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A STROLL WITH ALEXIA

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Abstract

This paper revisits the Alexia value, a recent solution concept for cooperative transferable utility games. We introduce the dual Alexia value and show that it coincides with the Alexia value for several classes of games. We demonstrate the importance of the notion of compromise stability for characterizing the Alexia value.

Keywords: Alexia value, dual Alexia value, compromise stability, bankruptcy

JEL Classification: C71

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1 Introduction

There are several ways to provide one allocation as a solution for broad classes of transferable utility (TU) games: the Shapley value (Shapley (1953)) for general TU-games, the compromise value (Tijs (1981)) for compromise admissible TU-games and the nucleolus (Schmeidler (1969)) for TU-games which allow for efficient and individually rational allocations. For all games with a non-empty core the nucleolus provides a core-element as a solution. For the Shapley value and the compromise value this does not hold in general.

A recent one-point solution concept for games with a non-empty core is the Alexia value, introduced by Tijs (2005). The Alexia value is defined as the average of all so-called 'lexinals'. Here, a lexinal is defined as the lexicographical maximum of the core, with respect to an arbitrary order on the players. In a lexinal, the amount allocated to a player is the maximum he can obtain within the core, under the restriction that the players before him in the corresponding order recursively obtain their restricted maximum. As an alternative we introduce the dual Alexia value which is defined as the average over the lexicographical minima of the core.

An important notion with respect to identifying the Alexia value turns out to be compromise stability. A game is called compromise stable if it has a non-empty core, and the core coincides with the core cover (Quant, Borm, Reijnierse, and Van Velzen (2005)). First of all it is established that for compromise stable games, the Alexia value and the dual Alexia value coincide. Secondly, we show that the Alexia value of a compromise stable game coincides with the compromise extension of the run-to-the-bank rule (O'Neill (1982)) for bankruptcy situations as introduced by Quant, Borm, Hendrickx, and Zwikker (2006). This generalizes one of the main theorems of Tijs (2005). Finally we show that for the class of strongly compromise admissible games á la Driessen (1988), which form a specific subclass of compromise stable games, the Alexia value and the nucleolus coincide.

The outline of this paper is as follows: section two recalls the definition of the Alexia value, introduces the dual Alexia value and establishes connections. An application on simple flow games is provided. Section three investigates the Alexia value for compromise stable games.

2 The Alexia value

A transferable utility game (N, v) is defined by a finite player set N and a function v on the set 2^N of all subsets of N assigning to each coalition $S \in 2^N$ a value $v(S)$, such that $v(\emptyset) = 0$. The core (Gillies (1953)) is defined as the set of those allocations of $v(N)$, for which no coalition has an incentive to split off:

$$C(v) = \Big\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \Big\}.$$

A game is called balanced if its core is non-empty.

The Alexia value was introduced as a one point solution concept for balanced games by Tijs (2005). Where the Shapley value is the average over marginal vectors, the Alexia value can be found by averaging over the lexicographic maxima of the core with respect to all possible orders on the players. An order σ is a bijective function $\sigma : \{1, ..., |N|\} \to N$. The class of all orders of N is denoted with $\Pi(N)$. Let $\sigma \in \Pi(N)$. With $\sigma(k)$ we denote the player at position $k \in \{1, ..., |N|\}$ in the order σ. For a balanced game (N, v), the lexinal $\lambda^\sigma(v) \in \mathbb{R}^N$ is defined as the lexicographic maximum with respect to σ, i.e.,

$$\lambda^\sigma_{\sigma(k)}(v) = \max\Big\{x_{\sigma(k)} \mid x \in C(v),\ \lambda^\sigma_{\sigma(l)}(v) = x_{\sigma(l)} \text{ for all } l \in \{1, .., k-1\}\Big\},$$

for all $k \in \{1, ..., |N|\}$. The lexinal is recursively defined such that every player gets the maximum he can obtain inside the core under the restriction that the players before him in the corresponding order obtain their restricted maxima.

