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NECESSARY AND SUFFICIENT CONDITIONS FOR SOLVING COOPERATIVE DIFFERENTIAL GAMES

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Necessary and Sufficient Conditions for Solving Cooperative Differential Games.

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Abstract In this note we present as well necessary as sufficient conditions for existence of a Pareto optimum for general non-convex differential games. The obtained results are used to analyze the non-convex regular indefinite linear quadratic differential game. For the scalar case an algorithm is devised to find all Pareto efficient solutions.

Keywords: Dynamic Optimization, Pareto Efficiency, Cooperative Differential Games, LQ theory.
Jel-codes: C61, C71, C73.

1 Introduction

In this paper we consider the problem to find the set of Pareto efficient solutions in case either there is one individual who has multiple objectives, or, there is more than one person affecting the system and, in order to minimize their cost, these persons decide to coordinate their actions. The system is described by a (set of) differential equation(s) and the information structure of the game is of the open-loop type (see e.g. [4]). Every player i may choose his control trajectory, $u_i(.)$, arbitrarily from the set $\mathcal{U}$ of piecewise continuous functions[1]. Formally, the players are assumed to minimize the performance criteria:

$$J_i(t_0, u_1, u_2) := \int_{t_0}^{T} g_i(t, x(t), u_1(t), u_2(t))dt + h_i(x(T)),\ i = 1, 2, \tag{1}$$

(or in shorthand $J_i(u_1, u_2)$ if $t_0 = 0$) where $x(t)$ is the solution of the differential equation

$$\dot{x}(t) = f(t, x(t), u_1(t), u_2(t)),\ x(t_0) = x_0. \tag{2}$$

To have a well-posed problem we make the assumptions that $f(t, x, u)$ and $g_i(t, x, u)$ are continuous functions on $I\!R^{1+n+m}$; for both f and g_i all partial derivatives w.r.t. x and u exist and are continuous; and $h_i(x)$ is continuously differentiable.

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[1] see e.g. [8] (or [4, p.134]) for a generalization.

By cooperation, in general, the cost a specific player incurs is not uniquely determined anymore. We consider only solutions that are such that they can not be improved upon by all players simultaneously. The so-called Pareto efficient solutions. Formally, a set of control actions $\hat{u}$ (in the sequel: a control) is called *Pareto efficient* if the set of inequalities $J_i(u) \le J_i(\hat{u}),\ i = 1, \cdots, N$, where at least one of the inequalities is strict, does not allow for any solution $u \in \mathcal{U}$. The corresponding point $(J_1(\hat{u}), \cdots, J_N(\hat{u})) \in I\!R^N$ is called a *Pareto solution*. The set of all Pareto solutions is called the *Pareto frontier*.

In literature a well-known way to find Pareto solutions is to solve a parameterized optimal control problem (see e.g. [9], [14]). However, in general, it is unclear whether in this way one obtains all Pareto solutions. Here, in Section 2, we present for problem (1,2) necessary conditions for a control to be Pareto efficient and discuss additional conditions from which one can conclude that these necessary conditions are sufficient too. As far as we know these conditions have not been stated explicitly in the literature before. The closest reference we could track is [1] where, using geometric properties of Pareto surfaces, under different assumptions about the cost and admissible strategies also necessary conditions in the spirit of the maximum principle are derived. Our results are of this type too and resemble the corresponding results for the static optimization problem as reported, e.g., in [12]. Section 3 considers the regular, non-convex, indefinite finite-planning horizon linear quadratic differential game. The corresponding convex case was recently solved for an arbitrary planning horizon in [5]. Most results presented in Section 3 are for a finite planning horizon. An algorithm is provided to compute all Pareto efficient solutions when the system is scalar. Section 4 concludes and mentions a number of open problems.

2 Necessary and Sufficient Conditions for the General Case

In the subsequent analysis the "unit-simplex", $\mathcal{A} := \{\alpha = (\alpha_1, \cdots, \alpha_N) \mid \alpha_i \ge 0 \text{ and } \sum_{i=1}^N \alpha_i = 1\}$, plays a crucial role. Lemmas 2.1 and 2.2 provide a characterization of Pareto efficient controls. A proof of them can be copied from the finite dimensional case as considered, e.g., in [12, Chapter 22] or [6].

Lemma 2.1 *(see also [9] or [14]) Let $\alpha_i \in (0, 1)$, with $\sum_{i=1}^{N} \alpha_i = 1$. Assume $\hat{u} \in \mathcal{U}$ is such that*

$$\hat{u} \in \arg\min_{u \in \mathcal{U}} \{\sum_{i=1}^{N} \alpha_i J_i(u)\}. \tag{3}$$

Then $\hat{u}$ is Pareto efficient. □

Lemma 2.2 *$\hat{u} \in \mathcal{U}$ is Pareto efficient if and only if for all i $\hat{u}(.)$ minimizes J_i on the constrained set*

$$\mathcal{U}_i := \{u \mid J_j(u) \le J_j(\hat{u}),\ j = 1, \cdots, N,\ j \ne i\},\ \text{for } i = 1, \cdots, N. \tag{4}$$

Corollary 2.3 *If $\hat{u}[0, T] \in \mathcal{U}$ is a Pareto efficient control for $x(0) = x_0$ in (1,2), then for any $t_0 > 0$, $\hat{u}[t_0, T]$ is a Pareto efficient control for $x(t_0) = \hat{x}(t_0)$ in (1,2). Here $\hat{x}(t_0) = x(t, 0, \hat{u}[0, t_0])$ is the value of the state at time t_0 induced by $\hat{u}[0, t_0]$.*

Proof. Consider for $x(0) = x_0$ (see (4)) $\mathcal{U}_1(0) := \{u \mid J_2(u) \leq J_2(\hat{u}[0,T])\}$.
Let $t_0 > 0$. We will next show (see Lemma 2.2) that $\hat{u}[t_0,T]$ minimizes $J_1(t_0,u)$ on the constrained set

$$\tilde{\mathcal{U}_1}(t_0) := \{u \mid J_2(u) \leq J_2(t_0, \hat{u}[t_0,T]), \text{ subject to (21) with } x(t_0) = \hat{x}(t_0).$$

