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A Short Survey on Semidefinite Programming*

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Abstract

Semidefinite programming (SDP) is one of the fastest developing branches of mathematical programming. The reason is twofold: efficient solution algorithms for SDP have come to light in the past few years, and SDP finds applications in combinatorial optimization and engineering. In this short survey we show how SDP duality theory can be used to prove classical results, and review the development of interior point algorithms for SDP.

1 Introduction

One could easily be led to believe that the field of semidefinite programming (SDP) originated in this decade. A glance at a bibliography of SDP papers indeed indicates an explosion of research effort, starting around 1991. A closer look reveals that interest in this class of problems is somewhat older, and dates back to the 1960's (see e.g. [6]). A paper on SDP from 1981 is descriptively named *Linear Programming with Matrix Variables* [11], and this apt title may be the best way to introduce the problem.

The goal is to minimize the inner product

$$\langle C, X \rangle := \text{Tr}(CX),$$

of two $n \times n$ symmetric matrices, a constant matrix C and a variable matrix X , subject to a set of constraints, where 'Tr' denotes the trace (sum of diagonal elements) of a matrix.¹ The first of the constraints are linear:

$$\langle A_i, X \rangle = b_i, \quad i = 1, \dots, m,$$

*Some authors prefer to use the term 'optimization' instead of 'programming'.

¹This inner product corresponds to the familiar Euclidean inner product of two vectors – if the columns of the two matrices C and X are stacked to form vectors $\text{vec}(X)$ and $\text{vec}(C)$, then $\text{vec}(C)^T \text{vec}(X) = \text{Tr}(CX)$.

where the A_i 's are given symmetric matrices, and the b_i 's given scalars. Up to this point, the stated problem is merely a linear programming (LP) problem with the entries of X as variables. We now add the convex, nonlinear constraint that X must be symmetric positive semidefinite, denoted by $X \succeq 0$.² The convexity follows from the convexity of the cone of positive semidefinite matrices. The problem under consideration is therefore

$$\min_X \{ \text{Tr}(CX) : \text{Tr}(A_i X) = b_i \ (i = 1, \dots, m), \ X \succeq 0 \}$$

The Lagrangian dual of our problem takes the form

$$\max_{y, S} \left\{ b^T y : \sum_{i=1}^m y_i A_i + S = C, \ S \succeq 0, \ y \in \mathbb{R}^m \right\}.$$

The *weak duality theorem* therefore implies that $\text{Tr}(CX) - b^T y \geq 0$ for all feasible solutions. Equality holds at optimality if both problems have feasible sets with nonempty interiors (*strong duality*) (see e.g. [5]). The duality theory for SDP is weaker than that of LP. Difficulties associated with general convex programming can occur if the strict feasibility condition is not met; thus a problem can be solvable although its Lagrangian dual is infeasible, or both problems can be solvable but with a positive duality gap at optimality.

SDP problems are of interest for a number of reasons, including

- SDP contains important classes of problems as special cases;
- important applications exist in combinatorial optimization and engineering;
- efficient solution strategies have emerged in the past few years (explaining the resurgence in research interest).

Each of these considerations will be discussed briefly. The discussion will be such as to limit the overlap with previous surveys. We will mainly focus on recent developments of interior point methods for SDP, and on the usefulness of SDP duality theory as a technique of proof.

An excellent survey by Vandenberghe and Boyd [58] deals with basic theory, diverse applications, and potential reduction algorithms (up to 1995). Two more recent surveys which focus more on applications of SDP in combinatorial optimization are by Alizadeh [2] and Ramana and Pardalos [48]. The former also deals with interior point methodology, whilst the latter contains surveys of geometric properties of the SDP feasible set (so-called spectrahedra), as well as complexity and duality theory.