For a balanced game (N, v), the Alexia value $\alpha(v)$ is defined as the average over the lexinals:

$$\alpha(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \lambda^\sigma(v).$$

It is readily verified that the Alexia value satisfies the basic properties of relative invariance with respect to strategic equivalence and symmetry. The following example demonstrates the Alexia value on the class of simple flow games, as introduced by Reijnierse, Maschler, Potters, and Tijs (1996).

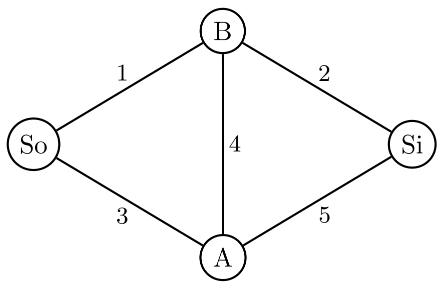

Figure 1: A simple flow network

Example 2.1. *In the simple flow network of Figure 1, all edges have capacity* 1 *and are undirected. The players want to generate a flow from the source So to the sink Si. The number next to an edge denotes the player that owns the edge: e.g. player* 3 *owns the edge* $\{So, A\}$*. The player set is* $N = \{1, ..., 5\}$*. For every coalition* $S \subseteq N$ *it can be computed how much they can transport from the* *source to the sink without using edges owned by players outside the coalition. This leads to the value $v_f(S)$ of the coalition in the flow game v_f corresponding with the simple flow network. One readily checks that the core of this game equals Conv* $\{(1,0,1,0,0),(0,1,0,0,1)\}$, *while* $\alpha(v_f) = (\frac{1}{2},\frac{1}{2},\frac{1}{2},0,\frac{1}{2})$.

The result of Example 2.1 can be generalized. By Reijnierse et al. (1996), the extreme points of the core coincide with the so-called minimum cut vectors. A minimum cut is a coalition that owns a set of edges with minimal total capacity, of which at least one is needed to generate a positive flow from source to sink. A minimum cut vector is the indicator vector of a minimum cut. As every minimum cut vector consists of the same number of ones and zeros, every minimum cut vector occurs the same number of times as lexinal. Therefore, the Alexia value equals the average over all minimal cut vectors.

Proposition 2.1. *Let v_f be a simple flow game. Then the Alexia value of v_f equals the average over the minimal cut vectors.*

The definition of the Alexia value allows for a natural modification. For a balanced game (N, v), we define the dual Alexia value $\overline{\alpha}(v)$ as the average over the lexicographical minima of the core. Formally,

$$\overline{\alpha}(v) = \frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} \overline{\lambda}^{\sigma}(v),$$

where for all $\sigma \in \Pi(N)$ the dual lexinal $\overline{\lambda}^{\sigma}(v) \in \mathbb{R}^N$ is defined by

$$\overline{\lambda}^{\sigma}_{\sigma(k)}(v) = \min\left\{x_{\sigma(k)} \mid \ x \in C(v),\ \overline{\lambda}^{\sigma}_{\sigma(l)}(v) = x_{\sigma(l)} \text{ for all } l \in \{1,...,k-1\}\right\},$$

for all $k \in \{1, ..., |N|\}$.

It turns out that for several classes of games the Alexia value and the dual Alexia value coincide. We start with the class of compromise stable games. Compromise stability uses the notion of the core cover, as introduced by Tijs and Lipperts (1982). The core cover is given by:

$$CC(v) = \left\{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N),\ m(v) \leq x \leq M(v)\right\},$$

where

$$M_i(v) = v(N) - v(N\backslash\{i\}),$$

and

$$m_i(v) = \max_{S:i\in S}\left\{v(S) - \sum_{j\in S\backslash\{i\}} M_j(v)\right\},$$

for all $i \in N$.