To that end we first note that $\hat{u}[t_0,T] \in \tilde{\mathcal{U}_1}(t_0)$.
Next we show that every element $u \in \tilde{\mathcal{U}_1}(t_0)$ can be viewed as an element $u^e \in \mathcal{U}_1(0)$ restricted to the time interval $[t_0,T]$. That is, $\forall u \in \tilde{\mathcal{U}_1}(t_0)$ there exists $u^e \in \mathcal{U}_1(0)$ such that $u^e[t_0,T] = u$. For, let $u^e[0,T]$ be the concatenation of $\hat{u}[0,t_0)$ with $u[t_0,T]$. Then, clearly, u^e is such that $x(t_0) = \hat{x}(t_0)$. Furthermore,

$$\begin{aligned} J_2(u^e) &= \int_0^T g_2(t,x(t),u^e(t))dt + h_2(x(T)) = \int_0^{t_0} g_2(t,\hat{x}(t),\hat{u}(t))dt + \int_{t_0}^T g_2(t,x(t),u(t))dt \\ &+ h_2(x(T)) \leq \int_0^{t_0} g_2(t,\hat{x}(t),\hat{u}(t))dt + \int_{t_0}^T g_2(t,\hat{x}(t),\hat{u}(t))dt + h_2(\hat{x}(T)) = J_2(\hat{u}). \end{aligned}$$

So, by definition, $u^e \in \mathcal{U}_1(0)$. From the dynamic programming principle it follows now directly that $\hat{u}[t_0,T]$ has to minimize $J_1(t_0,u)$ on $\tilde{\mathcal{U}_1}(t_0)$. In the same way one can show that $\hat{u}[t_0,T]$ also minimizes $J_2(t_0,u)$ on the corresponding constrained set $\tilde{\mathcal{U}_2}(t_0)$, which proves the claim. □

Corollary 2.4 *Assume $J_1(u)$ has a minimum which is uniquely attained at $\hat{u}$. Then $(J_1(\hat{u}), \cdots, J_N(\hat{u}))$ is a Pareto solution.*

Proof. Using the notation of Lemma 2.2 it follows directly from the fact that the minimum of $J_1(u)$ is uniquely attained at $\hat{u}$ that $\mathcal{U}_i = \{\hat{u}\}$, for $i = 2, \cdots, N$. Furthermore, it is clear that $\hat{u} \in \mathcal{U}_1$.
Consequently, $\min_{u \in \mathcal{U}_i} J_i = J_i(\hat{u})$, $i = 2, \cdots, N$ and, since $J_1(u) \geq J_1(\hat{u})$ for arbitrary u, $\min_{u \in \mathcal{U}_1} J_1(u) = J_1(\hat{u})$ too. Lemma 2.2 yields then the advertized result. □

Remark 2.5 From Lemma 2.1 it follows now that in case $J_i(u)$ has a unique minimum location for all i, then every control such that for some $\alpha_i \in [0,1]$, with $\sum_{i=1}^N \alpha_i = 1$, $\hat{u} \in \arg\min_{u \in \mathcal{U}} \{\sum_{i=1}^N \alpha_i J_i(u)\}$ is Pareto efficient. □

Theorem 2.6 *Let $T < \infty$. Assume $(J_1(\hat{u}), J_2(\hat{u}))$ is a Pareto solution for problem (1,2). Then, there exists an $\alpha \in [0,1]$, a costate function $\lambda^T(t) : [0,T] \to I\!R^n$ (which is continuous and piecewise continuously differentiable) such that, with $H(t,x,u,\lambda) := \alpha g_1(t,x,u) + (1-\alpha) g_2(t,x,u) + \lambda f(t,x,u)$, $\hat{u}$ satisfies*

$$\begin{aligned} H(t,\hat{x}(t),\hat{u}(t),\lambda(t)) &\leq H(t,\hat{x}(t),u(t),\lambda(t)), \text{ at each } t \in [0,T], && (5) \\ \dot{\lambda}(t) &= -[\alpha \frac{\partial g_1}{\partial x} + (1-\alpha)\frac{\partial g_2}{\partial x} + \lambda(t)\frac{\partial f}{\partial x}];\ \lambda(T) = \frac{\partial(\alpha h_1 + (1-\alpha)h_2)}{\partial x}, && (6) \\ \dot{\hat{x}}(t) &= f(t,\hat{x}(t),\hat{u}_1(t),\hat{u}_2(t)),\ \hat{x}(0) = x_0. && (7) \end{aligned}$$

Proof. Introduce the state variable $\dot{x}_2 := g_2(t,x,u),\ x_2(0)=0$, and $x_2^* := \int_0^T g_2(t,\hat{x},\hat{u})dt + h_2(\hat{x}(T))$, where $\hat{x}$ solves $\dot{\hat{x}} = f(t,\hat{x},\hat{u}),\ \hat{x}(0)=x_0$. From Lemma 2.2 we have then that in particular

$$\min_u \int_0^T g_1(t,x,u)dt + h_1(x(T)) \text{ such that } x_2(T) + h_2(x(T)) - x_2^* \geq 0$$

$$\text{subject to } \begin{bmatrix} \dot{x} \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f(t,x,u) \\ g_2(t,x,u) \end{bmatrix},\ \begin{bmatrix} x(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix},$$

has a solution. From the maximum principle we conclude next (see e.g. [10] and [11, p.61]) that there exist (continuous, piecewise continuously differentiable) costate functions $\lambda_i(t)$ and a constant $p_1 \in \{0,1\}$ (with $[p_1, \lambda_1(t),\ \lambda_2(t)] \neq [0,0,0]$ for all $t \in [0,T]$ **(i)**) such that the Hamiltonian $H^1 := p_1 g_1(t,x,u) + \lambda_1(t) f(t,x,u) + \lambda_2(t) g_2(t,x,u)$ is minimized at $u = \hat{u}$. Furthermore the costate variables satisfy:

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = -\begin{bmatrix} H_x^1 \\ H_{x_2}^1 \end{bmatrix};\ \begin{bmatrix} \lambda_1(T) \\ \lambda_2(T) \end{bmatrix} = \begin{bmatrix} p_1 h_1'(x(T)) + \alpha_1 h_2'(x(T)) \\ \alpha_1 \end{bmatrix}, \tag{8}$$

where $\alpha_1 \geq 0$, and $\alpha_1(x_2(T) + h_2(x(T)) - x_2^*) = 0$. Since $H_{x_2}^1 = 0$, it follows that $\lambda_2(t) = \alpha_1$, for all $t \in [0,T]$. Substitution of this into the "first-order" condition of H^1, gives then

$$p_1 g_1(t,\hat{x},\hat{u}) + \lambda_1(t) f(t,\hat{x},\hat{u}) + \alpha_1 g_2(t,\hat{x},\hat{u}) \leq p_1 g_1(t,\hat{x},u) + \lambda_1(t) f(t,\hat{x},u) + \alpha_1 g_2(t,\hat{x},u),\ t \in [0,T]. \tag{9}$$