2 Special cases of SDP

If the matrix X is restricted to be diagonal, then the requirement $X \succeq 0$ reduces to the requirement that the diagonal elements of X must be nonnegative. In other words, we once

²By definition, for symmetric X one has $X \succeq 0$ if $z^T X z \geq 0, \forall z \in \mathbb{R}^n$, or equivalently, if all eigenvalues of X are nonnegative.

again have an LP problem. Optimization problems with convex quadratic constraints are likewise special cases of SDP.³ This follows from the well-known *Shur complement* trick: if

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

for an invertible A , then the matrix

$$S_{hur} := C - B^T A^{-1} B$$

is called the Shur complement of A in X . One has

$$\begin{aligned} X \succ 0 & \text{ if and only if } A \succ 0 \text{ and } S_{hur} \succ 0 \\ \text{If } A \succ 0, & \text{ then } X \succeq 0 \text{ if and only if } S_{hur} \succeq 0. \end{aligned}$$

It follows that we can represent the quadratic constraint

$$(Az + b)^T(Az + b) - (c^T z + d) \leq 0, \quad z \in \mathbb{R}^n$$

by the semidefinite constraint

$$\begin{bmatrix} I & Az + b \\ (Az + b)^T & c^T z + d \end{bmatrix} \succeq 0.$$

In the same way, we can represent the *second order cone*, $\|x\|^2 \leq (\sum_{i=1}^n x_i)^2$, by

$$\begin{bmatrix} (\sum_{i=1}^n x_i) I & x \\ x^T & \sum_{i=1}^n x_i \end{bmatrix} \succeq 0.$$

Another nonlinear example which arises frequently is

$$\min \left\{ \frac{(c^T x)^2}{d^T x} : Ax \geq b \right\},$$

where it is known that $d^T x > 0$ if $Ax \geq b$. An equivalent SDP problem is:⁴

$$\min \left\{ t : \begin{bmatrix} t & c^T x & 0 \\ c^T x & d^T x & 0 \\ 0 & 0 & \text{diag}(Ax - b) \end{bmatrix} \succeq 0 \right\}.$$

Several problems involving matrix norm or eigenvalue minimization may be stated as SDP's. An extensive list of such problems may be found in [58]. A simple example is the classical problem of finding the largest eigenvalue $\lambda_{\max}(A)$ of a symmetric matrix A . The key observation here is that $t \geq \lambda_{\max}(A)$ if and only if $tI - A \succeq 0$. The SDP problem therefore becomes

$$\min \{ t : tI - A \succeq 0, \quad t \in \mathbb{R} \}.$$

An SDP algorithm for this problem is described in [26].

³This includes the well-known convex quadratic programming (QP) problem.

⁴We use the notation 'diag' as follows: for a matrix X , $\text{diag}(X)$ is the vector obtained by extracting the diagonal of X ; for a vector x , $\text{diag}(x)$ is the diagonal matrix with the coordinates of x as diagonal elements.

3 Applications in combinatorial optimization

General quadratic optimization problems allow SDP relaxations. The key observation is that

$$x^T Q x = \text{Tr} (Q x x^T),$$

for a given matrix Q and vector x . The rank one matrix $X = x x^T$ is positive semidefinite. We can therefore relax the condition $X := x x^T$ to $X \succeq 0$. This relaxation is originally due to Shor [53].

Combinatorial optimization problems can in turn be written as quadratic optimization problems. The condition $x_i \in \{-1, 1\}$ is equivalent to $x_i^2 = 1$, for example.

Lovász and Schrijver [35] considered the generic combinatorial problem

$$q^{\max} = \max \{x^T Q x : x_i \in \{-1, 1\} \ (\forall i)\} \quad (1)$$

and suggested the relaxation

$$\bar{q} = \max \{\text{Tr} (Q X) : \text{diag} (X) = e, X \succeq 0\}. \quad (2)$$

For this general relaxation Nesterov [42] recently proved that

$$\bar{q} - q \geq q^{\max} - q^{\min} \geq \frac{4 - \pi}{\pi} (\bar{q} - q)$$

where (q^{\min}, q^{\max}) is the range of feasible objective values in (1), and (q, \bar{q}) is the range of feasible values in the relaxation problem (2). Moreover, a random feasible solution x to (1) can be computed from the solution to the relaxation. The expected objective value of x , say $E(x)$, satisfies⁵

$$\frac{q^{\max} - E(x)}{q^{\max} - q^{\min}} < \frac{4}{7}.$$

For specific problems this bound can be improved. The showcase example is the *maximal cut problem*, i.e. the problem of finding a cut of maximal weight through a graph with weighted edges. In a pioneering article, Goemans and Williamson [21] proved that $\bar{q} \leq 1.14 q^{\max}$ in this case. They moreover devised a randomized algorithm which produces a cut with expected value greater than $0.878 q^{\max}$. Similar improvements were also reported in [21] for satisfiability problems.

