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Halická, M.; de Klerk, E.; Roos, C.

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ON THE CONVERGENCE OF THE CENTRAL PATH IN SEMIDEFINITE OPTIMIZATION*

M. HALICKÁ†, E. DE KLERK‡, AND C. ROOS§

Abstract. The central path in linear optimization always converges to the analytic center of the optimal set. This result was extended to semidefinite optimization in [D. Goldfarb and K. Scheinberg, *SIAM J. Optim.*, 8 (1998), pp. 871–886]. In this paper we show that this latter result is not correct in the absence of strict complementarity. We provide a counterexample, where the central path converges to a different optimal solution. This unexpected result raises many questions. We also give a short proof that the central path always converges in semidefinite optimization by using ideas from algebraic geometry.

Key words. semidefinite optimization, linear optimization, interior point method, central path, analytic center

AMS subject classifications. 90C51, 90C22

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1. Introduction.

The central path is of fundamental importance in the study of interior point algorithms. The geometric view of the central path is that of an analytic curve which converges to an optimal solution. Most interior point methods "follow" the central path approximately to reach the optimal set. In this paper we will re-examine the convergence property of the central path for semidefinite optimization (SDO). We will show that the characterization of the limit point of the central path as found in [1] is not correct in the absence of strict complementarity. This negative result raises the question of whether the central path always converges. Since there does not seem to be any simple proof of the convergence property in the literature, we include a complete proof as an appendix to this paper.

We first formulate SDO problems in standard form and recall the definition of the central path and some of its properties.

1.1. The central path in SDO.

By S^n we denote the space of all real symmetric $n \times n$ matrices, and for any $M, N \in S^n$ we define

$$M \bullet N = \operatorname{trace}(MN) = \sum_{i,j} m_{ij} n_{ij}.$$

The convex cones of symmetric positive semidefinite matrices and positive definite matrices will be denoted by S^n_+ and S^n_{++}, respectively; $X \succeq 0$ and $X \succ 0$ mean that a symmetric matrix X is positive semidefinite and positive definite, respectively.

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†Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynska dolina, 842 48 Bratislava, Slovakia (halicka@fmph.uniba.sk). The research of this author was supported in part by VEGA grant 1/7675/20 from the Slovak Scientific Grant Agency.

‡Faculty of Information Technology and Systems, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands (E.deKlerk@ITS.TUDelft.nl).

§Department TWI/SSOR, Mekelweg 4, Delft University of Technology, P.O. Box 5031, 2628 CD Delft, The Netherlands (C.Roos@ITS.TUDelft.nl).

We will consider the following primal-dual pair of semidefinite programs in the standard form:

$$\text{(P)} \qquad \min_{X \in S^n} \left\{ C \bullet X \;:\; A_i \bullet X = b_i \ (i = 1, \ldots, m) \ X \succeq 0 \right\},$$

$$\text{(D)} \qquad \max_{y \in \mathbb{R}^m, S \in S^n} \left\{ b^T y \;:\; \sum_{i=1}^{m} A_i y_i + S = C, \ S \succeq 0 \right\},$$

where $A_i \in S^n$ $(i = 1, \ldots, m)$ and $C \in S^n$, $b \in \mathbb{R}^m$. We assume that A_i $(i = 1, \ldots, m)$ are linearly independent. The solutions X and (y, S) will be referred to as feasible solutions if they satisfy the primal and dual constraints, respectively.

We assume that both (P) and (D) satisfy the interior point condition; i.e., there exists (X^0, S^0, y^0) such that

$$A_i \bullet X^0 = b_i \ (i = 1, \ldots, m), \ X^0 \succ 0, \quad \text{and} \quad \sum_{i=1}^{m} A_i y_i^0 + S^0 = C, \ S^0 \succ 0.$$

The primal and dual feasible sets will be denoted by $\mathcal{P}$ and $\mathcal{D}$, respectively, and $\mathcal{P}^*$ and $\mathcal{D}^*$ will denote the respective optimal sets. It is well known that under our assumptions both $\mathcal{P}^*$ and $\mathcal{D}^*$ are nonempty and bounded. The optimality conditions for (P) and (D) are

$$\text{(1)} \qquad \begin{array}{rcl} A_i \bullet X &=& b_i, \quad X \succeq 0 \quad (i = 1, \ldots, m), \\ \sum_{i=1}^{m} A_i y_i + S &=& C, \quad S \succeq 0, \\ XS &=& 0. \end{array}$$

A strictly complementary solution can be defined as an optimal solution pair (X, S) satisfying the rank condition: rank X + rank $S = n$. Contrary to linear optimization (LO), for SDO the existence of the strictly complementary solution is not generally ensured.