In a similar way it follows also from Lemma 2.2 that

$$\min_u \int_0^T g_2(t,x,u)dt + h_2(x(T)) \text{ such that } x_1(T) + h_1(x(T)) - x_1^* \geq 0$$

$$\text{subject to } \begin{bmatrix} \dot{x} \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} f(t,x,u) \\ g_1(t,x,u) \end{bmatrix},\ \begin{bmatrix} x(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix},$$

has a solution, where $x_1^* := \int_0^T g_1(t,\hat{x},\hat{u})dt + h_1(\hat{x}(T))$. Analogously one obtains then from the necessary conditions the existence of an $\alpha_2 \geq 0$, a costate function $\mu_1(t)$ and a constant $p_2 \in \{0,1\}$ that satisfy:

$$\alpha_2 g_1(t,\hat{x},\hat{u}) + p_2 g_2(t,\hat{x},\hat{u}) + \mu_1(t) f(t,\hat{x},\hat{u}) \leq \alpha_2 g_1(t,\hat{x},u) + p_2 g_2(t,\hat{x},u) + \mu_1(t) f(t,\hat{x},u),\ t \in [0,T], \tag{10}$$

$$\dot{\mu}_1 = -(\alpha_2 \frac{\partial g_1}{\partial x} + p_2 \frac{\partial g_2}{\partial x} + \mu_1(t) \frac{\partial f}{\partial x});\ \mu_1(T) = \alpha_2 h_1'(x(T)) + p_2 h_2'(x(T)). \tag{11}$$

Adding (9) and (10) yields

$$\begin{aligned} &(p_1+\alpha_2) g_1(t,\hat{x},\hat{u}) + (p_2+\alpha_1) g_2(t,\hat{x},\hat{u}) + (\lambda_1(t)+\mu_1(t)) f(t,\hat{x},\hat{u}) \leq \\ &\quad (p_1+\alpha_2) g_1(t,\hat{x},u) + (p_2+\alpha_1) g_2(t,\hat{x},u) + (\lambda_1(t)+\mu_1(t)) f(t,\hat{x},u). \end{aligned} \tag{12}$$

Addition of $\lambda_1(t)$ from (8) to $\mu_1(t)$ from (11) shows that $\tilde{\lambda}(t) := \lambda_1(t) + \mu_1(t)$ satisfies

$$\dot{\tilde{\lambda}}(t) = -[(p_1+\alpha_2)\frac{\partial g_1}{\partial x} + (p_2+\alpha_1)\frac{\partial g_2}{\partial x} + \tilde{\lambda}(t)\frac{\partial f}{\partial x}];\ \tilde{\lambda}(T) = (p_1+\alpha_2)h_1' + (p_2+\alpha_1)h_2'. \tag{13}$$

Next notice that if $p_1 = \alpha_1 = 0$ it follows straightforwardly from (8) that $\lambda_1(t) = \lambda_2(t) = p_1 = 0$, which violates the maximum principle condition (i). So either p_1 or α_1 differs from zero (and similarly either p_2 or α_2 is larger than zero too). Therefore $d := p_1 + p_2 + \alpha_1 + \alpha_2 > 0$. Introducing finally $\alpha := \frac{p_1+\alpha_2}{d}$ and $\lambda(t) := \frac{\tilde{\lambda}(t)}{d}$, one obtains directly by division of (12) and (13) by d the conditions (5-7). □

Remark 2.7 The necessary conditions from Theorem 2.6 are closely related to the minimization of $\alpha J_1 + (1-\alpha)J_2$ subject to (21). By considering the Hamiltonian for this problem $H := \alpha J_1 + (1-\alpha)J_2 + \lambda f$, we obtain from the maximum principle the conditions stated in Theorem 2.6. Unfortunately the maximum principle conditions just provide necessary conditions. So, in case all conditions from Theorem 2.6 are met, we still can not conclude that this parameterized problem has a solution. □

Remark 2.8 From the proof of Theorem 2.6 it is clear that without any complications one can also consider problem (1,2) subject to inequality constraints $m_j(x(t), u_1(t), u_2(t)) \leq 0$, where m_j is continuously differentiable in all its arguments, $j = 1, \cdots, k$. By restricting the set of admissible controls to those for which these inequalities are satisfied and under the assumption that the constraint qualification[2] is met, the following necessary conditions result.
Assume $(J_1(\hat{u}), J_2(\hat{u}))$ is a Pareto solution for this problem. Then, there exists an $\alpha \in [0,1]$, a (continuous and piecewise continously differentiable) costate function $\lambda^T(t) : [0,T] \to I\!R^n$ and (continuous) nonnegative Lagrange parameters $\mu_i(t)$ such that, with the Hamiltonian H as in Theorem 2.6, $\hat{u}$ satisfies

$$\begin{aligned} H(t,\hat{x}(t),\hat{u}(t),\lambda(t)) \ &\leq \ H(t,\hat{x}(t),u(t),\lambda(t)), \text{ at each } t \in [0,T] \text{ for all} \\ &\quad\ u \text{ satisfying } m_j(\hat{x}(t),u_1(t),u_2(t)) \leq 0; \end{aligned} \tag{14}$$

$$0 \ = \ \alpha\frac{\partial g_1}{\partial u} + (1-\alpha)\frac{\partial g_2}{\partial u} + \lambda(t)\frac{\partial f}{\partial u} + \sum_{j=1}^{k}\mu_j(t)\frac{\partial m_j}{\partial u}; \tag{15}$$

$$\dot{\lambda}(t) = -[\alpha\frac{\partial g_1}{\partial x} + (1-\alpha)\frac{\partial g_2}{\partial x} + \lambda(t)\frac{\partial f}{\partial x} + \sum_{j=1}^{k}\mu_j(t)\frac{\partial m_j}{\partial x}],\ \lambda(T) = \frac{\partial(\alpha h_1 + (1-\alpha)h_2)}{\partial x}; \tag{16}$$

$$\dot{\hat{x}}(t) \ = \ f(t,\hat{x}(t),\hat{u}_1(t),\hat{u}_2(t)),\ x(0) = x_0. \tag{17}$$

$$\mu_j(t) \ \geq \ 0;\ \mu_j(t)m_j(\hat{x}(t),\hat{u}_1(t),\hat{u}_2(t)) = 0;\ m_j(\hat{x}(t),\hat{u}_1(t),\hat{u}_2(t)) \leq 0. \tag{18}$$

□

[2] The most convenient constraint qualification is the following rank condition. If p of the inequalities are satisfied with equality then the matrix of partial derivatives of these p constraints w.r.t. $u(t)$ must have rank p.