The SDP relaxations are not always useful, though. Cases where the SDP relaxation is no stronger than the usual LP relaxation are reviewed in [48].

SDP offers more than just a numerical tool to generate lower and upper bounds on optimal values. It also provides a technique of proof via duality theory. We consider the classical *sandwich theorem*, and give a proof (which is new to the best of our knowledge) using strong duality theory. The theorem relates three characterizing numbers of a graph: the colouring number⁶ $\chi(G)$, the maximal clique number⁷ $\omega(G)$, and the Lovász number $\theta(G)$, which will be defined presently.

⁵The same bounds were obtained by Ye [60] for the 'box-constrained' problem where $x_i \in \{-1, 1\}$ is replaced by $-1 \leq x_i \leq 1$ in problem (1).

⁶Number of colours needed to colour all vertices so that no two adjacent vertices share the same colour.

⁷The cardinality of the maximal clique (connected subgraph).

For a graph $G = (V, E)$, a maximal clique is a subset $C \subset V$ with

$$\forall i, j \in C (i \neq j) : \{i, j\} \in E,$$

such that $|C|$ is maximal. The Lovász number $\theta(G)$ can be defined⁸ as the optimal value of the SDP relaxation (see [34, 23]):

$$\theta(G) := \max \operatorname{Tr} (ee^T X) = e^T X e \quad (3)$$

subject to

$$\left. \begin{aligned} X_{ij} &= 0, \{i, j\} \notin E (i \neq j) \\ \operatorname{Tr} (X) &= 1 \\ X &\succeq 0. \end{aligned} \right\} \quad (4)$$

The sandwich theorem states the following.

Theorem 3.1 (Lovász's Sandwich Theorem) *For any graph $G = (V, E)$ one has*

$$\omega(G) \leq \theta(G) \leq \chi(G).$$

Proof:

In order to prove the first inequality of the theorem, let x_C denote a 0-1 vector which defines a clique C of size k in G , i.e:

$$(x_C)_i = \begin{cases} 1 & \text{if } i \in C \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that the rank one matrix

$$X := \frac{1}{k} x_C x_C^T$$

is feasible in (4) with objective value

$$e^T X e = \frac{1}{k} (e^T x_C)^2 = \frac{k^2}{k} = k.$$

We therefore have $\omega(G) \leq \theta(G)$, which is the first part of the sandwich theorem. The second part is to prove $\theta(G) \leq \chi(G)$. To this end, we write down the Lagrangian dual of the SDP relaxation (4) to obtain

$$\theta(G) = \min \lambda \quad (5)$$

subject to

$$\left. \begin{aligned} Y + ee^T &\preceq \lambda I \\ Y_{ij} &= 0, \{i, j\} \in E (i \neq j) \\ Y_{ii} &= 0, i \in V. \end{aligned} \right\} \quad (6)$$

⁸Strictly speaking, the definition given here is of the Lovász number of the complement of G (nodes in the complement of G are connected if and only if they are not connected in G).

Given a colouring of G with k colours, we must construct a feasible solution for (6) with $\lambda \leq k$. Such a colouring defines a partition $V = \cup_{i=1}^k C_i$ where the C_i 's are subsets of nodes sharing the same colour. In other words, the C_i 's must be disjoint stable sets (co-cliques). Now let $\gamma_i = |C_i|$ and define

$$M_i := k(I_{\gamma_i} - J_{\gamma_i}), \quad i = 1, \dots, k,$$

where I_{γ_i} is the $(\gamma_i \times \gamma_i)$ identity matrix, and J_{γ_i} the all-one matrix of the same size.

We will show that the block diagonal matrix

$$Y = \begin{pmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_k \end{pmatrix} \quad (7)$$

is feasible in (6) if $\lambda = k$. By construction, Y satisfies the last two constraints in (6). We must still show that $Y + ee^T \preceq kI$, i.e. the largest eigenvalue of $Y + ee^T$ must be at most k .