We now relax the optimality conditions (1) to

$$\text{(2)} \qquad \begin{array}{rcl} A_i \bullet X &=& b_i, \quad X \succeq 0 \quad (i = 1, \ldots, m), \\ \sum_{i=1}^{m} A_i y_i + S &=& C, \quad S \succeq 0, \\ XS &=& \mu I, \end{array}$$

where I is the identity matrix and $\mu \geq 0$. It is easy to see that for $\mu = 0$ (2) gives (1), and hence it may have more than one solution. On the other hand, it is well known that for $\mu > 0$ system (2) has a unique solution, denoted by $(X(\mu), S(\mu), y(\mu))$ (see, e.g., [6]). As for LO, this solution is seen as the parametric representation of an analytic curve (the *central path*) in terms of the parameter $\mu > 0$.

It has been shown that the central path for SDO shares many properties with the central path for LO. First, the basic property was established that the central path restricted to $0 < \mu \leq \bar{\mu}$ for some $\bar{\mu} > 0$ is bounded, and thus it has limit points as $\mu \downarrow 0$ in the optimal set [9], [5]. Then it was shown that the limit points are in the relative interior of the optimal set [5], [1]. Finally, it was claimed by Goldfarb and Scheinberg [1] that the central path converges for $\mu \downarrow 0$ to the so-called analytic center of the optimal solution set. Although this result has been widely cited in the recent literature, we will show in this paper that it is not correct in the absence of strict complementarity. Let us mention that the correct proofs of this fact—however, only under the assumption of strict complementarity—were given in [9] and later in [4].

Since the central path does not converge to the analytic center in general, it is natural to ask whether it always converges. The convergence property seems to be a "folkloric" result that is already mentioned on page 74 of the review paper [10] (without supplying references or a proof). In [7] the convergence of the central path for the linear complementarity problem (LCP) is proven with the help of some results from algebraic geometry. In [6], Kojima, Shindoh, and Hara mention that this proof for LCP can be extended to the monotone semidefinite complementarity problem (which is equivalent to SDO) without giving a formal proof. A more general result was shown in [2], where convergence is proven for a class of convex SDO problems that includes SDO.

We include a complete convergence proof in an appendix to this paper, which also uses some ideas from the theory of algebraic sets, but in a different manner from [7]. It is also much shorter, and requires fewer auxiliary results, than the proof in [2].

1.2. Analytic center of the optimal solution set. A pair of optimal solutions $(X, S) \in \mathcal{P}^* \times \mathcal{D}^*$ is called a *maximally complementary solution pair* to the pair of problems (P) and (D) if it maximizes rank (X)+rank (S) over all optimal solution pairs. The set of maximally complementary solutions coincides with the relative interior of $(\mathcal{P}^* \times \mathcal{D}^*)$. Another characterization is as follows: $(\bar{X}, \bar{S}) \in \mathcal{P}^* \times \mathcal{D}^*$ is maximally complementary if and only if

$$\mathcal{R}(\hat{X}) \subseteq \mathcal{R}(\bar{X}) \quad \forall \hat{X} \in \mathcal{P}^*, \quad \mathcal{R}(\hat{S}) \subseteq \mathcal{R}(\bar{S}) \quad \forall \hat{S} \in \mathcal{D}^*,$$

where $\mathcal{R}$ denotes the range space. For proofs of these characterizations see [5] and [1] and the references therein.

Let $\bar{X}$ and $\bar{S}$ be a pair of maximally complementary optimal solutions. Denote

$$|B| := \text{rank } \bar{X}, \quad \text{and} \quad |N| := \text{rank } \bar{S}.$$

Obviously, $|B| + |N| \leq n$. Without loss of generality (applying an orthonormal transformation of problem data, if necessary) we can assume that

$$\bar{X} = \begin{bmatrix} \bar{X}^B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{S} = \begin{bmatrix} \bar{0} & 0 & 0 \\ 0 & \bar{S}^N & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\bar{X}^B \in S_{++}^{|B|}$ and $\bar{S}^N \in S_{++}^{|N|}$. Therefore, each optimal solution pair $(\hat{X}, \hat{S})$ is of the form

$$\hat{X} = \begin{bmatrix} \hat{X}^B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} \hat{0} & 0 & 0 \\ 0 & \hat{S}^N & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\hat{X}^B \in S_{+}^{|B|}$ and $\hat{S}^N \in S_{+}^{|N|}$, since $\mathcal{R}(\hat{X}) \subseteq \mathcal{R}(\bar{X})$ and $\mathcal{R}(\hat{S}) \subseteq \mathcal{R}(\bar{S})$.