A game is called compromise admissible if the core cover is non-empty. For a compromise admissible game (N,v) a characterization of the core cover involves the use of larginals. For all $\sigma \in \Pi(N)$, the larginal $\ell^\sigma(v)$ is defined by

$$\ell^\sigma_{\sigma(k)}(v) = \begin{cases} M_{\sigma(k)}(v) \quad \text{if} \quad \sum_{j=1}^{k} M_{\sigma(j)}(v) + \sum_{j=k+1}^{|N|} m_{\sigma(j)}(v) \leq v(N), \\ m_{\sigma(k)}(v) \quad \text{if} \quad \sum_{j=1}^{k-1} M_{\sigma(j)}(v) + \sum_{j=k}^{|N|} m_{\sigma(j)}(v) \geq v(N), \\ v(N) - \sum_{j=1}^{k-1} M_{\sigma(j)}(v) - \sum_{j=k+1}^{|N|} m_{\sigma(j)}(v) \quad \text{otherwise}, \end{cases}$$

for all $k \in \{1, \ldots, |N|\}$.
For each compromise admissible game (N,v) the core cover coincides with the convex hull of all larginal vectors:

$$CC(v) = \mathrm{Conv}\{\ell^\sigma(v) \mid \sigma \in \Pi(N)\}.$$

A compromise admissible game (N,v) is called compromise stable if the core cover equals the core. For compromise stable games, the dual Alexia value coincides with the Alexia value.

Theorem 2.1. *If (N,v) is compromise stable, then $\overline{\alpha}(v) = \alpha(v)$.*

Proof. Let (N,v) be compromise stable. Then

$$C(v) = CC(v) \text{ and } CC(v) \neq \emptyset.$$

Take $\sigma \in \Pi(N)$. It is readily verified that $\lambda^\sigma(v) = \ell^\sigma(v)$. Moreover, for all $k \in \{1, ..., |N|\}$,

$$\begin{aligned} \overline{\lambda}^\sigma_{\sigma(k)}(v) &= \min\left\{x_{\sigma(k)} \mid x \in C(v),\ \overline{\lambda}^\sigma_{\sigma(l)}(v) = x_{\sigma(l)} \text{ for all } l \in \{1, ..., k-1\}\right\} \\ &= \min\left\{x_{\sigma(k)} \mid x \in C(v),\ x_{\sigma(l)} = \ell^{\overline{\sigma}}_{\overline{\sigma}(l)}(v) \text{ for all } l \in \{1, ..., k-1\}\right\} \\ &= \ell^{\overline{\sigma}}_{\overline{\sigma}(k)}(v), \end{aligned}$$

where $\overline{\sigma} \in \Pi(N)$ is defined by $\overline{\sigma}(l) = \sigma(|N| - l + 1)$ for all $l \in \{1, ..., |N|\}$.
This leads to

$$\overline{\alpha}(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^{\overline{\sigma}}(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^\sigma(v) = \alpha(v),$$

which concludes the proof. □

For $\sigma \in \Pi(N)$, the vector $m^\sigma(v) \in \mathbb{R}^N$ of marginal contributions with respect to σ is defined by

$$m^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \sigma(2), ..., \sigma(k)\}) - v(\{\sigma(1), \sigma(2), ..., \sigma(k-1)\}),$$

for every $k \in \{1, ..., |N|\}$. The Shapley value (Shapley (1953)) is defined as the average over these marginal vectors:

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v),$$

while the Weber set (Weber (1988)) is the convex hull of all marginal vectors:

$$W(v) = \text{Conv}\left\{m^\sigma(v) \mid \sigma \in \Pi(N)\right\}.$$

A game is called convex if the core and the Weber set coincide (*cf* Shapley (1971) and Ichiishi (1981)).

Theorem 2.2. *Let (N, v) be convex. Then*

(i) $\alpha(v) = \Phi(v)$ (Tijs (2005)),

(ii) $\overline{\alpha}(v) = \alpha(v)$.