The Raleigh-Ritz theorem states that for any symmetric matrix A , one has:

$$\lambda_{\max}(A) = \max \{x^T A x : \|x\| = 1\}. \quad (8)$$

It follows that the maximal eigenvalue of Y is given by

$$\lambda_{\max}(Y) = \max \left\{ \sum_{i=1}^k \alpha_i \lambda_{\max}(M_i), \sum_{i=1}^k \alpha_i = 1, \alpha_i \geq 0 (\forall i) \right\}. \quad (9)$$

Moreover one has $\lambda_{\max}(M_i) = k$, so that (9) yields $\lambda_{\max}(Y) = k$. The eigenvector corresponding to k is orthogonal to the all-one vector e . To see this, note that $Yx = \lambda x$ implies

$$-k(\gamma_i - 1) \sum_{j \in C_i} x_j = \lambda \sum_{j \in C_i} x_j, \quad i = 1, \dots, k,$$

so that $\sum_{j \in C_i} x_j = 0$ ($i = 1, \dots, k$) if $\lambda > 0$. In particular, $e^T x = 0$ from which it follows that k is also an eigenvalue of $Y + ee^T$. Assuming that k is not the largest eigenvalue of $Y + ee^T$, then the largest eigenvalue must have an eigenspace orthogonal to the eigenspace of k . The orthogonal complement of the eigenspace of k is spanned by the vectors

$$(x_{C_i})_j := \begin{cases} 1 & \text{if } j \in C_i \\ 0 & \text{otherwise,} \end{cases}$$

where $i = 1, \dots, k$. The maximal eigenvalue of $Y + ee^T$ can therefore be computed from (8):

$$\begin{aligned} \lambda_{\max}(Y + ee^T) &= \max_x \{x^T (Y + ee^T) x : x \in \text{span} \{x_{C_1}, \dots, x_{C_k}\}, \|x\| = 1\} \\ &= \max_{\alpha} \left\{ x^T Y x + (e^T x)^2 : x = \sum_{i=1}^k \alpha_i x_{C_i}, \sum_{i=1}^k \gamma_i \alpha_i^2 = 1 \right\} \end{aligned}$$

Substituting the expression for x , and using the construction of Y simplifies this to

$$\begin{aligned}\lambda_{\max}(Y + ee^T) &= \max_{\alpha} \left\{ -k \sum_{i=1}^k \alpha_i^2 (\gamma_i^2 - \gamma_i) + \left(\sum_{i=1}^k \gamma_i \alpha_i \right)^2 : \sum_{i=1}^k \gamma_i \alpha_i^2 = 1 \right\} \\ &= k + \max_{\alpha} \left\{ -k \sum_{i=1}^k (\alpha_i \gamma_i)^2 + \left(\sum_{i=1}^k \gamma_i \alpha_i \right)^2 : \sum_{i=1}^k \gamma_i \alpha_i^2 = 1 \right\}.\end{aligned}$$

The expression in brackets is nonpositive, since it is of the form

$$-kz^T z + (e^T z)^2 \leq -kz^T z + (\|e\| \|z\|)^2 = -kz^T z + k\|z\|^2 = 0,$$

where $z_i = \alpha_i \gamma_i$, ($i = 1, \dots, k$). This leads to the contradiction $\lambda_{\max}(Y + ee^T) \leq k$.

We conclude that $\lambda_{\max}(Y + ee^T) = k$, as required. \square

Moreover, we have given a proof of the equivalence of two different definitions of $\theta(G)$ via (3) and (5).⁹

4 Engineering applications

The richest field of application of SDP is currently *system and control theory*. The standard reference for these problems is Boyd et al. [10]. Introductory examples are given in [58] and [45].

An application which receives less attention is *structural design*, where the best known SDP problem involves optimal truss¹⁰ design. Two variants are:

1. minimize the weight of the structure such that its fundamental frequency remains above a critical value;
2. minimize the worst-case compliance ('stored energy') of the truss given a set of forces which the structure has to withstand.