In what follows we consider the partition of any $M \in S^n$ corresponding to the above optimal partition so that

$$M = \begin{bmatrix} M^B & M^{BN} & M^{BT} \\ M^{NB} & M^N & M^{NT} \\ M^{TB} & M^{TN} & M^T \end{bmatrix}.$$

We denote by $\mathcal{I} = \{B, BN, BT, NB, N, NT, TB, TN, T\}$ the index set corresponding to the optimal partition. If we refer to all the blocks of M except M^B, we will write M^i $(i \in \mathcal{I} - B)$.

Now, the optimal solutions sets can be characterized by using the block partition:

$$\mathcal{P}^* = \left\{ X \;:\; A_i^B \bullet X^B = b_i \;(\; i = 1, \ldots, n),\; X^B \in S_+^{|B|},\; X^k = 0 \;(k \in \mathcal{I} - B) \right\},$$

$$\mathcal{D}^* = \left\{ (S, y) \;:\; \sum_{i=1}^m A_i^N y_i + S^N = C^N,\; S^N \in S_+^{|N|}, \right.$$
$$\left. \sum_{i=1}^m A_i^k y_i = C^k,\; S^k = 0 \;(k \in \mathcal{I} - N) \right\}.$$

The analytic centers of these sets are defined as follows: $X^a \in \mathcal{P}^*$ is the analytic center of $\mathcal{P}^*$ if

$$(X^a)^B = \arg \max_{X^B \in S_{++}^{|B|}} \left\{ \ln \det X^B \;:\; A_i^B \bullet X^B = b_i,\; i = 1, \ldots, m \right\},$$

and $(y^a, S^a) \in \mathcal{D}^*$ is the analytic center of $\mathcal{D}^*$ if

$$\left(y^a, (S^a)^N \right) = \arg \max_{y \in \mathbb{R}^m, S^N \in S_{++}^{|N|}} \left\{ \ln \det S^N \;: \sum_{i=1}^m A_i^N y_i + S^N = C^N, \right.$$
$$\left. \sum_{i=1}^m A_i^k y_i = C^k,\; k \in \; \mathcal{I} - N \right\}.$$

We end this section with two known results about the central path.

LEMMA 1.1 (see [5]). *Any limit point (X^*, S^*) of the central path is a maximally complementary optimal solution; i.e., it satisfies*

$$X^{*B} \succ 0 \quad \textit{and} \quad S^{*N} \succ 0.$$

LEMMA 1.2 (see, e.g., [3, Lemma 2.3.2]). *For any $\mu > 0$ the central path $X(\mu), S(\mu), y(\mu)$ is the analytic center of the level set of the duality gap*

$$\left\{ (X, S, y) \;:\; A_i \bullet X = b_i \;(i = 1, \ldots, m),\; \sum_{i=1}^m A_i y_i + S = C, \right.$$
$$\left. C \bullet X - b^T y = \mu n,\; X \in S_+^n,\; S \in S_+^n \right\}.$$

As a corollary we see that the primal μ-center $X(\mu)$ is the analytic center of the set

$$\{X \ : \ C \bullet X = C \bullet X(\mu), \quad A_i \bullet X = b_i \ (i = 1, \ldots, m), \quad X \succ 0\}.$$

We will use this observation in the next section.

The last two lemmas make it plausible that the central path converges to the analytic center of the optimal set, but in the next section we show that this is not true.

2. Counterexamples. Let $n = 4$, $m = 4$, $b = [1\ 0\ 0\ 0]^T$, and

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The primal problem (P) can be simplified to the following: Minimize x_{44} such that

$$X = \begin{bmatrix} 1 - x_{22} & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & -\frac{1}{2}x_{44} & -\frac{1}{2}x_{33} \\ x_{13} & -\frac{1}{2}x_{44} & x_{33} & 0 \\ x_{14} & -\frac{1}{2}x_{33} & 0 & x_{44} \end{bmatrix} \succeq 0.$$

The optimal set of (P) is given by all the positive semidefinite matrices of the form

$$X^* = \begin{bmatrix} 1 - x_{22} & x_{12} & 0 & 0 \\ x_{12} & x_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Solutions of the form X^* are clearly optimal, since $C \succeq 0$ and therefore $\mathrm{Tr}\,(CX) \geq 0$ $\forall X \in \mathcal{P}$.

