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**MULTICRITERIA DYNAMIC OPTIMIZATION PROBLEMS  
AND COOPERATIVE DYNAMIC GAMES**

By Jacob Engwerda

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# Multicriteria Dynamic Optimization Problems and Cooperative Dynamic Games

Jacob Engwerda\*

June, 2007

**Abstract** We survey some recent research results in the field of dynamic cooperative differential games with non-transferable utilities. Problems which fit into this framework occur for instance if a person has more than one objective he likes to optimize or if several persons decide to combine efforts in trying to realize their individual goals. We assume that all persons act in a dynamic environment and that no side-payments take place.

For these kind of problems the notion of Pareto efficiency plays a fundamental role. In economic terms, an allocation in which no one can be made better-off without someone else becoming worse-off is called Pareto efficient. In this paper we present as well necessary as sufficient conditions for existence of a Pareto optimum for general non-convex games. These results are elaborated for the special case that the environment can be modeled by a set of linear differential equations and the objectives can be modeled as functions containing just affine quadratic terms. Furthermore we will consider for these games the convex case.

In general there exists a continuum of Pareto solutions and the question arises which of these solutions will be chosen by the participating persons. We will flash some ideas from the axiomatic theory of bargaining, which was initiated by Nash [16, 17], to predict the compromise the persons will reach.

**Keywords :** *Dynamic Optimization, Pareto Efficiency, Cooperative Differential Games, LQ Theory, Riccati Equations, Bargaining*

**Jel-codes:** C61, C71, C73.

## 1 Introduction

In this paper we consider the problem to find an "optimal" strategy in case either there is one individual who has multiple objectives, or, there is more than one person affecting a dynamic system and in order to minimize their cost these persons decide to coordinate their actions. As an example one can think of a government who likes to realize an acceptable income level, full-fledged employment for everyone, a just income distribution, a stable price level, a stable currency exchange rate, a sustainable balance of payment, and a lasting economic growth. One can imagine that the realization of all these goals simultaneously will be a tough or, more probably, impossible task. From that perspective it is more realistic to classify all simultaneously feasible realizations of the objectives and

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next let the government choose one of these options. An example illustrating a situation where more than one player is involved in a game and the players cooperate in order to realize their objectives is for instance the next advertising game (see [13]). Consider two competing divisions in a conglomerate company which wish to maximize their individual profits by choosing an optimal advertising policy. Being divisions of a parent company, they are not out to "hurt" each other. Therefore it seems reasonable to assume that they will coordinate their policies in order to maximize their profits.

The above examples are characterized by the fact that there is either one or more player who can affect the outcome of the state of a system. We will assume that the evolution of this system over time is described by a differential equation. Since players simultaneously affect the outcome of the system the performance of the players depends on the actions taken by the other players. To realize an outcome which is as good as possible for all of them simultaneously, we assume in this paper that players decide to coordinate their actions and engage in a cooperative game. It is assumed that all players can communicate and enter into binding agreements. However, no side-payments take place. Moreover, it is assumed that every player has all information on as well the state dynamics as the cost functions of his opponents, and all players are able to implement their decisions. In literature this information structure is known as the open-loop information game. Concerning the strategies used by the players we assume that there are no restrictions. That is, every control trajectory chosen by player  $i$ ,  $u_i(\cdot)$ , may be chosen arbitrarily from a set  $\mathcal{U}$  (which depends on the problem setting and will be specified later) in order to have a well-posed problem. More formally, we consider the minimization of the performance criteria:

$$J_i(t_0, u_1, u_2) := \int_{t_0}^T g_i(t, x(t), u_1(t), u_2(t))dt + h_i(x(T)), \quad i = 1, 2, \quad (1)$$

where  $x(t)$ , is the solution of the differential equation

$$\dot{x}(t) = f(t, x(t), u_1(t), u_2(t)), \quad x(t_0) = x_0. \quad (2)$$

For the moment we do not explicitly state conditions on the functions  $g_i$  and  $f$  and the set of control actions  $\mathcal{U}$ . We just assume that they are such that the above integrals are well-defined and the differential equation has a solution on  $[t_0, T]$  in the extended sense (see e.g. [4], [12] or [5]). Later on, when needed, additional assumptions will be made on these functions and set. Furthermore we make the convention that in case we consider this problem for the initial time  $t_0 = 0$ , we use the shorthand notation  $J(u_1, u_2)$  instead of (1).

By cooperation, in general, the cost a specific player incurs is not uniquely determined anymore. If, within the above two-player context, both players decide for instance to use their control variables to minimize the performance of player 1 as much as possible, a different minimum is attained for player 1 than in case both players agree to help collectively player 2 to minimize his performance. So, depending on how the players choose to "divide" their control efforts, a player incurs different "minima". Therefore, in general, each player is confronted with a whole set of possible outcomes from which somehow one outcome (which in general does not coincide with a player's overall lowest cost) is cooperatively selected. Now, if there are two control strategies  $u_1$  and  $u_2$  such that every player has a lower cost if strategy  $u_1$  is played, then it seems reasonable to assume that all players will prefer this strategy. We say that the solution induced by strategy  $u_1$  *dominates* the solution induced by the strategy  $u_2$ . So, dominance means that the outcome is better for all players. Proceeding in this line of thinking it seems reasonable to consider only those cooperative outcomes which have the property that, if a different strategy than the one corresponding with this cooperative outcome is chosen then

at least one of the players has higher costs. Or, stated differently, to consider only solutions that are such that they can not be improved upon by all players simultaneously. This motivates the concept of Pareto efficiency.

**Definition 1.1** Let  $\mathcal{U}$  denote the set of admissible strategies. A set of control actions  $\hat{u}$  (in the sequel: a control) is called *Pareto efficient* if the set of inequalities

$$J_i(u) \leq J_i(\hat{u}), \quad i = 1, \dots, N,$$

where at least one of the inequalities is strict, does not allow for any solution  $u \in \mathcal{U}$ . The corresponding point  $(J_1(\hat{u}), \dots, J_N(\hat{u})) \in \mathbb{R}^N$  is called a *Pareto solution*. The set of all Pareto solutions is called the *Pareto frontier*.  $\square$

A Pareto solution is therefore never dominated, and for that reason called an *undominated* solution. Usually there is more than one Pareto solution, because dominance is a property which generally does not provide a total ordering. Lemma 2.1, below, gives a sufficient condition under which one can conclude that a control is Pareto efficient. This result is well-known in literature (see e.g. [13], [5] or [31]). This lemma holds without using any convexity conditions on the  $J_i$ 's nor any convexity assumptions regarding the strategy space. Its converse, Theorem 2.12, was proved by Fan et al in [9] under some convexity assumptions on the cost functions. This result was exploited in [7] to solve the regular cooperative indefinite linear quadratic game. In this paper we present in Section 2 for general non-convex problems necessary conditions for a control to be Pareto efficient and next discuss additional conditions from which one can conclude that these necessary conditions are sufficient too. As far as we know these conditions have not been stated explicitly in the literature. The obtained results resemble the corresponding results for the static optimization problem as reported, e.g., in [22]. Section 3 deals with the regular, non-convex, indefinite finite-planning horizon linear quadratic differential game. Most results are presented here for a finite planning horizon. In particular we present here an algorithm to compute all Pareto efficient solutions when the system is scalar. Unfortunately, it remains unclear whether this algorithm can also be used for non-scalar systems to find all Pareto efficient solutions. In Section 4 we consider the special case that both the set of admissible control functions and cost functions are convex. We show that in that case the algorithm presented in Section 3 can be used too to calculate all Pareto efficient solutions as well in case the planning horizon is finite as in case it is infinite.

Assuming an open-loop information structure it seems reasonable that players will also make a decision at the start of the planning horizon which Pareto efficient control they will use. In section 5 we present a number of candidate solutions that are motivated from the axiomatic bargaining literature. Section 6 concludes and mentions a number of open problems. This overview is to a large extent based on results reported in [5], [6], [7] and [8]. This paper contains some more detailed results and proofs.

## 2 The general case

In this section we first present some preliminary results and properties about Pareto efficient solutions (frontier) which hold in general. After this, we present then the necessary conditions followed by a discussion of some sufficient conditions.

In the subsequent analysis the following set of parameters,  $\mathcal{A}$  (the "unit-simplex"), plays a crucial role.

$$\mathcal{A} := \{\alpha = (\alpha_1, \dots, \alpha_N) \mid \alpha_i \geq 0 \text{ and } \sum_{i=1}^N \alpha_i = 1\}.$$

The following two lemmas provide a characterization of Pareto efficient controls. To emphasize the role of the assumptions, we include of both results a proof which is a straightforward copy of the proof for the finite dimensional case (see e.g. [5] and [22]).

**Lemma 2.1** *Let  $\alpha_i \in (0, 1)$ , with  $\sum_{i=1}^N \alpha_i = 1$ . Assume  $\hat{u} \in \mathcal{U}$  is such that*

$$\hat{u} \in \arg \min_{u \in \mathcal{U}} \left\{ \sum_{i=1}^N \alpha_i J_i(u) \right\}. \quad (3)$$

*Then  $\hat{u}$  is Pareto efficient.* □

**Proof.** Let  $\alpha_i \in (0, 1)$ , with  $\sum_{i=1}^N \alpha_i = 1$  and  $\hat{u} \in \arg \min_{u \in \mathcal{U}} \left\{ \sum_{i=1}^N \alpha_i J_i(u) \right\}$ . Assume  $\hat{u}$  is not Pareto efficient. Then, there exists an  $N$ -tuple of strategies  $\bar{u}$  such that

$$J_i(\bar{u}) \leq J_i(\hat{u}), \quad i = 1, \dots, N,$$

where at least one of the inequalities is strict. But then

$$\sum_{i=1}^N \alpha_i J_i(\bar{u}) < \sum_{i=1}^N \alpha_i J_i(\hat{u}),$$

which contradicts the fact that  $\hat{u}$  is minimizing. □

**Lemma 2.2**  *$\hat{u} \in \mathcal{U}$  is Pareto efficient if and only if for each  $i$   $\hat{u}(\cdot)$  minimizes  $J_i$  on the constrained set*

$$\mathcal{U}_i := \{u \mid J_j(u) \leq J_j(\hat{u}), \quad j = 1, \dots, N, \quad j \neq i\}, \quad \text{for } i = 1, \dots, N. \quad (4)$$

**Proof.** "  $\Rightarrow$  " Suppose  $\hat{u}$  is Pareto efficient. If  $\hat{u}$  does not minimize  $J_k$  on the constrained set  $\mathcal{U}_k$  for some  $k$ , then there exists a  $u$  such that  $J_j(u) \leq J_j(\hat{u})$  for all  $j \neq k$  and  $J_k(u) < J_k(\hat{u})$ . This contradicts the Pareto efficiency of  $\hat{u}$ .