The second of these problems allows another nice application of SDP duality theory. The problem may be stated as

Displacement formulation

$$\min_{t, x_1, \dots, x_k} \max_j x_j^T f_j, \quad j = 1, \dots, k$$

subject to

$$\begin{aligned}\left(\sum_{i=1}^m t_i b_i b_i^T \right) x_j &= f_j, \quad j = 1, \dots, k \\ \sum_{i=1}^m t_i &= V, \quad t \geq 0,\end{aligned}$$

⁹These and other equivalent definitions of $\theta(G)$ are discussed in [23].

¹⁰A truss is here defined as a structure of bars which connect a fixed ground structure of nodes. The design is fixed once the sizes of the bars have been decided.

where the t_i 's are the bar volumes (design variables), and the f_j 's are the set of forces which the truss has to withstand. The displacement of the nodes subject to force f_j is given by the vector x_j . The fixed vectors b_i depend only on the layout of the nodes and on the material properties (Young moduli) of the bars. The first constraint requires equilibrium of the structure and the second fixes its total volume. The objective is to minimize the worst-case compliance.

The name 'displacement' formulation stems from the displacement variables x_j . From engineering considerations, the problem may also be stated by using the forces in the bars as variables.

Bar forces formulation

$$\min_{\beta, t} \max_{j=1, \dots, k} \sum_{i=1}^m \frac{\beta_{ij}^2}{t_i}$$

subject to

$$\begin{aligned} \sum_{i=1}^m t_i &= V \\ f_j &= \sum_{i=1}^m \beta_{ij} b_i, \quad j = 1, \dots, k \\ t_i &\geq 0, \quad i = 1, \dots, m \\ \beta_{ij} &= 0 \quad \text{if } t_i = 0, \quad i = 1, \dots, m, \quad j = 1, \dots, k, \end{aligned}$$

where β_{ij} is proportional to the reaction force in bar i due to f_j . The second constraint simply requires static equilibrium, i.e. a 'balance of forces'.

From a purely mathematical point of view it is far from obvious that the two formulations are equivalent. This equivalence can be shown using SDP duality. We will sketch the proof here. Using the Shur complement trick, the displacement formulation can be written as an SDP problem (for details, see [12]).

SDP reformulation of the displacement formulation

$$\min \tau$$

subject to

$$\begin{aligned} \begin{bmatrix} \tau & f_j^T \\ f_j & \left(\sum_{i=1}^m t_i b_i b_i^T \right) \end{bmatrix} &\succeq 0, \quad j = 1, \dots, k \\ \sum_{i=1}^m t_i &= V, \quad t \geq 0. \end{aligned}$$

The equivalence proof is now done in three steps:

- 1: write down the dual of the SDP reformulation and simplify it;
- 2: obtain the dual of the resulting problem from Step 1;
- 3: reduce the problem obtained in Step 2 to the 'bar forces' formulation.

This sequence of steps is described detail in [41]. A survey of these and related formulations is given in [12], with emphasis on SDP formulations. Other structural design problems which may be formulated as SDP's include sandwich plate design [8], optimization of variable thickness sheets [51], and minimal compliance design with optimized materials [49]. Good reviews of interior point methods in truss topology design are Bendsøe et al. [9], and Jarre et al. [28] (see also [7]).

Other engineering applications of SDP include: VLSI transistor sizing, pattern recognition using ellipsoids, and logarithmic Chebychev approximation (see [58]).

5 Efficient solution strategies

Bearing the links between LP and SDP in mind, it may come as little surprise that interior point algorithms for LP have been successfully extended to SDP.

The field of interior point methods for LP more or less started with the famous paper by Karmarkar [30] in 1984, and in the following decade more than a thousand papers appeared on this topic. Some recent review papers include [20] and [26]. Several new books on the subject have also appeared recently, including [50] and [59].