The dual problem is to maximize y_1 such that

$$S = \begin{bmatrix} -y_1 & 0 & 0 & 0 \\ 0 & -y_1 & -y_3 & -y_2 \\ 0 & -y_3 & -y_2 & -y_4 \\ 0 & -y_2 & -y_4 & 1 - y_3 \end{bmatrix} \succeq 0.$$

Thus the dual problem has a unique optimal solution

$$y_i^* = 0 \quad (i = 1, 2, 3, 4), \quad S^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

It is also easy to see from (3) and (4) that strict complementary does *not* hold. The central path is well defined for this example, since the matrices $A_1, \ldots, A_4$ are clearly linearly independent and strictly feasible solutions exist for both the primal and the dual problem. Indeed,

$$x_{22} = \frac{1}{2}, \quad x_{33} = x_{44} = \frac{1}{4}, \quad x_{ij} = 0 \ \ (i \neq j)$$

defines a positive definite feasible solution for (P), and $y_1 = -1$, $y_2 = -\frac{1}{2}$, and $y_3 = y_4 = 0$ defines a strictly feasible solution of (D).

The analytic center of $\mathcal{P}^*$ is obviously given by

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

However, we will show that the limit point of the primal central path satisfies

$$X(\mu) \to \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{as } \mu \downarrow 0.$$

Due to the structure of feasible $S \in \mathcal{D}$ and the fact that $X(\mu) = \mu S(\mu)^{-1}$, the primal central path has the following structure:

$$X(\mu) = \begin{bmatrix} 1 - x_{22}(\mu) & 0 & 0 & 0 \\ 0 & x_{22}(\mu) & -\frac{1}{2}x_{44}(\mu) & -\frac{1}{2}x_{33}(\mu) \\ 0 & -\frac{1}{2}x_{44}(\mu) & x_{33}(\mu) & 0 \\ 0 & -\frac{1}{2}x_{33}(\mu) & 0 & x_{44}(\mu) \end{bmatrix}.$$

By Lemma 1.2, the point on the central path $X(\mu)$ is, for any $\mu > 0$, the analytic center of a level set. The level set is given by the primal feasibility and a level condition, which is $x_{44} = x_{44}(\mu) > 0$ in our case. This implies that $X(\mu)$ maximizes

$$\det \begin{bmatrix} 1 - x_{22} & 0 & 0 & 0 \\ 0 & x_{22} & -\frac{1}{2}x_{44}(\mu) & -\frac{1}{2}x_{33} \\ 0 & -\frac{1}{2}x_{44}(\mu) & x_{33} & 0 \\ 0 & -\frac{1}{2}x_{33} & 0 & x_{44}(\mu) \end{bmatrix} \tag{5}$$

under the conditions

$$x_{22} \in (0,1), \quad x_{33} > 0, \quad x_{22}x_{33}x_{44}(\mu) - \frac{x_{33}^3 + x_{44}^3(\mu)}{4} > 0.$$

Setting the gradient (with respect to x_{22} and x_{33}) of the determinant in (5) to zero, we obtain the two equations

$$x_{33}(\mu)x_{44}(\mu) - 2x_{22}(\mu)x_{33}(\mu)x_{44}(\mu) + \frac{1}{4}x_{44}(\mu)^3 + \frac{1}{4}x_{33}(\mu)^3 = 0, \tag{6}$$

$$(1 - x_{22}(\mu))\left(x_{22}(\mu)x_{44}(\mu) - \frac{3}{4}x_{33}(\mu)^2\right) = 0. \tag{7}$$

Using $x_{22}(\mu) \in (0,1)$, we deduce from (7) that

$$x_{33}(\mu) = \frac{2}{\sqrt{3}}\sqrt{x_{22}(\mu)x_{44}(\mu)}.$$

Substituting this expression in (6) and simplifying, we obtain

$$\frac{2}{\sqrt{3}}\sqrt{x_{22}(\mu)} - \frac{10}{3\sqrt{3}}x_{22}(\mu)^{3/2} + \frac{1}{4}x_{44}(\mu)^{3/2} = 0.$$

In the limit where $\mu \downarrow 0$, we have $x_{44}(\mu) \to 0$. Moreover, we can assume that $x_{22}(\mu)$ is positive in the limit, since the limit point of the central path is maximally complementary (Lemma 1.1). Denoting $\lim_{\mu\downarrow 0} x_{22}(\mu) := x_{22}(0) > 0$, we have

$$\frac{2}{\sqrt{3}}\sqrt{x_{22}(0)} - \frac{10}{3\sqrt{3}}x_{22}(0)^{3/2} = 0,$$

which implies $x_{22}(0) = 0.6$.