Next we derive some sufficient conditions for a control function to be Pareto efficient. The first well-known result, proved by [7], states that under convexity assumptions on the performance functions one can derive all Pareto efficient controls from the minimization of a parameterized optimal control problem. This property was used in [5] to obtain for a specific class of linear quadratic differential games both necessary and sufficient conditions for existence of Pareto efficient controls.

Theorem 2.9 *If $\mathcal{U}$ is convex and $J_i(u)$ is convex for all $i = 1, \cdots, N$, then for all Pareto efficient $\hat{u}$ there exist $\alpha \in \mathcal{A}$, such that $\hat{u} \in \arg\min_{u \in \mathcal{U}}\{\sum_{i=1}^N \alpha_i J_i(u)\}$.* $\square$

Theorem 2.10 gives sufficient conditions under which one can conclude from a solution of (5-7) that it will be Pareto efficient. The conditions and proof are inspired by Arrow's theorem. Since a formal proof is along the lines of that of Arrow's theorem we skip the proof here. A detailed proof can be found in [6].

Theorem 2.10 *Let $T < \infty$. Assume there exist an $\alpha \in (0,1)$, a costate function $\lambda^{*^T}(t) : [0,T] \to \mathbb{R}^n$, u^* and x^* that satisfy (5-7). Introduce the Hamiltonian $H(t,x,u,\lambda^*) := \alpha g_1 + (1-\alpha)g_2 + \lambda^* f$. Assume that $H(t,x,u,\lambda^*)$ has a minimum w.r.t. u for all x. Let $H^0(t,x,\lambda^*) := \min_u H(t,x,u,\lambda^*)$. Then, if both $H^o(t,x,\lambda^*)$ and $h(x) := \alpha h_1(x) + (1-\alpha)h_2(x)$ are convex in x, u^* is Pareto efficient.* $\square$

Remark 2.11 For the constrained problem considered in Remark 2.8 the conditions (15-18) are also sufficient to conclude that $\hat{u}$ is Pareto efficient if $h(x)$ is convex and either i) $H^0(x,\lambda,t) := \min_{\{u \mid m_j(x,u) \le 0\}} H(t,x,u,\lambda)$ (as defined in Theorem 2.10) exists and is convex on the convex hull of the set $\mathcal{B} := \{x \mid \text{for some } u,\ m_j(x,u) \le 0,\ j = 1, \cdots, k\}$ or ii) the Hamiltonian $H(t,x,u,\lambda^*)$ is simultaneously convex in (x,u) and the constraints $m_j(x,u)$ are simultaneously (quasi-)convex in (x,u). Details and extensions on this point can be found in [10, Chapter 4.3] or [11]. $\square$

Example 2.12 Consider the following advertising game of two competing divisions in a conglomerate company which wish to maximize their individual profits by choosing an optimal advertising policy (see [9, Example 3.5]). Being divisions of a parent company, they are not out to "hurt" each other; thus, a cooperative game solution seems reasonable. With x_i the gross revenue of the i^{th} division (with the initial revenue $x_i(0) = x_{i0}$ given) and $u_i \ge 0$ the rate of expenditure for advertising it is assumed that the changes in the gross revenues are given by

$$\dot{x}_1(t) = 12u_1(t) - 2u_1^2(t) - x_1(t) - u_2(t);\ \dot{x}_2(t) = 12u_2(t) - 2u_2^2(t) - x_2(t) - u_1(t) \tag{19}$$

The profits, to be maximized, are $J_i = \int_0^1 \{\frac{1}{3}x_i(t) - u_i(t)\}dt,\ i = 1,2.$
The with this problem corresponding Hamiltonian is

$$H := -\alpha(\frac{x_1}{3} - u_1) - (1-\alpha)(\frac{x_2}{3} - u_2) + \lambda_1(12u_1 - 2u_1^2 - x_1 - u_2) + \lambda_2(12u_2 - 2u_2^2 - x_2 - u_1).$$

From Theorem 2.6 we conclude that every Pareto efficient control $\hat{u} \ge 0$ satisfies for some $\alpha \in [0,1]$

$$H(t,\hat{x}(t),\hat{u}(t),\lambda_1(t),\lambda_2(t)) \le H(t,\hat{x}(t),u(t),\lambda_1(t),\lambda_2(t)),\ \text{at each } t \in [0,1],$$
$$\dot{\lambda}_1(t) = \lambda_1(t) + \frac{\alpha}{3},\ \lambda_1(1) = 0;\ \dot{\lambda}_2(t) = \lambda_2(t) + \frac{1-\alpha}{3},\ \lambda_2(1) = 0.$$

It is straightforwardly verified that for $\alpha \in (0,1)$ we obtain all the actions from which Leitmann showed in [9, Example 3.5] that they are Pareto efficient. Next consider $\alpha = 0$. Then $\lambda_1(t) = 0$ and $\lambda_2(t) = \frac{1-e^{t-1}}{3}$. Consequently, H is minimized for $\hat{u}_1(t) = 0$ and $\hat{u}_2(t) = \max\{0, \frac{3}{4}(e^{t-1}-1)^{-1}+3\}$. Obviously, the minimized Hamiltonian H^0 is linear in the state variables x_i. So in particular H^0 is a convex function of these variables. Therefore, by Theorem 2.10, additional to the above mentioned solutions the case $\alpha = 0$ yields an appropriate solution too. Similarly also $\alpha = 1$ yields a Pareto efficient solution $\hat{u}_1(t) = \max\{0, \frac{3}{4}(e^{t-1}-1)^{-1}+3\}$ and $\hat{u}_2(t) = 0$. From Theorem 2.6 again we conclude finally that with these additional two strategies we found all Pareto efficient controls for this game. □