The first extension of interior point algorithms from LP to SDP was by Nesterov and Nemirovskii [43], and independently by Alizadeh [1] in 1991. Nesterov and Nemirovskii actually considered a more general class of convex optimization problems, where the nonlinearity is 'banished' to a convex cone, like $X \succeq 0$. They show that such conic optimization problems can be solved by sequential minimization techniques, where the conic constraint is discarded and a barrier term is added to the objective. Suitable barriers are called *self-concordant*. These barriers go to infinity as the boundary of the cone is approached, and can be minimized efficiently by Newton's method.¹¹ The function

$$f_{\text{bar}}(X) = -\log \det(X)$$

is such a barrier for the cone of semidefinite matrices. Using this barrier, several classes of algorithms may be formulated which have polynomial worst-case iteration bounds for the computation of ϵ -optimal solutions.

5.1 Logarithmic barrier methods

Primal log-barrier methods use Newton's method to solve a sequence of problems of the form

$$\min_X \{ \text{Tr}(CX) - \mu \log \det(X) : \text{Tr}(A_i X) = b_i \ (i = 1, \dots, m) \}$$

where the parameter μ is sequentially decreased to zero. Such algorithms were analysed by Faybusovich in [17, 18] and later by other authors in [24] and [4]. Note that the condition $X \succeq 0$ has been replaced by adding a 'barrier term' to the objective.¹² The condition $X \succeq 0$

¹¹The definition of self-concordant barriers is omitted here; a well-written introductory text is [27].

¹²This idea actually dates back to the 1960's and the work of Fiacco and McCormick [19]; the implications for complexity theory only became clear almost three decades later.

is maintained by controlling the Newton process carefully – large decreases of μ necessitate damped Newton steps (see e.g. [4]), while small updates allow full Newton steps (see e.g. [24]).

Following the trend in LP, so-called *primal-dual* methods soon became more popular. These methods minimize the duality gap

$$\text{Tr}(CX) - b^T y = \text{Tr}(XS)$$

and employ the combined primal-dual barrier function

$$f_{pd} := -(\log \det(X) + \log \det(S)) = -\log \det(XS).$$

This means that a sequence of problems of the following form are solved

$$\min_{X, y, S} \left\{ \text{Tr}(XS) - \mu \log \det(XS) : \text{Tr}(A_i X) = b_i \ (i = 1, \dots, m), \sum_{i=1}^m y_i A_i + S = C \right\}. \quad (10)$$

The first order optimality conditions for (10) are

$$\left. \begin{aligned} \text{Tr}(A_i X) &= b_i, \quad i = 1, \dots, m \\ \sum_{i=1}^m y_i A_i + S &= C \\ XS &= \mu I \\ X, S &\succ 0. \end{aligned} \right\} \quad (11)$$

This system has a unique positive definite solution pair, denoted by $X(\mu) \succ 0$ and $S(\mu) \succ 0$.¹³ Primal-dual log-barrier methods solve the system (11) approximately, followed by a reduction in μ . Ideally, the goal is to obtain primal and dual steps ΔX and ΔS , respectively, which satisfy $X + \Delta X \succeq 0$, $S + \Delta S \succeq 0$ and

$$\begin{aligned} \text{Tr}(A_i \Delta X) &= 0, \quad i = 1, \dots, m \\ \sum_{i=1}^m \Delta y_i A_i + \Delta S &= 0 \\ (X + \Delta X)(S + \Delta S) &= \mu I. \end{aligned} \quad (12)$$

The last equation is nonlinear, and primal-dual methods differ with regard to how it is linearized. Moreover, care must be taken that the solution matrices ΔX and ΔS are symmetrical. Zhang [61] suggested to replace the nonlinear equation by

$$L(\Delta X S + X \Delta S) L^{-1} + [L(\Delta X S + X \Delta S) L^{-1}]^T = 2\mu I - L(XS) L^{-1} + [L(XS) L^{-1}]^T,$$

where the matrix L determines the symmetrization strategy. Some popular choices for L are listed in Table 1. The proof of the existence and uniqueness of each of the resulting search directions was done by Shidah et al. in [52].¹⁴ Other properties (such as scale-invariance) are compared by Todd et al. in [56].

¹³These solutions give a parametric representation of a smooth curve, called the *central path*, which tends to the *analytic center* of the primal-dual optimal sets as $\mu \rightarrow 0$. This was proved by Goldfarb and Scheinberg in [22].