"  $\Leftarrow$  " Suppose  $\hat{u}$  minimizes each  $J_k$  on  $\mathcal{U}_k$ . If  $\hat{u}$  does not provide a Pareto optimum, then there exists a  $u(\cdot) \in \mathcal{U}$  and an index  $k$  such that  $J_i(u) \leq J_i(\hat{u})$  for all  $i$  and  $J_k(u) < J_k(\hat{u})$ . This contradicts the minimality of  $\hat{u}$  for  $J_k$  on  $\mathcal{C}_k$ . □

A direct consequence of this lemma is that in the two-player case the Pareto frontier can be, loosely speaking, visualized as a decreasing function.

**Corollary 2.3** ( $N = 2$ ) Assume that the Pareto frontier consists of more than one point. If both  $\hat{u}$  and  $\tilde{u}$  are two Pareto efficient controls with  $J_1(\hat{u}) \leq J_1(\tilde{u})$ , then  $J_2(\hat{u}) \geq J_2(\tilde{u})$ .

**Proof.** Assume that  $\hat{u}$  and  $\tilde{u}$  are two Pareto efficient controls with  $J_1(\hat{u}) \leq J_1(\tilde{u})$ . Let  $\hat{\mathcal{U}}_1 := \{u \mid J_1(u) \leq J_1(\hat{u})\}$  and  $\tilde{\mathcal{U}}_1 := \{u \mid J_1(u) \leq J_1(\tilde{u})\}$ . Then, according Lemma 2.2,  $J_2(u) \geq J_2(\hat{u})$ ,  $\forall u \in \hat{\mathcal{U}}_1$  and, (i),  $J_2(u) \geq J_2(\tilde{u})$ ,  $\forall u \in \tilde{\mathcal{U}}_1$ . Next notice that  $\hat{u} \in \tilde{\mathcal{U}}_1$ . From (i) it follows then directly that in particular  $J_2(\hat{u}) \geq J_2(\tilde{u})$ .  $\square$

**Corollary 2.4** Assume  $\mathcal{U}$  is such that with  $u_i(\cdot) \in \mathcal{U}$ ,  $i = 1, 2$  also any concatenation of  $u_i(\cdot)$  belongs to  $\mathcal{U}$ . That is for every  $t_0$  we have that with  $u[0, t_0] := u_1[0, t_0]$ ,  $u[t_0, T] := u_2[t_0, T]$ ,  $u \in \mathcal{U}$ .

If  $\hat{u}[0, T]$  is a Pareto efficient control for  $x(0) = x_0$  in (1,2), then for any  $t_0 > 0$ ,  $\hat{u}[t_0, T]$  is a Pareto efficient control for  $x(t_0) = \hat{x}(t_0)$  in (1,2). Here  $\hat{x}(t_0) = x(t_0, 0, \hat{u}[0, t_0])$  is the value of the state at time  $t_0$  induced by  $\hat{u}[0, t_0]$ .

**Proof.** Consider for  $x(0) = x_0$  (see (4))

$$\mathcal{U}_1(0) := \{u \mid J_2(u) \leq J_2(\hat{u}[0, T])\}. \quad (5)$$

Let  $t_0 > 0$ . We will next show (see Lemma 2.2) that  $\hat{u}[t_0, T]$  minimizes  $J_1(t_0, u)$  on the constrained set

$$\tilde{\mathcal{U}}_1(t_0) := \{u \mid J_2(u) \leq J_2(t_0, \hat{u}[t_0, T])\}, \text{ subject to (30) with } x(t_0) = \hat{x}(t_0). \quad (6)$$

To that end we first note that  $\hat{u}[t_0, T] \in \tilde{\mathcal{U}}_1(t_0)$ .

Next we show that every element  $u \in \tilde{\mathcal{U}}_1(t_0)$  can be viewed as an element  $u^e \in \mathcal{U}_1(0)$  restricted to the time interval  $[t_0, T]$ . That is,  $\forall u \in \tilde{\mathcal{U}}_1(t_0)$  there exists  $u^e \in \mathcal{U}_1(0)$  such that  $u^e[t_0, T] = u$ . For, let  $u^e[0, T]$  be the concatenation of  $\hat{u}[0, t_0]$  with  $u[t_0, T]$ . Then, clearly,  $u^e$  is such that  $x(t_0) = \hat{x}(t_0)$ . Furthermore,

$$\begin{aligned} J_2(u^e) &= \int_0^T g_2(t, x(t), u^e(t))dt + h_2(x(T)) \\ &= \int_0^{t_0} g_2(t, \hat{x}(t), \hat{u}(t))dt + \int_{t_0}^T g_2(t, x(t), u(t))dt + h_2(x(T)) \\ &\leq \int_0^{t_0} g_2(t, \hat{x}(t), \hat{u}(t))dt + \int_{t_0}^T g_2(t, \hat{x}(t), \hat{u}(t))dt + h_2(\hat{x}(T)) \\ &= J_2(\hat{u}). \end{aligned}$$

So, by definition,  $u^e \in \mathcal{U}_1(0)$ .

From the dynamic programming principle it follows now directly that  $\hat{u}[t_0, T]$  has to minimize  $J_1(t_0, u)$  on  $\tilde{\mathcal{U}}_1(t_0)$ .

In the same way one can show that  $\hat{u}[t_0, T]$  also minimizes  $J_2(t_0, u)$  on the corresponding constrained set  $\tilde{\mathcal{U}}_2(t_0)$ , which proves the claim.  $\square$

**Corollary 2.5** Assume  $J_1(u)$  has a minimum which is uniquely attained at  $\hat{u}$ . Then  $(J_1(\hat{u}), \dots, J_N(\hat{u}))$  is a Pareto solution.

**Proof.** Using the notation of Lemma 2.2 it follows directly from the fact that the minimum of  $J_1(u)$  is uniquely attained at  $\hat{u}$  that  $\mathcal{U}_i = \{\hat{u}\}$ , for  $i = 2, \dots, N$ . Furthermore, it is clear that  $\hat{u} \in \mathcal{U}_1$ . Consequently,  $\min_{u \in \mathcal{U}_i} J_i = J_i(\hat{u})$ ,  $i = 2, \dots, N$  and, since  $J_1(u) \geq J_1(\hat{u})$  for arbitrary  $u$ ,  $\min_{u \in \mathcal{U}_1} J_1(u) = J_1(\hat{u})$  too. Lemma 2.2 yields then the advertized result.  $\square$

**Remark 2.6** From Lemma 2.1 it follows now that in case  $J_i(u)$  has a unique minimum location for all  $i$ , then every control such that for some  $\alpha_i \in [0, 1]$ , with  $\sum_{i=1}^N \alpha_i = 1$ ,  $\hat{u} \in \arg \min_{u \in \mathcal{U}} \{\sum_{i=1}^N \alpha_i J_i(u)\}$  is Pareto efficient.  $\square$

**Lemma 2.7** ( $N = 2$ ) Assume  $\alpha_1 < \alpha_2$  and  $\hat{u}_i \in \mathcal{U}$  is such that

$$\hat{u}_i \in \arg \min_{u \in \mathcal{U}} \{\alpha_i J_1(u) + (1 - \alpha_i) J_2(u)\}, \quad i = 1, 2. \quad (7)$$

Then, for all  $\alpha \in (\alpha_1, \alpha_2)$  there exists a  $\bar{J}(\alpha)$  such that for all  $u$

$$\alpha J_1(u) + (1 - \alpha) J_2(u) \geq \bar{J}(\alpha). \quad (8)$$

**Proof.** Note that

$$\begin{aligned} \alpha J_1(u) + (1 - \alpha) J_2(u) &= \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} (\alpha_2 J_1(u) + (1 - \alpha_2) J_2(u)) + \frac{\alpha_2 - \alpha}{\alpha_2 - \alpha_1} (\alpha_1 J_1(u) + (1 - \alpha_1) J_2(u)) \\ &\geq \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} (\alpha_2 J_1(\hat{u}_2) + (1 - \alpha_2) J_2(\hat{u}_2)) + \frac{\alpha_2 - \alpha}{\alpha_2 - \alpha_1} (\alpha_1 J_1(\hat{u}_1) + (1 - \alpha_1) J_2(\hat{u}_1)). \end{aligned}$$

$\square$

Using Lemma 2.2 we derive next our main result on necessary conditions for a Pareto solution. For that purpose we impose some additional smoothness conditions on our functions. Furthermore we will for simplicity assume that  $\mathcal{U}$  consists of the set of piecewise continuous functions<sup>1</sup>.

