $a_{ij}^t := w_{ij}^{\sigma_t, v_t}$ for each $t \in \mathcal{M}_+^N$, $i \in N$ and $j \in M_i^t$, where σ_t is the restriction of σ to t (obtained via the procedure described above and illustrated in Example 4.1). We claim that this scheme is a limas extension of $w^{\sigma,v}$.

Clearly, $a_{ij}^m = w_{ij}^{\sigma,v}$ for each $i \in N$ and $j \in M_i^+$ since $v_m = v$. Further, each multi-choice subgame v_t , $t \in \mathcal{M}_+^N$, is a convex game, and since $m^{\sigma_t, v_t} \in W(v_t) \subset C(v_t)$ (cf. Theorem 11.9 in Branzel, Dimitrov and Tijs (2005)), it follows that $(a_{ij}^t)_{i \in N, j \in M_i^t} \in C(v_t)$. Hence, the scheme satisfies the stability condition.

To prove the participation monotonicity condition, take $s, t \in \mathcal{M}_+^N$ with $s \leq t$, $i \in \text{car}(s)$, and $j \in M_i^s \subset M_i^t$. We have to show that $a_{ij}^s \leq a_{ij}^t$. Now, $a_{ij}^s = w_{ij}^{\sigma_s, v_s} = v(u_{-i}, j) - v(u_{-i}, j - 1)$, where (u_{-i}, j) is the intermediary multi-choice coalition in the maximal chain generated by the restriction of σ to s , when player i increased his participation level from $j - 1$ to j . Similarly, $a_{ij}^t = w_{ij}^{\sigma_t, v_t} = v(\bar{u}_{-i}, j) - v(\bar{u}_{-i}, j - 1)$.

Note that, since $s \leq t$, in the maximal chain generated by σ_s the turn of i to increase his participation level from $j - 1$ to j will come not later than the same turn in the maximal chain generated by σ_t , implying that $(u_{-i}, j) \leq (\bar{u}_{-i}, j)$. Furthermore, $(\bar{u}_{-i}, j) \leq m$. Then,

$$a_{ij}^s = v(u_{-i}, j) - v(u_{-i}, j - 1) \leq v(\bar{u}_{-i}, j) - v(\bar{u}_{-i}, j - 1) = a_{ij}^t,$$

where the inequality follows from the convexity of v .

Specifically, we used relation (2.2) with $(u_{-i}, j - 1)$ in the role of \bar{s} , $(\bar{u}_{-i}, j - 1)$ in the role of s , and $(0_{-i}, 1)$ in the role of t . Hence, $[a_{ij}^t]_{i \in N, j \in M_i^t}^{t \in \mathcal{M}_+^N}$ is a limas extension of $w^{\sigma, v}$. \blacksquare

Further, the total Shapley value (cf. Nouweland et al., 1995) of a convex multi-choice game, which is the scheme $[\Phi_{ij}^t]_{i \in N, j \in M_i^t}^{t \in \mathcal{M}_+^N}$ with the Shapley value of the multi-choice subgame v_t in each row t , is a limas.

Example 4.2 Consider the convex multi-choice game $\langle N, m, v \rangle$ with $N = \{1, 2\}$, $m = (2, 1)$, $v((0, 0)) = 0$, $v((1, 0)) = 5$, $v((2, 0)) = 6$, $v((0, 1)) = 3$, $v((1, 1)) = 9$, $v((2, 1)) = 13$.

There are three orderings on $M^+ = \{(1, 1), (1, 2), (2, 1)\}$: $\sigma^1 = (f^1, f^1, f^2)$, $\sigma^2 = (f^1, f^2, f^1)$ and $\sigma^3 = (f^2, f^1, f^1)$. The corresponding marginal vectors $m^{\sigma^1, v}$, $m^{\sigma^2, v}$, $m^{\sigma^3, v}$ are extendable to the following level-increase monotonic schemes:

$$\begin{array}{l} (2, 1) \\ (1, 1) \\ (2, 0) \\ (0, 1) \\ (1, 0) \end{array} \begin{array}{|c|c|c|} \hline 5 & 1 & 7 \\ \hline 5 & * & 4 \\ \hline 5 & 1 & * \\ \hline * & * & 3 \\ \hline 5 & * & * \\ \hline \end{array} ; \begin{array}{|c|c|c|} \hline 5 & 4 & 4 \\ \hline 5 & * & 4 \\ \hline 5 & 1 & * \\ \hline * & * & 3 \\ \hline 5 & * & * \\ \hline \end{array} ; \begin{array}{|c|c|c|} \hline 6 & 4 & 3 \\ \hline 6 & * & 3 \\ \hline 5 & 1 & * \\ \hline * & * & 3 \\ \hline 5 & * & * \\ \hline \end{array} .$$

Then, the total Shapley value $\Phi(v)$ generates the limas

(2, 1)	16/3	3	14/3
(1, 1)	16/3	*	11/3
(2, 0)	5	1	*
(0, 1)	*	*	3
(1, 0)	5	*	*

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