¹⁴For $L = I$ uniqueness is not always guaranteed; a sufficient condition for uniqueness is $XS + SX \succeq 0$.

L	Reference
$\left[X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}}\right]^{\frac{1}{2}}$	Nesterov and Todd [44];
$X^{-\frac{1}{2}}$	Monteiro [38], Kojima et al. [33];
$S^{\frac{1}{2}}$	Monteiro [38], Helmberg et al. [25], Kojima et al. [33];
I	Alizadeh, Haeberley and Overton [3];

Table 1: Choices for the linearization matrix L .

The conspicuous entry $L = \left[X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}}\right]^{\frac{1}{2}}$ in Table 1 warrants some comment. Nesterov and Todd [44] showed¹⁵ that for each pair $X \succ 0$, $S \succ 0$ there exists a matrix D such that

$$f''_{\text{bar}}(D)X = S.$$

It can be shown that $f''_{\text{bar}}(D)$ is the linear operator which satisfies $f''_{\text{bar}}(D) : X \mapsto D^{-1}XD^{-1}$. It follows that $X = DSD$, from which it easily follows that $D = L^2$. In this way we obtain the *symmetric primal-dual scaling* $L^{-1}XL^{-1} = LSL$. This symmetry explains the usefulness of D in symmetrization.

Algorithms differ in how μ is updated, and how the symmetrized equations are solved. Methods which use large reductions of μ followed by several damped Newton steps are called long step (or large update) methods. These are analysed in [29], [38], and [55].

Methods which use dynamic updates of μ include the popular *predictor-corrector* methods. References include [3, 31, 32, 46, 54]. Other dynamic μ -updates are described in [14]. Superlinear convergence properties of predictor-corrector schemes are studied in [31, 37].

5.2 Primal-dual potential reduction methods

These algorithms are based on the potential function

$$\phi(X, S) = (n + \nu\sqrt{n})\text{Tr}(XS) - \log \det(XS) - n \log n,$$

where $\nu \geq 1$. In order to obtain a polynomial complexity bound it is sufficient to show that ϕ can be reduced by an absolute constant at each iteration [57]. A survey of algorithms which achieve such a reduction is given in [58].

¹⁵This result was proved in the more general setting of optimization problems where the variable is restricted to a self-dual cone which allows a special type of self-concordant barrier, namely self-scaled barriers. The interested reader is referred to [44].

5.3 Affine-scaling methods

The primal affine-scaling direction for SDP minimizes the primal objective over an ellipsoid which is inscribed in the primal feasible region. Surprisingly, Muramatsu [39] has shown that an algorithm using this search direction may converge to a non-optimal point, regardless of which step length is used. This is in sharp contrast to the LP case, and shows that extension of algorithms from LP to SDP cannot always be taken for granted.

Two primal-dual variants of the affine scaling methods were extended by De Klerk et al. in [15] from LP to SDP. These algorithms minimize the duality gap over ellipsoids in the scaled primal-dual space, where the matrix $L = D^{\frac{1}{2}}$ is used for the scaling. The primal-dual method fails if either of the scalings $L = X^{\frac{1}{2}}$ or $L = S^{\frac{1}{2}}$ from Table 1 is used [40].

5.4 Infeasible start methods

Several infeasible start algorithms have been suggested. A review of traditional big-M initialization strategies may be found in [58]. One of the first infeasible-start predictor-corrector algorithms was by Potra and Sheng [46]. Other references include [31, 37].

The idea of embedding the SDP problem in a self-dual problem with known feasible starting point was investigated for SDP in [13] and [36]. A solution of the self-dual embedding gives information about the solution of the original problem. This analysis was extended in [16] to include pathological cases caused by the weaker duality theory of SDP (as compared to LP). In the latter case the stronger ELSD (extended Lagrange-Slater) dual problem is used in the embedding. These duals have better properties than the usual Lagrangean duals, and were formulated by Ramana [47].

6 Further information

An up-to-date list of publications dealing with SDP may be found in the *semidefinite programming homepage*, maintained by Christolph Helmberg. The address is

<http://www.zib-berlin.de/~bzfhelmb/semidef.html>

Available SDP software can also be accessed via this address.

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