Proof of (ii). Let $\sigma \in \Pi(N)$. By induction on the position $k \in \{1, ..., |N|\}$, we prove that $\overline{\lambda}^\sigma(v) = m^\sigma(v)$. Consider $k = 1$. Then

$$\begin{aligned}
\overline{\lambda}^\sigma_{\sigma(1)}(v) &= \min\left\{x_{\sigma(1)} \mid x \in C(v)\right\} \\
&= \min\left\{x_{\sigma(1)} \mid x \in W(v)\right\} \\
&= v(\sigma(1)) \\
&= m^\sigma_{\sigma(1)}(v).
\end{aligned}$$

Now consider $k \in \{2, ..., |N|\}$ and, assume that $\overline{\lambda}^\sigma_{\sigma(l)}(v) = m^\sigma_{\sigma(l)}(v)$ for $l \in \{1, ..., k-1\}$. Then

$$\sum_{l=1}^{k-1} \overline{\lambda}^\sigma_{\sigma(l)}(v) = \sum_{l=1}^{k-1} m^\sigma_{\sigma(l)}(v) = v(\{\sigma(1), ..., \sigma(k-1)\}).$$

Let $S = \{\sigma(1), ..., \sigma(k)\}$. By definition of the core, $\min\left\{\sum_{i \in S} x_i \mid \; x \in C(v)\right\} \geq v(S)$. Therefore, $\overline{\lambda}^\sigma_{\sigma(k)}(v) \geq v(S) - v(S \backslash \{\sigma(k)\}) = m^\sigma_{\sigma(k)}$. However, by convexity $C(v) = W(v)$. As $m^\sigma(v) \in W(v)$ we obtain $\overline{\lambda}^\sigma_{\sigma(k)}(v) = m^\sigma_{\sigma(k)}(v)$. We may conclude that $\overline{\lambda}^\sigma(v) = m^\sigma(v)$. This gives us

$$\overline{\alpha}(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v) = \Phi(v) = \alpha(v).$$

The last equality follows from part (i). □

3 Alexia and bankruptcy

A bankruptcy situation is defined as a pair (E, d), where $E \in \mathbb{R}_+$ is the estate which has to be divided among a set of players N. The claim vector is denoted by $d \in \mathbb{R}^N$, where $d_i \geq 0$ represents the claim of player $i \in N$. It is assumed that $E \leq \sum_{i\in N} d_i$. With a bankruptcy situation one can associate a bankruptcy game $(N, v_{E,d})$, where the value of a coalition equals the amount of the estate not claimed by the players outside the coalition: $v_{E,d}(S) = \max\Big\{0, E - \sum_{i\in N\setminus S} d_i\Big\}$ for all $S \subseteq N$. It is well known that every bankruptcy game is convex and compromise stable.

O'Neill (1982) introduces the run-to-the-bank (RTB) rule to divide the estate among the claimants. For a bankruptcy situation (E, d),

$$RTB(E, d) = \frac{1}{|N|!} \sum_{\sigma\in\Pi(N)} r^{\sigma}(E, d),$$

where for all $\sigma \in \Pi(N)$ and $k \in \{1, ..., |N|\}$,

$$r^{\sigma}_{\sigma(k)}(E, d) = \max\Big\{ \min\big\{d_{\sigma(k)}, E - \sum_{l=1}^{k-1} d_{\sigma(l)}\big\}, 0\Big\}.$$

The interpretation of the $r^{\sigma}_{\sigma(k)}$ is as follows: the players arrive at the bank in the order σ. Upon arrival, a player receives his total claim or, if there is not enough money left to satisfy his claim, the maximum amount that is available. Importantly, for every bankruptcy situation (E, d) it holds that $RTB(E, d) = \Phi(v_{E,d})$.