**Assumption 2.8** Assume that

- (i)  $f(t, x, u)$  and  $g_i(t, x, u)$  are continuous functions on  $\mathbb{R}^{1+n+m}$ . Moreover, for both  $f$  and  $g_i$  all partial derivatives w.r.t.  $x$  and  $u$  exist and are continuous.
- (ii)  $h_i(x)$  is continuously differentiable.  $\square$

**Theorem 2.9** Let Assumptions 2.8 be satisfied. Assume  $(J_1(\hat{u}), J_2(\hat{u}))$  is a Pareto solution for problem (1,2). Then, there exists an  $\alpha \in [0, 1]$ , a costate function  $\lambda^T(t) : [0, T] \rightarrow \mathbb{R}^n$  (which is continuous and piecewise continuously differentiable) such that, with  $H(t, x, u, \lambda) := \alpha g_1(t, x, u) + (1 - \alpha) g_2(t, x, u) + \lambda f(t, x, u)$ ,  $\hat{u}$  satisfies

$$H(t, \hat{x}(t), \hat{u}(t), \lambda(t)) \leq H(t, \hat{x}(t), u(t), \lambda(t)), \quad \text{at each } t \in [0, T], \quad (9)$$

$$\dot{\lambda}(t) = -\left[\alpha \frac{\partial g_1}{\partial x} + (1 - \alpha) \frac{\partial g_2}{\partial x} + \lambda(t) \frac{\partial f}{\partial x}\right]; \quad \lambda(T) = \frac{\partial(\alpha h_1 + (1 - \alpha) h_2)}{\partial x}, \quad (10)$$

$$\dot{\hat{x}}(t) = f(t, \hat{x}(t), \hat{u}_1(t), \hat{u}_2(t)), \quad \hat{x}(0) = x_0. \quad (11)$$

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<sup>1</sup>see e.g. [12] (or [5, p.134]) for a generalization.

**Proof.** Introduce the state variable  $\dot{x}_2 := g_2(t, x, u)$ ,  $x_2(0) = 0$ , and  $x_2^* := \int_0^T g_2(t, \hat{x}, \hat{u})dt + h_2(\hat{x}(T))$ , where  $\hat{x}$  solves  $\dot{\hat{x}} = f(t, \hat{x}, \hat{u})$ ,  $\hat{x}(0) = x_0$ . From Lemma 2.2 we have then that in particular the optimization problem

$$\min_u \int_0^T g_1(t, x, u)dt + h_1(x(T)) \text{ such that } x_2(T) + h_2(x(T)) - x_2^* \geq 0$$

subject to

$$\begin{bmatrix} \dot{x} \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f(t, x, u) \\ g_2(t, x, u) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix},$$

has a solution. From the maximum principle we conclude next (see e.g. [20] and [21, p.61]) that there exist (continuous, piecewise continuously differentiable) costate functions  $\lambda_i(t)$  and a constant  $p_1 \in \{0, 1\}$  (with  $[p_1, \lambda_1(t), \lambda_2(t)] \neq [0, 0, 0]$  for all  $t \in [0, T]$  **(i)**) such that the Hamiltonian  $H^1 := p_1 g_1(t, x, u) + \lambda_1(t) f(t, x, u) + \lambda_2(t) g_2(t, x, u)$  is minimized at  $u = \hat{u}$ . Furthermore the costate variables satisfy the set of differential equations:

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = - \begin{bmatrix} H_x^1 \\ H_{x_2}^1 \end{bmatrix}; \quad \begin{bmatrix} \lambda_1(T) \\ \lambda_2(T) \end{bmatrix} = \begin{bmatrix} p_1 h_1'(x(T)) + \alpha_1 h_2'(x(T)) \\ \alpha_1 \end{bmatrix}, \quad (12)$$

where  $\alpha_1 \geq 0$ , and  $\alpha_1(x_2(T) + h_2(x(T)) - x_2^*) = 0$ .

Since  $H_{x_2}^1 = 0$ , it follows that  $\lambda_2(t) = \alpha_1$ , for all  $t \in [0, T]$ . Substitution of this into the "first-order" condition of  $H^1$ , gives then

$$p_1 g_1(t, \hat{x}, \hat{u}) + \lambda_1(t) f(t, \hat{x}, \hat{u}) + \alpha_1 g_2(t, \hat{x}, \hat{u}) \leq p_1 g_1(t, \hat{x}, u) + \lambda_1(t) f(t, \hat{x}, u) + \alpha_1 g_2(t, \hat{x}, u), \quad t \in [0, T]. \quad (13)$$

In a similar way it follows also from Lemma 2.2 that the optimization problem

$$\min_u \int_0^T g_2(t, x, u)dt + h_2(x(T)) \text{ such that } x_1(T) + h_1(x(T)) - x_1^* \geq 0$$

subject to

$$\begin{bmatrix} \dot{x} \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} f(t, x, u) \\ g_1(t, x, u) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix},$$

has a solution, where  $x_1^* := \int_0^T g_1(t, \hat{x}, \hat{u})dt + h_1(\hat{x}(T))$ . Analogously as before one obtains then from the necessary conditions the existence of an  $\alpha_2 \geq 0$ , a costate function  $\mu_1(t)$  and a constant  $p_2 \in \{0, 1\}$  that satisfy:

$$\alpha_2 g_1(t, \hat{x}, \hat{u}) + p_2 g_2(t, \hat{x}, \hat{u}) + \mu_1(t) f(t, \hat{x}, \hat{u}) \leq \alpha_2 g_1(t, \hat{x}, u) + p_2 g_2(t, \hat{x}, u) + \mu_1(t) f(t, \hat{x}, u), \quad t \in [0, T], \quad (14)$$

$$\dot{\mu}_1 = -(\alpha_2 \frac{\partial g_1}{\partial x} + p_2 \frac{\partial g_2}{\partial x} + \mu_1(t) \frac{\partial f}{\partial x}); \quad \mu_1(T) = \alpha_2 h_1'(x(T)) + p_2 h_2'(x(T)). \quad (15)$$

Adding (13) and (14) yields

$$\begin{aligned} (p_1 + \alpha_2) g_1(t, \hat{x}, \hat{u}) + (p_2 + \alpha_1) g_2(t, \hat{x}, \hat{u}) + (\lambda_1(t) + \mu_1(t)) f(t, \hat{x}, \hat{u}) &\leq \\ (p_1 + \alpha_2) g_1(t, \hat{x}, u) + (p_2 + \alpha_1) g_2(t, \hat{x}, u) + (\lambda_1(t) + \mu_1(t)) f(t, \hat{x}, u) & \end{aligned} \quad (16)$$

Addition of  $\lambda_1(t)$  from (12) to  $\mu_1(t)$  from (15) shows that  $\tilde{\lambda}(t) := \lambda_1(t) + \mu_1(t)$  satisfies

$$\dot{\tilde{\lambda}}(t) = -[(p_1 + \alpha_2)\frac{\partial g_1}{\partial x} + (p_2 + \alpha_1)\frac{\partial g_2}{\partial x} + \tilde{\lambda}(t)\frac{\partial f}{\partial x}]; \quad \tilde{\lambda}(T) = (p_1 + \alpha_2)h'_1 + (p_2 + \alpha_1)h'_2. \quad (17)$$

Next notice that if  $p_1 = \alpha_1 = 0$  it follows straightforwardly from (12) that  $\lambda_1(t) = \lambda_2(t) = p_1 = 0$ , which violates the maximum principle condition (i). So either  $p_1$  or  $\alpha_1$  differs from zero (and similarly either  $p_2$  or  $\alpha_2$  is larger than zero too). Therefore  $p_1 + p_2 + \alpha_1 + \alpha_2 > 0$ . Introducing finally  $\alpha := \frac{p_1 + \alpha_2}{p_1 + p_2 + \alpha_1 + \alpha_2}$  and  $\lambda(t) := \frac{\tilde{\lambda}(t)}{p_1 + p_2 + \alpha_1 + \alpha_2}$ , one obtains directly by division of (16) and (17) by  $p_1 + p_2 + \alpha_1 + \alpha_2$  the conditions (9-11).  $\square$

**Remark 2.10** Notice that the above necessary conditions in Theorem 2.9 are closely related to the minimization of

$$\alpha J_1 + (1 - \alpha)J_2 \quad \text{subject to (30)}. \quad (18)$$

By considering the Hamiltonian for this problem  $H := \alpha J_1 + (1 - \alpha)J_2 + \lambda f$ , we obtain from the maximum principle in particular the conditions stated in Theorem 2.9. Unfortunately the maximum principle conditions just provide necessary conditions. So, in case all conditions from Theorem 2.9 are met, we still can not conclude (in general) that problem (18) will have a solution.  $\square$

**Remark 2.11** From the proof of Theorem 2.9 it is clear that without any complications one can also consider problem (1,2) subject to inequality constraints  $m_j(x(t), u_1(t), u_2(t)) \leq 0$ , where  $m_j$  is continuously differentiable in all its arguments,  $j = 1, \dots, k$ . By restricting the set of admissible controls to those for which these inequalities are satisfied and under the assumption that the constraint qualification<sup>2</sup> is met, the following necessary conditions result.