3 The General Linear Quadratic Case

In this section we consider the finite planning horizon linear quadratic differential game. That is

$$J_i(u_1,u_2) := \int_0^T [x^T(t),\ u_1^T(t),\ u_2^T(t)]M_i[x^T(t),\ u_1^T(t),\ u_2^T(t)]^T dt + x^T(T)Q_{iT}x(T),\ i=1,2, \tag{20}$$

where $M_i = \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix}$ is symmetric, $R_i := \begin{bmatrix} R_{1i} & N_i \\ N_i^T & R_{2i} \end{bmatrix} \geq 0,\ i=1,2,\ T<\infty$ and $x(t)$ is the solution of the linear differential equation

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t),\ x(0) = x_0. \tag{21}$$

Notice that we make no definiteness assumptions w.r.t. matrix Q_i. For notational convenience we introduce for $\alpha \in [0,1]$ the next matrices and vectors $M := \alpha M_1 + (1-\alpha)M_2$, $Q := \alpha Q_1 + (1-\alpha)Q_2$, $R := \alpha R_1 + (1-\alpha)R_2$, $V := \alpha V_1 + (1-\alpha)V_2$, $W := \alpha W_1 + (1-\alpha)W_2$, $Q_T := \alpha Q_{1T} + (1-\alpha)Q_{2T}$, $B := [B_1\ B_2]$, $u^T := [u_1^T\ u_2^T]$ and $G := \begin{bmatrix} A - BR^{-1}\begin{bmatrix} V^T \\ W^T \end{bmatrix} & -BR^{-1}B^T \\ -(Q + [V\ W]R^{-1}\begin{bmatrix} V^T \\ W^T \end{bmatrix}) & -A^T \end{bmatrix}$.

For this special case Theorem 2.6 can be specialized as follows.

Corollary 3.1 *If $(J_1(\hat{u}), J_2(\hat{u}))$ is a Pareto solution for problem (20,21) then there exists an $\alpha \in [0,1]$ such that*

$$[\hat{x}^T(t),\ \hat{u}^T(t)]M\begin{bmatrix} \hat{x}(t) \\ \hat{u}(t) \end{bmatrix} + \lambda B\hat{u} \leq [\hat{x}^T(t),\ u^T(t)]M\begin{bmatrix} \hat{x}(t) \\ u(t) \end{bmatrix} + \lambda Bu \tag{22}$$

$$\dot{\lambda}(t) = -2\hat{x}^T(t)Q - 2\hat{u}^T\begin{bmatrix} V^T \\ W^T \end{bmatrix} - \lambda A;\ \lambda(T) = 2\hat{x}^T(T)Q_T; \tag{23}$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{u}(t),\ \hat{x}(0) = x_0. \tag{24}$$

In case α is such that $R_i > 0,\ i=1,2$, (i.e. R_i is positive definite), the above formulae can be equivalently rephrased as that every Pareto efficient control satisfies $\hat{u}(t) = -R^{-1}(\begin{bmatrix} V^T \\ W^T \end{bmatrix} x(t) +$

$B^T\tilde{\lambda}(t))$*, where* $\tilde{\lambda}(t)$ *is the solution of the set of linear differential equations:*

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{\lambda}}^T(t) \end{bmatrix} = G \begin{bmatrix} x(t) \\ \tilde{\lambda}^T(t) \end{bmatrix}, \begin{bmatrix} x(0) \\ \tilde{\lambda}^T(T) \end{bmatrix} = \begin{bmatrix} x_0 \\ Q_T x(T) \end{bmatrix}. \tag{25}$$

□

In case $R > 0$, for all $\alpha \in [0, 1]$, it follows directly from this Corollary 3.1 that $(0, 0)$ is the only potential Pareto solution if $x_0 = 0$. For in that case $(x(.), \lambda(.)) = (0, 0)$ always solves (25).
Another property which follows from (25) is that if for some α for both the initial states x_0 and x_1 there exists a $\lambda_i(t)$ satisfying (25) then also for every linear combination of x_0 and x_1 (25) has a solution.

This property hints to the property that if for both the initial states x_0 and x_1 there exists a Pareto solution also for every linear combination of these initial states a Pareto solution will exist. Unfortunately this result does not hold, as will be demonstrated in Example 3.12. However, the following linearity property does hold.

Lemma 3.2 *Assume* $\hat{u}$ *is a Pareto efficient control for (20,21). Then* $\lambda\hat{u}$ *is a Pareto efficient solution for (20,21) with* $x(0) = \lambda x_0$*.*

Proof. Let $x(t, x_0, u)$ denote the solution of (21). Then elementary calculations show that $x(t, \lambda x_0, \lambda u) = \lambda x(t, x_0, u)$ and, consequently, $J_i(\lambda x_0, \lambda u) = \lambda^2 J_i(x_0, u)$.
By Lemma 2.2, $\hat{u}$ is Pareto efficient if and only if $\hat{u}$ minimizes $J_i(x_0, u)$ on the constrained set

$$\mathcal{U}_i(x_0, \hat{u}) := \{u \mid J_j(x_0, u) \leq J_j(x_0, \hat{u}),\ j = 1, 2,\ j \neq i\},\ \text{for } i = 1, 2. \tag{26}$$

We will next show that $\lambda\hat{u}$ minimizes $J_1(\lambda x_0, u)$ on the constrained set

$$\tilde{\mathcal{U}}_1 := \{u \mid J_2(\lambda x_0, u) \leq J_2(\lambda x_0, \lambda\hat{u})\}. \tag{27}$$

Let $u \in \tilde{\mathcal{U}}_1$. Then according (27) $J_2(\lambda x_0, u) \leq J_2(\lambda x_0, \lambda\hat{u})$ or, equivalently, $J_2(x_0, \frac{1}{\lambda}u) \leq J_2(x_0, \hat{u})$. So, by (26), $\frac{1}{\lambda}u \in \mathcal{U}_1$. But this implies that $J_1(x_0, \frac{1}{\lambda}u) \geq J_1(x_0, \hat{u})$ or, equivalently, $J_1(\lambda x_0, u) \geq J_1(\lambda x_0, \lambda\hat{u})$.
Similarly one can show that $\lambda\hat{u}$ also minimizes $J_2(\lambda x_0, u)$ on the corresponding constrained set $\tilde{\mathcal{U}}_2$.
□

It is well-known that existence of a solution of the linear quadratic control problem for an arbitrary initial state is equivalent to the existence of a solution of an associated Riccati equation. For that reason we consider below the problem under which conditions for an arbitrary initial state (20,21) has a Pareto solution. We have the following two preliminary results. A proof of Lemma 3.3 can be found, e.g., in [6], whereas more details on Lemma 3.4 can be found in [4, Corollary 5.13].