The compromise extension RTB* of the RTB-rule to the class of all compromise admissible games is introduced by Quant et al. (2006) and is given by

$$RTB^*(v) = m(v) + RTB\big(v(N) - \sum_{i\in N} m_i(v), M(v) - m(v)\big),$$

for each compromise admissible game (N, v).

Theorem 3.1 (Quant et al. 2006)**.** *Let (N, v) be compromise admissible. Then*

$$RTB^*(v) = \frac{1}{|N|!} \sum_{\sigma\in\Pi(N)} \ell^{\sigma}(v).$$

A balanced game (N, v) is called exact (Schmeidler (1972)) if for every coalition $S \subseteq N$ there exists an element $x \in C(v)$ such that $\sum_{i\in S} x_i = v(S)$. The exactification of an arbitrary game is the smallest exact game with the same core as the original game v:

$$v^E(S) = \min\Big\{ \sum_{i\in S} x_i \mid x \in C(v)\Big\}$$

for each $S \subseteq N$.

Since two and three person balanced games are compromise stable, as well as simplex and dual simplex games, the following theorem is a generalization of Theorem 3.1 of Tijs (2005).

Theorem 3.2. *Let (N, v) be compromise stable. Then*

(i) (N, v^E) is strategically equivalent[1] *to a bankruptcy game.*

(ii) $\alpha(v) = \Phi(v^E) = RTB^*(v)$.

Proof.
(i) Since (N, v) is compromise stable, $C(v) = CC(v)$. Hence, for all $S \subseteq N$

$$v^E(S) = \begin{cases} \sum_{i \in S} m_i(v), & \text{if } \sum_{i \in N \setminus S} M_i(v) + \sum_{i \in S} m_i(v) \geq v(N), \\ v(N) - \sum_{i \in N \setminus S} M_i(v), & \text{if } \sum_{i \in N \setminus S} M_i(v) + \sum_{i \in S} m_i(v) < v(N). \end{cases}$$

This means that (N, v^E) is strategically equivalent to the game (N, w) determined by

$$w(S) = \max\Big\{0, v(N) - \sum_{i \in N \setminus S} M_i(v) - \sum_{i \in S} m_i(v)\Big\},$$

since $v^E = w + m(v)$. Note that (N, w) is the bankruptcy game corresponding to the bankruptcy problem (E, d) with $E = v(N) - \sum_{i \in N} m_i(v)$ and $d_i = M_i(v) - m_i(v)$ for all $i \in N$.

(ii) Because (N, v^E) is strategically equivalent to a bankruptcy game, (N, v^E) is convex and compromise stable. Since $C(v) = C(v^E)$, we have $\alpha(v) = \alpha(v^E)$. So, with $E = v(N) - \sum_{i \in N} m_i(v)$ and $d_i = M_i(v) - m_i(v)$ for all $i \in N$, the proof of (i) implies

$$\begin{aligned} RTB^*(v) &= m(v) + RTB(v(N) - \sum_{i \in N} m(v), M(v) - m(v)) \\ &= m(v) + \Phi(v_{E,d}) \\ &= \Phi(v_{E,d} + m(v)) \\ &= \Phi(v^E) \\ &= \alpha(v), \end{aligned}$$

where the last equality holds by Theorem 2.2. □

The compromise value $\tau(v)$ for a compromise admissible game (N, v) is given by

$$\tau(v) = \gamma \cdot M(v) + (1 - \gamma) \cdot m(v),$$

[1] Two games (N, v) and (N, w) are strategically equivalent if there exist an additive game a and $k \in \mathbb{R}_{++}$ such that $w = k \cdot v + a$.

where $\gamma \in [0,1]$ is uniquely determined by $\sum_{i\in N} \tau_i(v) = v(N)$.
A compromise admissible game (N, v) is called strongly compromise admissible (Driessen (1988)) if for all $S \subseteq N, S \neq \emptyset$ it holds that:

$$\sum_{j\in N} M_j(v) - v(N) \leq \sum_{j\in S} M_j(v) - v(S). \tag{1}$$

From Theorem 3.1 of Quant et al. (2005) it follows that every strongly compromise admissible game is compromise stable.