Assume  $(J_1(\hat{u}), J_2(\hat{u}))$  is a Pareto solution for this problem. Then, there exists an  $\alpha \in [0, 1]$ , a (continuous and piecewise continuously differentiable) costate function  $\lambda^T(t) : [0, T] \rightarrow \mathbb{R}^n$  and (continuous) nonnegative Lagrange parameters  $\mu_i(t)$  such that, with the Hamiltonian  $H$  as in Theorem 2.9,  $\hat{u}$  satisfies

$$H(t, \hat{x}(t), \hat{u}(t), \lambda(t)) \leq H(t, \hat{x}(t), u(t), \lambda(t)), \quad \text{at each } t \in [0, T] \text{ for all} \quad (19)$$

$u$  satisfying  $m_j(\hat{x}(t), u_1(t), u_2(t)) \leq 0$ ;

$$0 = \alpha \frac{\partial g_1}{\partial u} + (1 - \alpha) \frac{\partial g_2}{\partial u} + \lambda(t) \frac{\partial f}{\partial u} + \sum_{j=1}^k \mu_j(t) \frac{\partial m_j}{\partial u}; \quad (20)$$

$$\dot{\lambda}(t) = -[\alpha \frac{\partial g_1}{\partial x} + (1 - \alpha) \frac{\partial g_2}{\partial x} + \lambda(t) \frac{\partial f}{\partial x} + \sum_{j=1}^k \mu_j(t) \frac{\partial m_j}{\partial x}], \quad (21)$$

$$\lambda(T) = \frac{\partial(\alpha h_1 + (1 - \alpha)h_2)}{\partial x};$$

$$\dot{\hat{x}}(t) = f(t, \hat{x}(t), \hat{u}_1(t), \hat{u}_2(t)), \quad x(0) = x_0. \quad (22)$$

$$\mu_j(t) \geq 0; \quad \mu_j(t)m_j(\hat{x}(t), \hat{u}_1(t), \hat{u}_2(t)) = 0; \quad m_j(\hat{x}(t), \hat{u}_1(t), \hat{u}_2(t)) \leq 0. \quad (23)$$

$\square$

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<sup>2</sup>The most convenient constraint qualification is the following rank condition. If  $p$  of the inequalities are satisfied with equality then the matrix of partial derivatives of these  $p$  constraints w.r.t.  $u(t)$  must have rank  $p$ .

Next we proceed with the derivation of some sufficient conditions for a control function to be Pareto efficient. The first well-known result (see e.g. [5, Theorem 6.4]) states that under convexity assumptions on the performance functions one can derive all Pareto efficient controls from the minimization of the parametrized optimal control problem (18). This property will be exploited in Section 4 to obtain for a specific class of linear quadratic differential games both necessary and sufficient conditions for existence of Pareto efficient controls.

**Theorem 2.12** *If  $\mathcal{U}$  is convex<sup>3</sup> and  $J_i(u)$  is convex for all  $i = 1, \dots, N$ , then for all Pareto efficient  $\hat{u}$  there exist  $\alpha \in \mathcal{A}$ , such that*

$$\hat{u} \in \arg \min_{u \in \mathcal{U}} \left\{ \sum_{i=1}^N \alpha_i J_i(u) \right\}. \quad \square$$

The next result gives sufficient conditions under which one can conclude from a solution of (9-11) that it will be Pareto efficient. The conditions and proof are inspired by Arrow's theorem.

**Theorem 2.13** *Let Assumptions 2.8 be satisfied. Assume there exist an  $\alpha \in (0, 1)$ , a costate function  $\lambda^{*T}(t) : [0, T] \rightarrow \mathbb{R}^n$ ,  $u^*$  and  $x^*$  that satisfy (9-11). Introduce the Hamiltonian  $H(t, x, u, \lambda^*) := \alpha g_1 + (1 - \alpha)g_2 + \lambda^* f$ . Assume that  $H(t, x, u, \lambda^*)$  has a minimum w.r.t.  $u$  for all  $x$ . Let  $H^0(t, x, \lambda^*) := \min_u H(t, x, u, \lambda^*)$ .*

*Then, if both  $H^0(t, x, \lambda^*)$  and  $h(x) := \alpha h_1(x) + (1 - \alpha)h_2(x)$  are convex in  $x$ ,  $u^*$  is Pareto efficient.*

**Proof.** Let  $\alpha$  be as in Theorem 2.9 and  $J(u) := \alpha J_1(u) + (1 - \alpha)J_2(u)$ . Then,

$$\begin{aligned} J(u^*) - J(u) &= \int_0^T \{H(t, x^*, u^*, \lambda^*) - \lambda^*(t)\dot{x}^*(t) - (H(t, x, u, \lambda^*) - \lambda^*(t)\dot{x}(t))\} dt + h(x^*(T)) - h(x(T)) \\ &= \int_0^T \{H(t, x^*, u^*, \lambda^*) + \dot{\lambda}^*(t)x^*(t) - H(t, x, u, \lambda^*) + \dot{\lambda}^*(t)x(t)\} dt + h(x^*(T)) - h(x(T)) \\ &\quad + \lambda^*(T)(x(T) - x^*(T)), \quad (\text{i}) \end{aligned}$$

where the last equality follows by integration by parts. From the convexity assumption on  $H^0$  and the fact that  $H^0$  is differentiable (see [5, Lemma 4.7]) we get

$$H^0(t, x, \lambda^*) - H^0(t, x^*, \lambda^*) \geq \frac{\partial H^0}{\partial x}(t, x^*, \lambda^*)(x - x^*).$$

Since  $H^0(t, x^*, \lambda^*) = H(t, x^*, u^*, \lambda^*)$ , using (10), we can rewrite the above inequality as

$$\begin{aligned} 0 &\leq H^0(t, x, \lambda^*) - H(t, x^*, u^*, \lambda^*) + \dot{\lambda}^*(x - x^*) \\ &\leq H(t, x, u, \lambda^*) - H(t, x^*, u^*, \lambda^*) + \dot{\lambda}^*(x - x^*). \quad (\text{ii}) \end{aligned}$$

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<sup>3</sup>Note that if  $\mathcal{U}_i$  is convex also the Cartesian product  $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2$  is a convex set.

The last inequality is due to the fact that by definition of  $H^0$ ,  $H^0(t, x, \lambda^*) \leq H(t, x, u, \lambda^*)$  for every choice of  $u$ .

On the other hand it follows from (10) and the convexity of  $h$

$$\begin{aligned} h(x^*(T)) - h(x(T)) + \lambda^*(T)(x(T) - x^*(T)) &= h(x^*(T)) - h(x(T)) + \frac{\partial h}{\partial x}(x^*(T))(x(T) - x^*(T)) \\ &\leq 0. \quad (\text{iii}) \end{aligned}$$

From the inequalities (ii) and (iii) it is then obvious that  $J(u^*) - J(u) \leq 0$  in (i). So,  $\alpha J_1 + (1 - \alpha)J_2$  is minimized at  $u^*$ . So according to Lemma 2.1  $u^*$  is Pareto efficient.  $\square$

**Remark 2.14** For the constrained problem considered in Remark 2.11 the conditions (20-23) are also sufficient to conclude that  $\hat{u}$  is Pareto efficient if  $h(x)$  is convex and either i)  $H^0(x, \lambda, t) := \min_{\{u | m_j(x, u) \leq 0\}} H(t, x, u, \lambda)$  (as defined in Theorem 2.13) exists and is convex on the convex hull of the set  $\mathcal{B} := \{x \mid \text{for some } u, m_j(x, u) \leq 0, j = 1, \dots, k\}$  or ii) the Hamiltonian  $H(t, x, u, \lambda^*)$  is simultaneously convex in  $(x, u)$  and the constraints  $m_j(x, u)$  are simultaneously (quasi-)convex in  $(x, u)$ . Details and extensions on this point can be found in [20, Chapter 4.3] or [21].  $\square$

We illustrate some of the theory presented above in an example that was presented in [13, Example 3.5].

**Example 2.15** Consider the following advertising game of two competing divisions in a conglomerate company which wish to maximize their individual profits by choosing an optimal advertising policy. Being divisions of a parent company, they are not out to "hurt" each other; thus, a cooperative game solution seems reasonable. With  $x_i$  the gross revenue of the  $i^{\text{th}}$  division and  $u_i$  the rate of expenditure for advertising it is assumed that the changes in the gross revenues are given by

$$\dot{x}_1(t) = 12u_1(t) - 2u_1^2(t) - x_1(t) - u_2(t) \quad (24)$$

$$\dot{x}_2(t) = 12u_2(t) - 2u_2^2(t) - x_2(t) - u_1(t) \quad (25)$$

with  $x_i(0) = x_{i0}$  and  $u_i(t) \geq 0$ .

The profits are

$$J_i = \int_0^1 \left\{ \frac{1}{3}x_i(t) - u_i(t) \right\} dt, \quad i = 1, 2,$$

and these are to be maximized for both players.

The with this problem corresponding Hamiltonian is

$$H := -\alpha \left( \frac{x_1}{3} - u_1 \right) - (1 - \alpha) \left( \frac{x_2}{3} - u_2 \right) + \lambda_1 (12u_1 - 2u_1^2 - x_1 - u_2) + \lambda_2 (12u_2 - 2u_2^2 - x_2 - u_1).$$

From Theorem 2.9 we conclude that every Pareto efficient control  $\hat{u} \geq 0$  satisfies for some  $\alpha \in [0, 1]$

$$H(t, \hat{x}(t), \hat{u}(t), \lambda_1(t), \lambda_2(t)) \leq H(t, \hat{x}(t), u(t), \lambda_1(t), \lambda_2(t)), \text{ at each } t \in [0, 1], \quad (26)$$

$$\dot{\lambda}_1(t) = \lambda_1(t) + \frac{\alpha}{3}; \quad \lambda_1(1) = 0, \quad (27)$$

$$\dot{\lambda}_2(t) = \lambda_2(t) + \frac{1 - \alpha}{3}; \quad \lambda_2(1) = 0, \quad (28)$$

and the system equations (24,25).