Lemma 3.3 *Consider the two-point boundary value problem*

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = G \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}, \begin{bmatrix} x(0) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} x_0 \\ Q_T x(T) \end{bmatrix}. \tag{28}$$

Let $[W_1\ W_2] := [I\ 0]e^{-GT}$*. Then, (28) has a solution for every* x_0 *if and only if* $U := W_1 + W_2 Q_T$ *is invertible.* □

Lemma 3.4 *Let G be as introduced before and $U(t) := [I\ 0]e^{G(T-t)}\begin{bmatrix} I \\ Q_T \end{bmatrix}$. Consider the with G corresponding Riccati differential equation*

$$\dot{K}(t) = -A^T K(t) - K(t)A + (K(t)B + [V\ W])R^{-1}(K(t) + [V\ W]^T) - Q,\ K(T) = Q_T. \tag{29}$$

Then, (29) has a solution on $[0, T]$ if and only if $U(t)$ is invertible on $[0, T]$. □

Remark 3.5 According to the fundamental existence-uniqueness theorem of differential equations there exists a maximum time interval $[0, T_1)$ where equation (29) has a unique solution. So by Lemma 2.1 and [4, Theorem 5.1] for a planning horizon $T < T_1$ for every initial state x_0 the game has a Pareto solution. □

Using the above lemmas we obtain for the scalar case, that is the case that the dimension of the state variable $x(t)$ is one, the following result.

Theorem 3.6 *Consider the scalar system with $R_i > 0,\ i = 1, 2$. Let $x_0 \neq 0$. Then,*

1. *if (20,21) has a Pareto efficient control $\hat{u}(x_0)$ there exists an $\alpha \in [0, 1]$ such that $\min \alpha J_1 + (1-\alpha)J_2$ subject to (21) has a solution for all initial states $x_0 \in I\!R$.*
2. *if $J_i,\ i = 1, 2$, attains a minimum* **i)** *for all $\alpha \in [0, 1]$, $\min \alpha J_1 + (1 - \alpha)J_2$ exists for all x_0;* **ii)** *all Pareto efficient controls of (20,21) are given by $\{\hat{u} \mid \hat{u} = \arg\min \alpha J_1 + (1-\alpha)J_2,\ \alpha \in [0, 1]\}$.*

Proof. "**1**." Assume $(J_1(\hat{u}), J_2(\hat{u}))$ is a Pareto solution for problem (20,21) for some $x_0 \neq 0$. Then, according to Lemma 3.2, for every initial state problem (20,21) has a Pareto solution. Let $x_0 \neq 0$ be fixed and α, $x(t)$ and $\lambda(t)$ a with $\hat{u}(x_0)$ corresponding solution that satisfies (25). Then it is easily verified that for the initial state μx_0, α, $\mu x(t)$ and $\mu\lambda(t)$ satisfy (25). In other words, for this α, $\forall x_0$ (25) has a solution. So, by Lemma 3.3, $U(t) := [I\ 0]e^{G(T-t)}\begin{bmatrix} I \\ Q_T \end{bmatrix}$ is invertible at $t = 0$. Furthermore, by Corollary 3.1, $\hat{u}[t_0, T]$ is Pareto efficient for problem (20,21) if we consider J_i on the interval $[t_0, T]$, instead of $[0, T]$, with initial state $x(t_0) = e^{Gt_0}x_0$. Again it is easily verified that also in this case α, $x(t)$ and $\lambda(t)$ satisfy (25). So similar as before it follows that $U(t_0)$ is invertible too. Since $t_0 \in [0, T]$ was chosen arbitrarily by Lemma 3.4 the Riccati differential equation (29) has a solution on $[0, T]$. But this implies (see e.g. [4, Theorem 5.1]) that the optimization problem $\min \alpha J_1 + (1 - \alpha)J_2$ subject to (21) has a solution for all initial states x_0. It is easily verified that this solution is actually attained by $\hat{u}(x_0)$.
"**2**." Since $J_i(\lambda x_0, \lambda u) = \lambda^2 J_i(x_0, u)$ and by assumption J_1 and J_2 have a minimum, it follows immediately that for all $\alpha \in [0, 1]$ there exists a $\bar{J}(x_0, \alpha)$ such that

$$\alpha J_1(\lambda x_0, \lambda u) + (1 - \alpha)J_2(\lambda x_0, \lambda u) = \lambda^2(\alpha J_1(x_0, u) + (1 - \alpha)J_2(x_0, u)) \geq \lambda^2 \bar{J}(x_0, \alpha).$$

From this it follows that for all $\alpha \in [0, 1]$, for all x_0 $\inf \alpha J_1(x_0, u) + (1 - \alpha)J_2(x_0, u)$ exists. But this implies (see e.g. [4, p.182,183]) that actually the optimization problem $\min \alpha J_1 + (1 - \alpha)J_2$ subject to (21) has a unique solution for all x_0. Part ii) follows then directly from Part i) and Remark 2.5. □

Remark 3.7

1. As already noticed in the proof of Theorem 3.6.1 it follows directly from Lemma 3.2 that, in case the scalar game has a Pareto solution for some initial state different from zero, the game (20,21) has a Pareto solution for every initial state.

2. From the proof of Theorem 3.6 we can in fact conclude the following result. If for $x_0 \neq 0$ there exists a Pareto solution and for $\alpha_0 \in [0,1]$ (25) has a solution with $x(0) = x_0$, then $\min \alpha_0 J_1 + (1-\alpha_0)J_2$ subject to (21) has a solution for all initial states $x_0 \in I\!R$.