Theorem 3.3. *If (N, v) is strongly compromise admissible, then*

(i) $C(v) = Conv\left(\{M(v) \cdot e^{N\backslash\{i\}} + m(v) \cdot e^{\{i\}}\}_{i\in N}\right)$[2],

(ii) $\tau(v) =$[3] $\frac{1}{|N|}[(|N|-1) \cdot M(v) + m(v)]$.

Proof of (i). Let (N, v) be strongly compromise admissible. Then

$$m_i(v) = \max_{S, i\in S}\{v(S) - \sum_{j\in S\backslash\{i\}} M_j(v)\} = v(N) - \sum_{j\in N\backslash\{i\}} M_j(v),$$

for all $i \in N$. Let $\sigma \in \Pi(N)$. Then

$$\ell^{\sigma}_{\sigma(k)}(v) = \begin{cases} M_{\sigma(k)} & \text{if } k \in \{1, ..., |N|-1\} \\ m_{\sigma(k)} & \text{if } k = |N|. \end{cases}$$

As (N, v) is compromise stable,

$$\begin{aligned} C(v) &= CC(v), \\ &= \text{Conv}\left(\{\ell^{\sigma}(v) \mid \sigma \in \Pi(N)\}\right), \\ &= \text{Conv}\left(\{M(v) \cdot e^{N\backslash\{i\}} + m(v) \cdot e^{\{i\}}\}_{i\in N}\right), \end{aligned}$$

which completes the proof.

Proof of (ii). Let (N, v) be strongly compromise admissible. By the proof of (i),

$$\sum_{j\in N\backslash\{i\}} M_j(v) + m_i(v) = v(N),$$

for all $i \in N$. Therefore, $\sum_{i\in N}(\sum_{j\in N\backslash\{i\}} M_j(v) + m_i(v)) = |N| \cdot v(N)$ and $(|N|-1)\sum_{j\in N} M_j(v) + \sum_{j\in N} m_j(v) = |N| \cdot v(N)$. Take $\gamma = \frac{|N|-1}{|N|}$. Then $\gamma \cdot \sum_{j\in N} M_j(v) + (1-\gamma) \cdot \sum_{j\in N} M_j(v) = v(N)$. Hence,

$$\tau(v) = \frac{1}{|N|}[(|N|-1) \cdot M(v) + m(v)],$$

which proves part (ii). □

[2] e^S denotes the indicator vector for coalition $S \subseteq N$: $e^S_i = 1$ if $i \in S$, $e^S_i = 0$ if $i \in N\backslash S$.
[3] By Driessen (1988) the τ-value equals the nucleolus for strongly compromise admissible games.

For strongly compromise admissible games, the Alexia value coincides with the compromise value.

Theorem 3.4. *If* (N, v) *is strongly compromise admissible, then* $\alpha(v) = \tau(v)$.

Proof. Let (N, v) be strongly compromise admissible. By part (i) of Theorem 3.3, $C(v) = \text{Conv}\left(\{M(v) \cdot e^{N\setminus\{i\}} + m(v) \cdot e^{\{i\}}\}_{i\in N}\right)$. Let $\sigma \in \Pi(N)$. Then $\lambda^{\sigma}_{\sigma(k)}(v) = M_{\sigma(k)}(v)$ for all $k < |N|$, and $\lambda^{\sigma}_{\sigma(|N|)}(v) = m_{\sigma(|N|)}(v)$. Therefore, $\alpha(v) = \frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} \lambda^{\sigma}(v) = \frac{1}{|N|!}(|N|! - (|N|-1)!) \cdot M(v) + (|N|-1)! \cdot m(v)] = \tau(v)$, where the last equality follows from Theorem 3.3. □

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