An example for the second order cone. The following example shows that the central path may already fail to converge to the analytic center of the optimal set in the special case of second order cone optimization.

Consider the problem of minimizing x_{12} subject to

$$\begin{bmatrix} x_{11} & x_{12} & 0 & 0 & 0 \\ x_{12} & x_{22} & 0 & 0 & 0 \\ 0 & 0 & x_{33} & x_{22} & 0 \\ 0 & 0 & x_{22} & x_{12} & 0 \\ 0 & 0 & 0 & 0 & 1-(x_{11}+x_{33}) \end{bmatrix} \succeq 0.$$

Note that this problem is equivalent to a second order cone optimization problem: the semidefiniteness constraint is on a block-diagonal matrix with all blocks of size 1×1 or 2×2; it is also easy to check that the notions of analytic center and central path coincide whether the example is viewed as an SDO or as a second order cone problem.

The optimal set is given by all matrices of the above form where $x_{12} = x_{22} = 0$, and the analytic center of the optimal set is given by the optimal solution where $x_{11} = x_{33} = \frac{1}{3}$.

Using exactly the same technique as in the previous example, one can show that the limit point for the central path is $x_{11} = 2/7$, $x_{33} = 3/7$. However, the proof is more technical for this example due to the larger number of variables, and is therefore omitted.

3. Conclusions and future work. The purpose of this paper was twofold:

- to show that the central path in SDO may converge to an optimal solution which is not the analytic center of the optimal set (in the absence of strict complementarity);
- to give a simplified yet rigorous proof that the central path always converges for SDO.

The first result raises some questions:

- Can we give a "geometrical" characterization of the limit point of the central path?
- For which subclasses of SDO problems can one guarantee convergence of the central path to the analytic center of the optimal set?

We therefore hope that the observations in this paper will lead to a renewed interest in the limiting behavior of the central path in SDO.

Appendix. Convergence proof for the central path. In this appendix we give a proof of the convergence of the central path for SDO by using a result from algebraic geometry.

DEFINITION A.1 (algebraic set). *A subset $V \in R^k$ is called an algebraic set if V is the locus of common zeros of some collection of polynomial functions on $\mathbb{R}^k$.*

LEMMA A.2 (curve selection lemma). *Let $V \subset \mathbb{R}^k$ be a real algebraic set, and let $U \subset \mathbb{R}^k$ be an open set defined by finitely many polynomial inequalities:*

$$U = \left\{x \in \mathbb{R}^k : g_1(x) > 0, \ldots, g_l(x) > 0\right\}.$$

If $U \cap V$ contains points arbitrarily close to the origin, then there exists an $\epsilon > 0$ and a real analytic curve

$$p : [0, \epsilon) \mapsto \mathbb{R}^k$$

with $p(0) = 0$ and with $p(t) \in U \cap V$ for $t > 0$.

A proof of the curve selection lemma is given in [8, Lemma 3.1, p. 25].

THEOREM A.3. *The central path in semidefinite optimization always converges.*

Proof. Let (X^*, y^*, S^*) be any limit point of the central path of (P) and (D).

With reference to Lemma A.2, let the real algebraic set V be defined via

$$V = \left\{ (\bar{X}, \bar{S}, \bar{y}, \mu) \;\middle|\; \begin{array}{rcl} A_i \bullet \bar{X} & = & 0 \quad (i = 1, \ldots, m), \\ \sum_i (\bar{y}_i) A_i + \bar{S} & = & 0, \\ (\bar{X} + X^*)(\bar{S} + S^*) - \mu I & = & 0, \end{array} \right\}$$

and let the open set U be defined as the set of all $(\bar{X}, \bar{S}, \bar{y}, \mu)$ such that all principal minors of $(\bar{X} + X^*)$ and $(\bar{S} + S^*)$ are positive and $\mu > 0$.

Now $V \cap U$ corresponds to the central path excluding its limit points, in the sense that if $(\bar{X}, \bar{S}, \bar{y}, \mu) \in U \cap V$ then $X(\mu) = (\bar{X} + X^*)$ and $S(\mu) = (\bar{S} + S^*)$, where $X(\mu)$ (respectively, $S(\mu)$) denotes the μ-center of (P) (respectively, (D)) as before.