3. The assumption that the minimum of J_i exists in Theorem 3.6.2 implies that J_i is convex. Therefore, this part of the theorem is a special case of those dealt with in [5, Theorems 2.8, 2.9]). □

Example 3.8 This example illustrates some of the subtleties of Theorem 3.6. We consider

$$\min_u J_i = \int_0^{\pi/2} \{-x^2 + \beta_i u^2\} dt \ \text{ subject to } \dot{x} = u, \ x(0) = x_0, \ i = 1,2.$$

To that end we first consider the minimization of

$$J_1 \text{ subject to } \dot{x} = u, \ x(0) = x_0. \tag{30}$$

In case $\beta_1 = 1$, it is well-known (see e.g. [4, Example 5.1]) that this problem (30) has a solution if and only if $x_0 = 0$. In case $x_0 = 0$, $u(.) = \gamma \cos(t)$ yields for every γ the optimal value 0.
If $\beta_1 > 1$, the Riccati differential equation (29), $\dot{k}(t) = \frac{1}{\beta_1} k^2(t) + 1; \ k(\frac{\pi}{2}) = 0$, has a solution on $[0, \frac{\pi}{2}]$. So in that case problem (30) has for every initial state x_0 a solution.
Finally, in case $\beta_1 < 1$ one can use e.g. the control sequence from [2, Remark 3.1.4] (which was used to show that for $\beta_1 = 1$ this example has no solution in case $x_0 \neq 0$) to construct also for $x_0 = 0$ a control sequence for which J_1 becomes negative (implying that $\inf J_1$ does not exist), yielding the conclusion that for $\beta_1 < 1$ for all initial states problem (30) has no solution.

Next we consider the cooperative game with $\beta_i = 1, \ i = 1,2$. Then, obviously, for every $\alpha \in [0,1]$ $\alpha J_1 + (1-\alpha) J_2 = J_1$. From the above considerations we have that for all $x_0 \neq 0$, for all $\alpha \in [0,1]$, this problem has no solution. So according Theorem 3.6.1 this cooperative game has no Pareto solution if $x_0 \neq 0$. On the other hand we conclude from Lemma 2.1 and the above considerations that for $x_0 = 0$, $u(t) = \gamma \cos(t)$ yield the Pareto solution $(0,0)$. □

Theorem 3.9 *Consider the scalar system with $R_i > 0, \ i = 1,2$. Then for every x_0 (20,21) has a Pareto efficient control $\hat{u}(x_0)$ if and only if there exists an $\alpha \in [0,1]$ such that for every x_0 $\min \alpha J_1 + (1-\alpha) J_2$ subject to (21) has a solution.*

Proof. " $\Rightarrow$ " In particular it follows that (20,21) has a Pareto efficient solution for some $x_0 \neq 0$. Theorem 3.6 yields then the advertized result.
" $\Leftarrow$ " Since the minimum exists for all x_0 it is well-known that the argument at which this minimum is attained is unique (see e.g. [4, Theorem 5.1]). Remark 2.5 yields then directly the conclusion. □

From Theorem 3.9 and Remark 3.7.2 we obtain the next procedure to find all Pareto efficient solutions for the scalar game.

Algorithm 3.10 Using the notation introduced in the beginning of this section, consider the scalar system with $R_i > 0,\ i = 1,2$. Let $\hat{\mathcal{U}}$ be the set of all Pareto efficient controls for which (20,21) has a Pareto solution. Next consider for $\alpha \in \mathcal{A}$ the Riccati differential equation

$$\begin{aligned} \dot{K}(t) &= -A^T K(t) - K(t)A + (K(t)B + [V(\alpha)\ W(\alpha)])R^{-1}(\alpha)(B^T K(t) + [V(\alpha)\ W(\alpha)]^T) - Q(\alpha), \\ K(T) &= Q_T(\alpha). \end{aligned} \tag{31}$$

Then all Pareto efficient controls $\hat{u} \in \hat{\mathcal{U}}$ are obtained as follows:
1) If $x_0 \neq 0$, determine $I(\alpha) := \{\alpha \in \mathcal{A} \mid (31) \text{ has a solution } K_\alpha(t) \text{ on } [0,T]\}$. Then for every $\hat{u}$ there exists an $\alpha \in I(\alpha)$ such that

$$\hat{u}(t) := -R^{-1}(\alpha)([V(\alpha)\ W(\alpha)]^T + B^T K_\alpha(t))\hat{x}(t),$$

where $\hat{x}(t)$ solves the differential equation $\dot{x}(t) = (A - BR^{-1}(\alpha)([V(\alpha)\,W(\alpha)]^T + B^T K_\alpha(t))x(t),\ x(0) = x_0$. The corresponding Pareto solution is $(J_1(\hat{u}), J_2(\hat{u}))$.
2) If $x_0 = 0$ and $I(\alpha)$ in item 1) is not empty, $\hat{u} = (0,0)$ is the unique Pareto efficient control. If $I(\alpha) = \emptyset$, $(0,0)$ is the only candidate Pareto solution. If in this case $(0,0)$ turns out to be a Pareto solution there may exist more than one Pareto efficient control. $\square$

Example 3.11 Consider the cooperative game with

$$J_1 = \int_0^{\frac{\pi}{2}} \{-x^2 + \frac{9}{10}u_1^2 + \frac{1}{10}u_2^2\}dt \text{ and } J_2 = \int_0^{\frac{\pi}{2}} \{-x^2 + \frac{1}{10}u_1^2 + \frac{9}{10}u_2^2\}dt$$

$$\text{subject to the system } \dot{x} = \frac{4}{10}(u_1 + u_2),\ x(0) = x_0.$$

Here player i controls u_i. According to Algorithm 3.10 this game has a Pareto efficient solution precisely for those $\alpha \in [0,1]$ for which

$$\dot{k} = sk^2 + 1,\ k(\frac{\pi}{2}) = 0,\ \text{where } s := \frac{16}{(1+8\alpha)(9-8\alpha)},\ \text{has a solution on } [0, \frac{\pi}{2}].$$

The solution of this differential equation is $k(t) = \frac{1}{\sqrt{s}} \tan \sqrt{s}(t - \frac{\pi}{2})$. So, $k(t)$ exists on $[0, \frac{\pi}{2}]$ if and only if $\sqrt{s} < 1$. It is easily verified that this is equivalent to the condition $\alpha \in (\frac{1}{8}, \frac{7}{8})$.
So, the set of all Pareto efficient controls is given by $\hat{u}(t) = -\frac{4k(t)}{10} \begin{bmatrix} \frac{10}{1+8\alpha} & 0 \\ 0 & \frac{10}{9-8\alpha} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \hat{x}(t)$, where $\hat{x}(t)$ solves $\dot{x}(t) = -sk(t)x(t),\ x(0) = x_0$, and $\alpha \in (\frac{1}{8}, \frac{7}{8})$. $\square$

The following example illustrates a two-dimensional state game where for all initial states in the interior of the second and fourth quadrant and the point $(0,0)$ a Pareto solution exists whereas for all other initial states there exists no Pareto solution.