It is straightforwardly verified that for  $\alpha \in (0, 1)$  we obtain the actions from which Leitmann showed in [13, Example 3.5] that they are Pareto efficient. That is:

$$\begin{aligned} \text{if } \alpha > \alpha_1, \quad \hat{u}_1 &= \begin{cases} \frac{3}{4}(e^{t-1} - 1)^{-1} + 3 - \frac{1-\alpha}{4\alpha} & \text{for } t \in [0, \tau_1] \\ 0 & \text{for } t \in (\tau_1, 1] \end{cases} \\ \text{if } 0 < \alpha \leq \alpha_1, \quad \hat{u}_1 &= 0, \end{aligned}$$

whereas

$$\begin{aligned} \text{if } 0 < \alpha < \alpha_2, \quad \hat{u}_2 &= \begin{cases} \frac{3}{4}(e^{t-1} - 1)^{-1} + 3 - \frac{\alpha}{4(1-\alpha)} & \text{for } t \in [0, \tau_2] \\ 0 & \text{for } t \in (\tau_2, 1] \end{cases} \\ \text{if } \alpha \geq \alpha_2, \quad \hat{u}_2 &= 0. \end{aligned}$$

Here  $\alpha_1 = 1/(3(e^{-1} - 1)^{-1} + 13)$ ,  $\alpha_2 = 1 - \alpha_1$ ,  $\tau_1 = 1 + \ln(1 + 3(\frac{1-\alpha}{\alpha} - 12)^{-1})$  and  $\tau_2 = 1 + \ln(1 + 3(\frac{\alpha}{1-\alpha} - 12)^{-1})$ .

On the other hand it is easily verified that if  $\alpha = 0$ ,  $\lambda_1(t) = 0$  and  $\lambda_2(t) = \frac{1-e^{t-1}}{3}$ . So  $H$  is minimized for  $\hat{u}_1(t) = 0$  and  $\hat{u}_2(t) = \max\{0, \frac{3}{4}(e^{t-1} - 1)^{-1} + 3\}$ . Obviously, the minimized Hamiltonian  $H^0$  is linear in the state variables  $x_i$ . So in particular  $H^0$  is a convex function of these variables. Therefore we conclude from Theorem 2.13 that the case  $\alpha = 0$  yields an appropriate solution too. Similarly one can also show that for  $\alpha = 1$  we obtain the Pareto efficient solution  $\hat{u}_1(t) = \max\{0, \frac{3}{4}(e^{t-1} - 1)^{-1} + 3\}$  and  $\hat{u}_2(t) = 0$ . From Theorem 2.9 again we conclude finally that there are no additional Pareto efficient controls than those advertized here.  $\square$

### 3 The General Linear Quadratic Case

In this section we consider the linear quadratic differential game. That is

$$J_i(u_1, u_2) := \int_0^T [x^T(t), u_1^T(t), u_2^T(t)] M_i \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} dt + x^T(T) Q_{iT} x(T), \quad i = 1, 2, \quad (29)$$

where  $M_i = \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix}$  is symmetric,  $R_i := \begin{bmatrix} R_{1i} & N_i \\ N_i^T & R_{2i} \end{bmatrix} \geq 0$ ,  $i = 1, 2$ , and  $x(t)$  is the solution of the linear differential equation

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), \quad x(0) = x_0. \quad (30)$$

Notice that we make no definiteness assumptions w.r.t. matrix  $Q_i$ . In this section we will assume that the planning horizon,  $T$ , is finite.

For notational convenience we introduce for  $\alpha \in [0, 1]$  the next matrices and vectors  $M := \alpha M_1 + (1 - \alpha) M_2$ ,  $Q := \alpha Q_1 + (1 - \alpha) Q_2$ ,  $R := \alpha R_1 + (1 - \alpha) R_2$ ,  $V := \alpha V_1 + (1 - \alpha) V_2$ ,  $W := \alpha W_1 + (1 - \alpha) W_2$ ,  $Q_T := \alpha Q_{1T} + (1 - \alpha) Q_{2T}$ ,  $B := [B_1 \ B_2]$ ,  $u^T := [u_1^T \ u_2^T]$  and  $G := \begin{bmatrix} A - BR^{-1} \begin{bmatrix} V^T \\ W^T \end{bmatrix} & -BR^{-1} B^T \\ -(Q + [V \ W] R^{-1} \begin{bmatrix} V^T \\ W^T \end{bmatrix}) & -A^T \end{bmatrix}$ .

For this special case Theorem 2.9 can be specialized as follows.

**Corollary 3.1** *If  $(J_1(\hat{u}), J_2(\hat{u}))$  is a Pareto solution for problem (29,30) then there exists an  $\alpha \in [0, 1]$  such that*

$$[\hat{x}^T(t), \hat{u}^T(t)]M \begin{bmatrix} \hat{x}(t) \\ \hat{u}(t) \end{bmatrix} + \lambda B\hat{u} \leq [\hat{x}^T(t), u^T(t)]M \begin{bmatrix} \hat{x}(t) \\ u(t) \end{bmatrix} + \lambda Bu \quad (31)$$

$$\dot{\lambda}(t) = -2\hat{x}^T(t)Q - 2\hat{u}^T \begin{bmatrix} V^T \\ W^T \end{bmatrix} - \lambda A; \quad \lambda(T) = 2\hat{x}^T(T)Q_T; \quad (32)$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{u}(t), \quad \hat{x}(0) = x_0. \quad (33)$$

In case  $\alpha$  is such that  $R_i > 0$ ,  $i = 1, 2$ , (i.e.  $R_i$  is positive definite), the above formulae can be equivalently rephrased as that every Pareto efficient control satisfies  $\hat{u}(t) = -R^{-1} \left( \begin{bmatrix} V^T \\ W^T \end{bmatrix} x(t) + B^T \tilde{\lambda}(t) \right)$ , where  $\tilde{\lambda}(t)$  is the solution of the set of linear differential equations:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{\lambda}}^T(t) \end{bmatrix} = G \begin{bmatrix} x(t) \\ \tilde{\lambda}^T(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ \tilde{\lambda}^T(T) \end{bmatrix} = \begin{bmatrix} x_0 \\ Q_T x(T) \end{bmatrix}. \quad (34)$$

□

In case  $R > 0$ , for all  $\alpha \in [0, 1]$ , it follows directly from this Corollary 4.5 that  $(0, 0)$  is the only potential Pareto solution if  $x_0 = 0$ . For in that case  $(x(\cdot), \lambda(\cdot)) = (0, 0)$  always solves the above set of differential equations (34).

Another property which readily follows from (34) is that if for some  $\alpha$  for both the initial states  $x_0$  and  $x_1$  there exists a  $\lambda_i(t)$  satisfying (34) then also for every linear combination of  $x_0$  and  $x_1$  (34) has a solution.

The above property hints to the property that if for both the initial states  $x_0$  and  $x_1$  there exists a Pareto solution also for every linear combination of these initial states a Pareto solution will exist. Unfortunately this property does not hold in general, as will be demonstrated in Example 3.12. However, the following linearity property does hold.

**Lemma 3.2** *Assume  $\hat{u}$  is a Pareto efficient control for (29,30). Then  $\lambda\hat{u}$  is a Pareto efficient solution for (29,30) with  $x(0) = \lambda x_0$ .*

**Proof.** Let  $x(t, x_0, u)$  denote the solution of (30). Then elementary calculations show that  $x(t, \lambda x_0, \lambda u) = \lambda x(t, x_0, u)$  and, consequently,  $J_i(\lambda x_0, \lambda u) = \lambda^2 J_i(x_0, u)$ .

From Lemma 2.2 we know that  $\hat{u}$  is Pareto efficient if and only if  $\hat{u}$  minimizes  $J_i(x_0, u)$  on the constrained set

$$\mathcal{U}_i(x_0, \hat{u}) := \{u \mid J_j(x_0, u) \leq J_j(x_0, \hat{u}), \quad j = 1, 2, \quad j \neq i\}, \quad \text{for } i = 1, 2. \quad (35)$$

We will next show that  $\lambda\hat{u}$  minimizes  $J_1(\lambda x_0, u)$  on the constrained set

$$\tilde{\mathcal{U}}_1 := \{u \mid J_2(\lambda x_0, u) \leq J_2(\lambda x_0, \lambda\hat{u})\}. \quad (36)$$

Let  $u \in \tilde{\mathcal{U}}_1$ . Then according (36)  $J_2(\lambda x_0, u) \leq J_2(\lambda x_0, \lambda \hat{u})$  or, equivalently,  $J_2(x_0, \frac{1}{\lambda}u) \leq J_2(x_0, \hat{u})$ . So, by (35),  $\frac{1}{\lambda}u \in \mathcal{U}_1$ . But this implies that  $J_1(x_0, \frac{1}{\lambda}u) \geq J_1(x_0, \hat{u})$  or, equivalently,  $J_1(\lambda x_0, u) \geq J_1(\lambda x_0, \lambda \hat{u})$ .

In a similar way one can show that  $\lambda \hat{u}$  also minimizes  $J_2(\lambda x_0, u)$  on the corresponding constrained set  $\tilde{\mathcal{U}}_2$ , from which the claim is obvious then.  $\square$

It is well-known that existence of a solution of the linear quadratic control problem for an arbitrary initial state is equivalent to the existence of a solution of an associated Riccati equation. For that reason we consider below the problem under which conditions for an arbitrary initial state (29,30) has a Pareto solution. We have the following two preliminary results.

**Lemma 3.3** *Consider the two-point boundary value problem*

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = G \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} x_0 \\ Q_T x(T) \end{bmatrix}. \quad (37)$$

Let  $[W_1 \ W_2] := [I \ 0]e^{-GT}$ . Then, (37) has a solution for every  $x_0$  if and only if  $U := W_1 + W_2 Q_T$  is invertible.