Moreover, the zero element is in the closure of $V \cap U$, by construction.

The required result now follows from the curve selection lemma. To see this, note that Lemma A.2 implies the existence of an $\epsilon > 0$ and an analytic function $p : [0, \epsilon) \mapsto \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m \times \mathbb{R}$ such that

$$p(t) = \left(\bar{X}(t), \bar{S}(t), \bar{y}(t), \mu(t)\right) \to (0_{n\times n}, 0_{n\times n}, 0_m, 0) \text{ as } t \downarrow 0, \tag{8}$$

and if $t > 0$, $\big(\bar{X}(t), \bar{S}(t), \bar{y}(t), \mu(t)\big) \in U \cap V$, i.e.,

$$
\begin{array}{rcl}
A_i \bullet \bar{X}(t) & = & 0 \quad (i = 1, \ldots, m), \\
\sum_i \bar{y}_i(t) A_i + \bar{S}(t) & = & 0, \\
(\bar{X}(t) + X^*)(\bar{S}(t) + S^*) - \mu(t) I & = & 0,
\end{array} \tag{9}
$$

and $\bar{X}(t) \succ 0$, $\bar{S}(t) \succ 0$, $\mu(t) > 0$.

Since the centrality system (2) has a unique solution, the system (9) also has a unique solution given by

$$\bar{X}(t) + X^* = X(\mu(t)), \quad \bar{S}(t) + S^* = S(\mu(t))$$

if $t > 0$. By (8), we therefore have

$$\lim_{t \downarrow 0} X(\mu(t)) = X^*, \quad \lim_{t \downarrow 0} S(\mu(t)) = S^*, \quad \lim_{t \downarrow 0} \mu(t) = 0.$$

Since $\mu(t) > 0$ on $(0, \epsilon)$, $\mu(0) = 0$, and μ is analytic on $[0, \epsilon)$, there exists an interval, say $(0, \epsilon')$, where $\frac{d\mu(t)}{dt} > 0$. Therefore the inverse function $\mu^{-1} : \mu(t) \mapsto t$ exists on the interval $(0, \mu(\epsilon'))$. Moreover, $\mu^{-1}(t) > 0 \ \forall\, t \in (0, \mu(\epsilon'))$ and $\lim_{t \downarrow 0} \mu^{-1}(t) = 0$.

This implies that

$$\lim_{t \downarrow 0} X(t) = \lim_{t \downarrow 0} X(\mu(\mu^{-1}(t))) = \lim_{t \downarrow 0} \bar{X}(\mu^{-1}(t)) + X^* = X^*.$$

Similarly, $\lim_{t \downarrow 0} S(t) = S^*$, which completes the proof. $\square$

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REFERENCES

[1] D. Goldfarb and K. Scheinberg, *Interior point trajectories in semidefinite programming*, SIAM J. Optim., 8 (1998), pp. 871–886.

[2] L. M. Graña Drummond and Y. Peterzil, *The central path in smooth convex semidefinite programs*, Optimization, to appear.

[3] E. de Klerk, *Interior Point Methods for Semidefinite Programming*, Ph.D. thesis, Delft University of Technology, Delft, The Netherlands, 1999.

[4] E. de Klerk, C. Roos, and T. Terlaky, *Infeasible-start semidefinite programming algorithms via self dual embeddings*, in Topics in Semidefinite and Interior-Point Methods, Fields Inst. Commun. 18, P. M. Pardalos and H. Wolkowicz, eds., AMS, Providence, RI, 1998, pp. 215–236.

[5] E. de Klerk, C. Roos, and T. Terlaky, *Initialization in semidefinite programming via a self-dual, skew-symmetric embedding*, Oper. Res. Lett., 20 (1999), pp. 213–221.

[6] M. Kojima, S. Shindoh, and S. Hara, *Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices*, SIAM J. Optim., 7 (1997), pp. 86–125.

[7] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems*, Lecture Notes in Comput. Sci. 538, Springer-Verlag, Berlin, Heidelberg, 1991.

[8] J. Milnor, *Singular Points of Complex Hypersurfaces*, Ann. Math. Stud., Princeton University Press, Princeton, NJ, 1968.

[9] Z.-Q. Luo, J. F. Sturm, and S. Zhang, *Superlinear convergence of a symmetric primal-dual path following algorithm for semidefinite programming*, SIAM J. Optim., 8 (1998), pp. 59–81.

[10] L. Vandenberghe and S. Boyd, *Semidefinite programming*, SIAM Rev., 38 (1996), pp. 49–95.