Example 3.12 Consider the cooperative minimization of

$$J_1 = \int_0^{\pi/2} \{x^T(t) \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} x(t) + u^2(t)\}dt \text{ and } J_2 = \int_0^{\pi/2} \{x^T(t) \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix} x(t) + u^2(t)\}dt$$

$$\text{subject to } \dot{x}_1 = u,\ x_1(0) = p,\ \dot{x}_2 = u,\ x_2(0) = q. \tag{32}$$

Next consider the minimization of

$$\alpha J_1 + (1-\alpha)J_2 = \int_0^{\frac{\pi}{2}} \{x^T(t) \begin{bmatrix} -3+2\alpha & 2-\alpha \\ 2-\alpha & -2 \end{bmatrix} x(t) + u^2(t)\}dt \tag{33}$$

subject to (32). From (32) it follows directly that $x_1(t) = x_2(t) + p - q$. Substitution of this into (33) shows that the minimization of (33) subject to (32) is equivalent to the minimization of

$$\alpha J_1 + (1-\alpha)J_2 = \int_0^{\frac{\pi}{2}} \{-(x_2(t) + (1-\alpha)(p-q))^2 + u^2(t) + (-2+\alpha^2)(p-q)^2\}dt$$

$$\text{subject to } \dot{x_2}(t) = u(t);\ x_2(0) = q.$$

From Example 3.8 it follows that this problem has a solution if and only if the initial state satisfies $-(1-\alpha)(p-q) = q$ or, equivalently, $(1-\alpha)p = -\alpha q$. From this it is easily verified that for every initial state (p,q) of (32) there is exact one $\alpha \in [0,1]$ such that the minimization of (33) has a solution if (p,q) is located either in the second quadrant, the fourth quadrant, on the line $p = 0$ (which corresponds with $\alpha = 0$) or on the line $q = 0$ (which corresponds with $\alpha = 1$). So by Lemma 2.1 for all the initial states (p,q) which are located in the interior of the second and fourth quadrant there is a Pareto solution.
Next consider the initial state $(p,0)$. Then $x_1(t) = x_2(t) + p$ and we can rewrite J_1 as follows:

$$J_1 = \int_0^{\frac{\pi}{2}} \{-(x_1(t) - x_2(t))^2 - x_2^2(t) + u^2(t)\}dt = \int_0^{\frac{\pi}{2}} \{-p^2 - x_2^2(t) + u^2(t)\}dt.$$

From Example 3.8 we recall again that the minimum value of J_1 is 0. Furthermore, $u_2(t) := \gamma\cos(t)$ belongs to the with this problem corresponding constrained set $\mathcal{U}_2$ (see Lemma 2.2). Using this control we see that

$$\begin{aligned} J_2 &= \int_0^{\frac{\pi}{2}} \{-2(x_1(t) - x_2(t))^2 - x_1^2(t) + u_2^2(t)\}dt \\ &= \int_0^{\frac{\pi}{2}} \{-2p^2 - (\gamma\sin(t) + p)^2 + \cos^2(t)\}dt = -\frac{3p^2\pi}{2} + 2p\gamma. \end{aligned}$$

Since γ is an arbitrary number it is clear from this that $\min_{u\in\mathcal{U}_2} J_2$ does not exist if $p \neq 0$. So, by Lemma 2.2, for the initial state $(p,0)$, there exists no Pareto efficient solution. Similarly it follows that also for the initial state $(0,q)$ the problem has no Pareto efficient solution. Finally, it is easily verified that for the initial state $(0,0)$, for every γ, $\hat{u} = \gamma\cos(t)$ is a Pareto efficient control yielding the same Pareto solution $(0,0)$.
Obviously, the with problem (33,32) corresponding Riccati differential equation has for every $\alpha \in$

$[0,1]$ not a solution on $[0, \frac{\pi}{2}]$.
Finally, for the initial state $x_0 = [p;\ q]^T$ there exists an $\alpha \in [0,1]$, $x(.)$, and $\lambda(.)$ satisfying (25) if and only if there exists an $\alpha \in [0,1]$ and $\bar{\lambda_1}$, $\bar{\lambda_2}$ such that the equation $e^{G\frac{\pi}{2}}[p;\ q;\ \bar{\lambda_1};\ \bar{\lambda_2}]^T = [x_1(\frac{\pi}{2});\ x_2(\frac{\pi}{2});\ 0;\ 0]^T$ has a solution. An elementary spelling of these equations shows that there exists a solution if and only if $p(1-\alpha) = -\alpha q$. So, in particular we conclude from this that for all initial states located in either the interior of the first or third quadrant there exists no Pareto solution. $\square$

4 Concluding Remarks

In this note we derived necessary conditions for the existence of a Pareto solution in a cooperative dynamic differential game. These conditions are in the spirit of the maximum principle conditions. Furthermore we presented some conditions, in line with Arrow's conditions, under which the necessary conditions are sufficient too. Finally we indicated the effects on the necessary conditions if the problem setting includes inequality constraints.
We elaborated these necessary conditions for the regular indefinite linear quadratic control problem. For the scalar case we presented an algorithm to calculate all Pareto efficient outcomes. Basically this proceeds by solving the corresponding well-known parameterized optimal control problem over the unit simplex. The generalization of these results to non-scalar systems remains an open problem. We illustrated in an example some pecularities that may occur. Within this context we recall that if the performance criteria of the players are convex, then such a generalization exists (see [5]), both for a finite and infinite planning horizon.
Finally, we believe that without too many complications the presented theory can be extended for an infinite planning horizon. The elaboration of the herewith involved technical problems remains however a point for future research. Another open problem that remains to be solved is the case that the performance criteria of both players are not directly affected by the control efforts used by the other player (giving rise to singular control problems).

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