**Proof.** Obviously, (37) has a solution for every  $x_0$  if and only if  $\forall x_0$  the following differential equation has a solution

$$\dot{\tilde{x}} = G\tilde{x}, \quad \text{with} \quad \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tilde{x}(0) + \begin{bmatrix} 0 & 0 \\ -Q_T & I \end{bmatrix} \tilde{x}(T) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

Clearly, this is the case if and only if  $\forall x_0$  the following equation has a solution

$$\left( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -Q_T & I \end{bmatrix} e^{GT} \right) \tilde{x}(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

Or, equivalently,  $\forall x_0$  the equation below is solvable

$$\left( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{-GT} + \begin{bmatrix} 0 & 0 \\ -Q_T & I \end{bmatrix} \right) e^{GT} \tilde{x}(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

Or, stated differently, with  $z := e^{GT} \tilde{x}(0)$  the following equation has a solution  $\forall x_0$

$$\begin{bmatrix} W_1 & W_2 \\ -Q_T & I \end{bmatrix} z = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

Premultiplication of both sides of this equation by matrix  $\begin{bmatrix} I & -W_2 \\ 0 & I \end{bmatrix}$  shows then that the above equation is solvable if and only if matrix  $U$  is invertible.  $\square$

Whereas from [5, Corollary 5.13] we recall the following result.

**Lemma 3.4** Let  $G$  be as introduced before and  $U(t) := [I \ 0]e^{G(T-t)} \begin{bmatrix} I \\ Q_T \end{bmatrix}$ . Consider the with  $G$  corresponding Riccati differential equation

$$\dot{K}(t) = -A^T K(t) - K(t)A + (K(t)B + [V \ W])R^{-1}(K(t) + [V \ W]^T) - Q, \quad K(T) = Q_T. \quad (38)$$

Then, (38) has a solution on  $[0, T]$  if and only if  $U(t)$  is invertible on  $[0, T]$ .  $\square$

**Remark 3.5** According to the fundamental existence-uniqueness theorem of differential equations there exists a maximum time interval  $[0, T_1)$  where equation (38) has a unique solution. So we conclude from 2.1 and [5, Theorem 5.1] that for a planning horizon  $T < T_1$  for every initial state  $x_0$  the game has a Pareto solution.  $\square$

Using the above lemmas we obtain for the scalar case, that is the case that the dimension of the state variable  $x(t)$  is one, the following result.

**Theorem 3.6** Consider the scalar system with  $R_i > 0$ ,  $i = 1, 2$ . Let  $x_0 \neq 0$ . Then,

1. if (29,30) has a Pareto efficient control  $\hat{u}(x_0)$  there exists an  $\alpha \in [0, 1]$  such that  $\min \alpha J_1 + (1 - \alpha)J_2$  subject to (30) has a solution for all initial states  $x_0 \in \mathbb{R}$ .
2. if  $J_i$ ,  $i = 1, 2$ , attains a minimum then
  - i) for all  $\alpha \in [0, 1]$ ,  $\min \alpha J_1 + (1 - \alpha)J_2$  exists for all  $x_0$ ;
  - ii) all Pareto efficient controls of (29,30) are given by  $\{\hat{u} \mid \hat{u} = \arg \min \alpha J_1 + (1 - \alpha)J_2, \alpha \in [0, 1]\}$ .

**Proof.** "1." Assume  $(J_1(\hat{u}), J_2(\hat{u}))$  is a Pareto solution for problem (29,30) for some  $x_0 \neq 0$ . Then, according to Lemma 3.2, for every initial state problem (29,30) has a Pareto solution. Let  $x_0 \neq 0$  be fixed and  $\alpha$ ,  $x(t)$  and  $\lambda(t)$  a with  $\hat{u}(x_0)$  corresponding solution that satisfies (34). Then it is easily verified that for the initial state  $\mu x_0$ ,  $\alpha$ ,  $\mu x(t)$  and  $\mu \lambda(t)$  satisfy (34). In other words, for this  $\alpha$ ,  $\forall x_0$  (34) has a solution. So, by Lemma 3.3,  $U(t) := [I \ 0]e^{G(T-t)} \begin{bmatrix} I \\ Q_T \end{bmatrix}$  is invertible at  $t = 0$ . Furthermore, by Corollary 4.5,  $\hat{u}[t_0, T]$  is Pareto efficient for problem (29,30) if we consider  $J_i$  on the interval  $[t_0, T]$ , instead of  $[0, T]$ , with initial state  $x(t_0) = e^{Gt_0}x_0$ . Again it is easily verified that also in this case  $\alpha$ ,  $x(t)$  and  $\lambda(t)$  satisfy (34). So similar as before it follows that  $U(t_0)$  is invertible too. Since  $t_0 \in [0, T]$  was chosen arbitrarily we conclude from Lemma 3.4 that the Riccati differential equation (38) has a solution on  $[0, T]$ . But this implies (see e.g. [5, Theorem 5.1]) that the optimization problem  $\min \alpha J_1 + (1 - \alpha)J_2$  subject to (30) has a solution for all initial states  $x_0$ . It is easily verified that this solution is actually attained by  $\hat{u}(x_0)$ .

"2." Since  $J_i(\lambda x_0, \lambda u) = \lambda^2 J_i(x_0, u)$  and by assumption  $J_1$  and  $J_2$  have a minimum, it follows immediately from Lemma 2.7 that for all  $\alpha \in [0, 1]$  there exists a  $\bar{J}(x_0, \alpha)$  such that

$$\alpha J_1(\lambda x_0, \lambda u) + (1 - \alpha)J_2(\lambda x_0, \lambda u) = \lambda^2(\alpha J_1(x_0, u) + (1 - \alpha)J_2(x_0, u)) \geq \lambda^2 \bar{J}(x_0, \alpha).$$

From this it follows that for all  $\alpha \in [0, 1]$ , for all  $x_0$   $\inf \alpha J_1(x_0, u) + (1 - \alpha)J_2(x_0, u)$  exists. But this implies (see e.g. [5, p.182,183]) that actually the optimization problem  $\min \alpha J_1 + (1 - \alpha)J_2$  subject

to (30) has a unique solution for all  $x_0$ .

Part ii) follows directly from Part i) and Remark 2.6. □

**Remark 3.7**

1. As already noticed in the proof of Theorem 3.6.1 it follows directly from Lemma 3.2 that, in case the scalar game has a Pareto solution for some initial state different from zero, the game (29,30) has a Pareto solution for every initial state.
2. From the proof of Theorem 3.6 we can in fact conclude the following result. If for  $x_0 \neq 0$  there exists a Pareto solution and for  $\alpha_0 \in [0, 1]$  (34) has a solution with  $x(0) = x_0$ , then  $\min \alpha_0 J_1 + (1 - \alpha_0) J_2$  subject to (30) has a solution for all initial states  $x_0 \in \mathbb{R}$ .
3. The assumption that the minimum of  $J_i$  exists in Theorem 3.6.2 implies that  $J_i$  is convex (see [7, Theorems 2.8, 2.9]). Therefore, this part of the theorem is a special case of those dealt with in Section 4. □

The next example illustrates some of the subtleties of Theorem 3.6.

**Example 3.8** Consider the minimization of

$$J_i = \int_0^{\pi/2} \{-x^2 + \beta_i u^2\} dt \quad \text{subject to } \dot{x} = u, \quad x(0) = x_0, \quad i = 1, 2.$$

First consider the minimization of

$$J_1 \text{ subject to } \dot{x} = u, \quad x(0) = x_0. \tag{39}$$

In case  $\beta_1 = 1$ , it is well-known (see e.g. [5, Example 5.1]) that this problem (39) has a solution if and only if  $x_0 = 0$ . In case  $x_0 = 0$ ,  $u(\cdot) = \gamma \cos(t)$  yields for every  $\gamma$  the optimal value 0.

Furthermore, if  $\beta_1 > 1$ , the with this problem corresponding Riccati differential equation (38),  $\dot{k}(t) = \frac{1}{\beta_1} k^2(t) + 1$ ;  $k(\frac{\pi}{2}) = 0$ , has a solution on  $[0, \frac{\pi}{2}]$ . So in that case problem (39) has for every initial state  $x_0$  a solution.

Finally, in case  $\beta_1 < 1$  one can use e.g. the control sequence from [3, Remark 3.1.4] (which was used to show that for  $\beta_1 = 1$  this example has no solution in case  $x_0 \neq 0$ ) to construct also for  $x_0 = 0$  a control sequence for which  $J_1$  becomes negative (implying that  $\inf J_1$  does not exist), yielding the conclusion that for  $\beta_1 < 1$  for all initial states problem (39) has no solution.

Next we consider a consequence of these conclusions for cooperative games in relationship with Theorem 3.6.

Consider the case  $\beta_i = 1$ ,  $i = 1, 2$ . Then, obviously, for every  $\alpha \in [0, 1]$   $\alpha J_1 + (1 - \alpha) J_2 = J_1$ . From the above consideration we have that for all  $x_0 \neq 0$ , for all  $\alpha \in [0, 1]$ , this problem has no solution. So from Theorem 3.6.1 we conclude that this cooperative game has no Pareto solution if  $x_0 \neq 0$ . On the other hand we conclude from Lemma 2.1 and the above considerations that for  $x_0 = 0$  there does exist a Pareto solution, i.e.  $(0, 0)$ . □

A direct consequence of the above Theorem 3.6 is the following theorem.

**Theorem 3.9** *Consider the scalar system with  $R_i > 0$ ,  $i = 1, 2$ . Then for every  $x_0$  (29,30) has a Pareto efficient control  $\hat{u}(x_0)$  if and only if there exists an  $\alpha \in [0, 1]$  such that for every  $x_0$   $\min \alpha J_1 + (1 - \alpha)J_2$  subject to (30) has a solution.*

**Proof.** "  $\Rightarrow$  " In particular it follows that (29,30) has a Pareto efficient solution for some  $x_0 \neq 0$ . Theorem 3.6 yields then the advertized result.

"  $\Leftarrow$  " Since the minimum exists for all  $x_0$  it is well-known that the argument at which this minimum is attained is unique (see e.g. [5, Theorem 5.1]). Consequently Remark 2.6 yields then directly the conclusion.  $\square$

Using Theorem 3.9 we arrive then at the next procedure to find all Pareto efficient solutions for the scalar game.

**Algorithm 3.10** Using the notation introduced in the beginning of this section, consider the scalar system with  $R_i > 0$ ,  $i = 1, 2$ .

Let  $\hat{\mathcal{U}}$  be the set of all Pareto efficient controls for which for every  $x_0$  (29,30) has a Pareto solution. Next consider the Riccati differential equation

$$\begin{aligned} \dot{K}(t) &= -A^T K(t) - K(t)A + (K(t)B + [V(\alpha) W(\alpha)])R^{-1}(\alpha)(B^T K(t) + [V(\alpha) W(\alpha)]^T) - Q(\alpha), \\ K(T) &= Q_T(\alpha). \end{aligned} \quad (40)$$

Then all Pareto efficient controls  $\hat{u}(\alpha) \in \hat{\mathcal{U}}$  are obtained by determining all  $\alpha \in [0, 1]$  for which (40) has a solution  $K_\alpha(t)$  on  $[0, T]$ . More in particular, if (40) has a solution  $K_\alpha(t)$  on  $[0, T]$  then

$$\hat{u}(\alpha)(t) := -R^{-1}(\alpha)([V(\alpha) W(\alpha)]^T + B^T K_\alpha(t))\hat{x}(t),$$

where  $\hat{x}(t)$  solves the differential equation  $\dot{x}(t) = (A - BR^{-1}(\alpha)([V(\alpha) W(\alpha)]^T + B^T K_\alpha(t)))x(t)$ ,  $x(0) = x_0$ , yields a Pareto efficient control. The corresponding Pareto solution is obtained by determining  $(J_1(\hat{u}(\alpha)), J_2(\hat{u}(\alpha)))$ .  $\square$

**Example 3.11** Consider the cooperative game with

$$J_1 = \int_0^{\frac{\pi}{2}} \left\{ -x^2 + \frac{9}{10}u_1^2 + \frac{1}{10}u_2^2 \right\} dt \text{ and } J_2 = \int_0^{\frac{\pi}{2}} \left\{ -x^2 + \frac{1}{10}u_1^2 + \frac{9}{10}u_2^2 \right\} dt$$

subject to the system

$$\dot{x} = \frac{4}{10}(u_1 + u_2), \quad x(0) = x_0.$$

Here player  $i$  controls  $u_i$ . According to Algorithm 3.10 this game has for every initial state a Pareto efficient solution precisely for those  $\alpha \in [0, 1]$  for which the following Riccati differential equation has a solution on  $[0, \frac{\pi}{2}]$ :

$$\dot{k} = sk^2 + 1, \quad k\left(\frac{\pi}{2}\right) = 0, \quad \text{where } s := \frac{16}{(1 + 8\alpha)(9 - 8\alpha)}.$$

The solution of this differential equation is  $k(t) = \frac{1}{\sqrt{s}} \tan \sqrt{s}(t - \frac{\pi}{2})$ . So,  $k(t)$  exists on  $[0, \frac{\pi}{2}]$  if and only if  $\sqrt{s} < 1$ . It is easily verified that this is equivalent to the condition  $\alpha \in (\frac{1}{8}, \frac{7}{8})$ .

So, the set of all Pareto efficient controls is given by

$$\hat{u}(t) = -\frac{4k(t)}{10} \begin{bmatrix} \frac{10}{1+8\alpha} & 0 \\ 0 & \frac{10}{9-8\alpha} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \hat{x}(t),$$

where  $\hat{x}(t)$  solves  $\dot{x}(t) = -sk(t)x(t)$ ,  $x(0) = x_0$ , and  $\alpha \in (\frac{1}{8}, \frac{7}{8})$ .

To complete the picture we will finally analyze the case  $\alpha \in I := [0, \frac{1}{8}] \cup [\frac{7}{8}, 1]$ . From Remark 3.7.2 and Theorem 3.9 it follows that for all  $\alpha \in I$  there does not exist a  $x_0 \neq 0$  such that the game has a Pareto solution for this initial state. So, only for  $x_0 = 0$  for these  $\alpha \in I$  an additional Pareto solution might occur. However, as already noticed directly after Corollary 4.5, this leads to the only candidate control  $(0, 0)$ , from which we know already that it is Pareto efficient.  $\square$

The following example illustrates a two-dimensional state game where for all initial states in the interior of the second and fourth quadrant and the point  $(0, 0)$  a Pareto solution exists whereas for all other initial states there exists no Pareto solution.

**Example 3.12** Consider the cooperative minimization of

$$J_1 = \int_0^{\pi/2} \{x^T(t) \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} x(t) + u^2(t)\} dt \quad \text{and} \quad J_2 = \int_0^{\pi/2} \{x^T(t) \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix} x(t) + u^2(t)\} dt$$

subject to

$$\dot{x}_1 = u, \quad x_1(0) = p, \quad \dot{x}_2 = u, \quad x_2(0) = q.$$

Next consider the minimization of

$$\alpha J_1 + (1 - \alpha) J_2 = \int_0^{\pi/2} \{x^T(t) \begin{bmatrix} -3 + 2\alpha & 2 - \alpha \\ 2 - \alpha & -2 \end{bmatrix} x(t) + u^2(t)\} dt \quad (41)$$

subject to

$$\dot{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t); \quad x(0) = \begin{bmatrix} p \\ q \end{bmatrix}. \quad (42)$$

From the above set of state differential equations it follows directly that  $x_1(t) = x_2(t) + p - q$ . Substitution of this into (41) shows that the minimization of (41) subject to (42) is equivalent to the minimization of

$$\alpha J_1 + (1 - \alpha) J_2 = \int_0^{\pi/2} \{-(x_2(t) + (1 - \alpha)(p - q))^2 + u^2(t) + (-2 + \alpha^2)(p - q)^2\} dt \quad (43)$$

subject to

$$\dot{x}_2(t) = u(t); \quad x_2(0) = q. \quad (44)$$

From Example 3.8 it follows that this problem has a solution if and only if the initial state satisfies  $-(1 - \alpha)(p - q) = q$  or, equivalently,  $(1 - \alpha)p = -\alpha q$ . From this it is easily verified that for every initial state  $(p, q)$  of (42) there is exact one  $\alpha \in [0, 1]$  such that the minimization of (41) has a solution if  $(p, q)$  is located either in the second quadrant, the fourth quadrant, on the line  $p = 0$  (which corresponds with  $\alpha = 0$ ) or on the line  $q = 0$  (which corresponds with  $\alpha = 1$ ). So we conclude from Lemma 2.1 that for all the initial states  $(p, q)$  which are located in the interior of the second and fourth quadrant there is a Pareto solution.

Next consider the initial state  $(p, 0)$ . Then  $x_1(t) = x_2(t) + p$  and we can rewrite  $J_1$  as follows:

$$\begin{aligned} J_1 &= \int_0^{\frac{\pi}{2}} \{-(x_1(t) - x_2(t))^2 - x_2^2(t) + u^2(t)\} dt \\ &= \int_0^{\frac{\pi}{2}} \{-p^2 - x_2^2(t) + u^2(t)\} dt. \end{aligned}$$

From Example 3.8 we recall again that the minimum value of  $J_1$  is 0. Furthermore,  $u_2(t) := \gamma \cos(t)$  belongs to the with this problem corresponding constrained set  $\mathcal{U}_2$  (see Lemma 2.2). Using this control we see that

$$\begin{aligned} J_2 &= \int_0^{\frac{\pi}{2}} \{-2(x_1(t) - x_2(t))^2 - x_1^2(t) + u_2^2(t)\} dt \\ &= \int_0^{\frac{\pi}{2}} \{-2p^2 - (\gamma \sin(t) + p)^2 + \cos^2(t)\} dt \\ &= -\frac{3p^2\pi}{2} + 2p\gamma. \end{aligned}$$

Since  $\gamma$  is an arbitrary number it is clear from this that  $\min_{u \in \mathcal{U}_2} J_2$  does not exist if  $p \neq 0$ . So, by Lemma 2.2, for the initial state  $(p, 0)$ , there exists no Pareto efficient solution. Similarly it follows that also for the initial state  $(0, q)$  the problem has no Pareto efficient solution. Finally, it is easily verified that for the initial state  $(0, 0)$ , for every  $\gamma$ ,  $\hat{u} = \gamma \cos(t)$  is a Pareto efficient control yielding the same Pareto solution  $(0, 0)$ .

Obviously, the with problem (41,42) corresponding Riccati differential equation has for every  $\alpha \in [0, 1]$  not a solution on  $[0, \frac{\pi}{2}]$ .

Finally, consider matrix

$$G := \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix},$$

from (34). It is easily verified that

$$G = SJS^{-1}, \text{ with } S = \begin{bmatrix} 0 & 1/\alpha & 0 & \frac{-\alpha}{2-\alpha^2} \\ 0 & 1/\alpha & 0 & \frac{1-\alpha}{2-\alpha^2} \\ \frac{1-\alpha}{\alpha} & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So for the initial state  $x_0 = [p; q]^T$  there exists an  $\alpha \in [0, 1]$ ,  $x(\cdot)$ , and  $\lambda(\cdot)$  satisfying (34) if and only if there exists an  $\alpha \in [0, 1]$  and  $\bar{\lambda}_1, \bar{\lambda}_2$  such that the equation

$$e^{G\frac{\pi}{2}} [p; q; \bar{\lambda}_1; \bar{\lambda}_2]^T = [x_1(\frac{\pi}{2}); x_2(\frac{\pi}{2}); 0; 0]^T$$

has a solution. An elementary spelling of these equations shows that there exists a solution if and only if  $p(1 - \alpha) = -\alpha q$ . So, in particular we conclude from this that for all initial states located in either the interior of the first or third quadrant there exists no Pareto solution.  $\square$

## 4 The Convex Linear Quadratic Case

Inspired by the results of Lemma 2.1 and Theorem 2.12 we consider in this section in detail the case when the linear quadratic cost functions are convex. The performance criteria of the players is again given by (29). An additional assumption will be that  $R_i > 0$  throughout this section. On the other hand we consider in this section a slightly more general system, i.e.

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) + c(t), \quad x(0) = x_0. \quad (45)$$

The variable  $c(\cdot) \in L_2$  is some given trajectory. The planning horizon,  $T$ , may be either finite or infinite.

Note, that whenever  $J_i$  are convex also  $\sum_{i=1}^N \alpha_i J_i(\gamma)$  is convex for an arbitrary  $\alpha \in \mathcal{A}$ .

Next we consider some sets of control functions that are relevant in our problem setting. It can be easily shown that each of these (nonempty<sup>4</sup>) sets is convex.

**Lemma 4.1** *Let  $\mathcal{U}$  be given by either:*

- 1)  $L_2[0, T] := \{(u_1, u_2) \mid u_i(\cdot) \text{ are square integrable functions on } [0, T]\}$ .
- 2)  $\{u \in L_2[0, T] \mid x(0) = x(T) = 0 \text{ in } \dot{x}(t) = Ax(t) + Bu(t)\}$ .
- 3)  $L_{2,e}^+ := \{(u_1, u_2) \mid u_i(\cdot) \in L_{2,loc} \text{ and } \lim_{T \rightarrow \infty} J_i(x_0, u) \text{ exists in } \mathbb{R} \cup \{-\infty, \infty\}, \forall x_0\}$ , where  $L_{2,loc}$  is the set of locally square-integrable functions, i.e.,

$$L_{2,loc} = \{u \mid \forall T > 0, \int_0^T u^T(s)u(s)ds < \infty\}.$$

- 4)  $L_{2,e,s}^+ := \{(u_1, u_2) \mid (u_1, u_2) \in L_{2,e}^+ \text{ and } \lim_{t \rightarrow \infty} x(t) = 0 \text{ in (45)}\}$ .

*Then each of these set of control functions is convex.*  $\square$

Next we consider the question under which conditions the cost functions  $J_i$  in (29) are convex. To that end we first derive some preliminary results.

**Lemma 4.2** *Assume  $\mathcal{U}$  is convex. Consider the linear quadratic cost function*

$$J(s, T, u, x_0) = \int_s^T \{[x^T(t) \ u^T(t)]M \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + 2[p_1^T(t) \ p_2^T(t)] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + c_2(t)\}dt + x^T(T)Q_Tx(T) \quad (46)$$

*subject to the state dynamics*

$$\dot{x}(t) = Ax(t) + Bu(t) + c_1(t), \quad x(s) = x_0, \quad (47)$$

---

<sup>4</sup>assuming for case 2) and 3) that  $(A, [B_1 \ B_2])$  is stabilizable

where  $u$ ,  $p_i$  and  $c_i$  are such that (46) and (47) have a solution;  $M = \begin{bmatrix} Q & V \\ V^T & R \end{bmatrix}$ .

Let  $x_0 \in \mathbb{R}^N$ . Then,  $J(0, T, u, x_0)$  is convex as a function of  $u$  if and only if  $\bar{J}(0, T, v, 0) \geq 0$  for all  $v$ , where

$$\bar{J}(s, T, v, 0) = \int_s^T [z^T(t) \ v^T(t)] M \begin{bmatrix} z(t) \\ v(t) \end{bmatrix} dt + z^T(T) Q_T z(T) \text{ with } \dot{z}(t) = Az(t) + Bv(t), \ z(s) = 0. \quad (48)$$

**Proof:** Let  $x_u(t)$  denote the state trajectory of (47) in case the control  $u(\cdot)$  is used. Then it is well-known that due to the linearity of the system

$$x_{\lambda u + (1-\lambda)w}(t) = \lambda x_u(t) + (1-\lambda)x_w(t). \quad (49)$$

From the definition of convexity (see (51)) it follows then that  $J(0, T, u, x_0)$  is convex if and only if

$$\begin{aligned} & \int_0^T \{ [x_{\lambda u + (1-\lambda)w}^T(t) \ (\lambda u + (1-\lambda)w)^T(t)] M \begin{bmatrix} x_{\lambda u + (1-\lambda)w}(t) \\ (\lambda u + (1-\lambda)w)(t) \end{bmatrix} + \\ & \quad 2[p_1^T(t) \ p_2^T(t)] \begin{bmatrix} x_{\lambda u + (1-\lambda)w}(t) \\ (\lambda u + (1-\lambda)w)(t) \end{bmatrix} + c_2(t) \} dt x_{\lambda u + (1-\lambda)w}^T(t) \\ & \leq \int_0^T \{ \lambda [x_u^T(t) \ u(t)] M \begin{bmatrix} x_u(t) \\ u(t) \end{bmatrix} + 2[p_1^T(t) \ p_2^T(t)] \begin{bmatrix} x_u(t) \\ u(t) \end{bmatrix} + c_2(t) \} + \\ & (1-\lambda) \{ [x_w^T(t) \ w(t)] M \begin{bmatrix} x_w(t) \\ w(t) \end{bmatrix} + 2[p_1^T(t) \ p_2^T(t)] \begin{bmatrix} x_w(t) \\ w(t) \end{bmatrix} + c_2(t) \} dt + \\ & \quad \lambda x_u^T(T) Q_T x_u(T) + (1-\lambda) x_w^T(T) Q_T x_w(T). \end{aligned}$$

Or equivalently, using (49), we obtain after some rewriting

$$\begin{aligned} & \lambda(1-\lambda) \left( \int_0^T \{ [x_u^T(t) - x_w^T(t) \ u^T(t) - w^T(t)] M \begin{bmatrix} x_u(t) - x_w(t) \\ u(t) - w(t) \end{bmatrix} \} dt + \right. \\ & \quad \left. (x_u^T(T) - x_w^T(T)) Q_T (x_u^T(T) - x_w^T(T)) \right) \geq 0, \end{aligned}$$

where

$$\dot{x}_u(t) = Ax_u(t) + Bu(t) + c(t), \ x_u(0) = x_0 \text{ and } \dot{x}_w(t) = Ax_w(t) + Bw(t) + c(t), \ x_w(0) = x_0.$$

With  $z := x_u - x_w$  and  $v := u - w$  the stated result then follows immediately.  $\square$

### Remark 4.3

1. Note that the second part of the equivalence does not depend on  $x_0$ . In particular it follows from this that if  $J$  is convex for one  $x_0$  then  $J$  is convex for all  $x_0$ . This property can also be verified by a direct elaboration of the convexity definition using the linearity property of the system again.

2. Consider a linear system in its state controllable canonical form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t), \quad x(0) = x_0, \quad \text{with } (A_1, B_1) \text{ controllable and} \\ J &= \int_0^T \{x^T(t) \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} x(t) + u^T(t) R u(t)\} dt + x^T(T) \begin{bmatrix} Q_{1T} & Q_{2T} \\ Q_{2T}^T & Q_{3T} \end{bmatrix} x(T). \end{aligned}$$

Then it follows immediately from Lemma 4.2 that  $J$  is convex if and only if the controllable part of this system is convex. That is if, with

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t), \quad x_1(0) = x_{10}, \text{ and} \\ J_1 &= \int_0^T \{x_1^T(t) Q_1 x_1(t) + u^T(t) R u(t)\} dt + x_1^T(T) Q_{1T} x_1(T), \end{aligned}$$

$J_1$  is convex. □

**Lemma 4.4** *Assume  $\mathcal{U}$  is convex. Consider the linear quadratic cost function (46) and (47). Then, if  $J(0, T, u, x_0)$  is convex for some  $x_0$ ,  $J(s, T, u, x_0)$  is convex for all  $s \geq 0$  and for all  $x_0$ .*

**Proof:** Let  $s > 0$ . Let  $u_i(t) = v(t) + w_i(t)$ , where  $v(t) = 0$ ,  $t \geq s$  and  $w_i(t) = 0$ ,  $0 \leq t \leq s$ . Then,

$$J(0, T, u_i, x_0) = J(0, s, v, x_0) + J(s, T, w_i, x(s, x_0)). \quad (50)$$

Since  $J(0, T, u, x_0)$  is convex, by definition, for an arbitrary  $u$  and  $w$  and  $\lambda \in (0, 1)$

$$J(0, T, \lambda u + (1 - \lambda)w, x_0) \leq \lambda J(0, T, u, x_0) + (1 - \lambda)J(0, T, w, x_0). \quad (51)$$

So in particular if we choose  $u = u_1$  and  $w = u_2$  as above we get, using (50),

$$\begin{aligned} J(0, s, v, x_0) + J(s, T, \lambda w_1 + (1 - \lambda)w_2, x(s, x_0)) &= J(0, T, \lambda u_1 + (1 - \lambda)u_2, x_0) \\ &\leq \lambda J(0, T, u_1, x_0) + (1 - \lambda)J(0, T, u_2, x_0) \\ &= \lambda(J(0, s, v, x_0) + J(s, T, w_1, x(s, x_0))) + (1 - \lambda)(J(0, s, v, x_0) + J(s, T, w_2, x(s, x_0))) \\ &= J(0, s, v, x_0) + \lambda J(s, T, w_1, x(s, x_0)) + (1 - \lambda)J(s, T, w_2, x(s, x_0)). \end{aligned}$$

Comparing both sides of this inequality shows then that  $J(s, T, w, x(s, x_0))$  is convex. The rest of the statement follows then directly from Remark 4.3, item 1. □

**Corollary 4.5** *Consider the linear quadratic cost function (46) and (47). Then,  $J(s, T, u, x_0)$  is convex for all  $s \geq 0$  and  $x_0$  if and only if  $J(0, T, u, x_0)$  is convex for all/an  $x_0$ , which holds if and only if  $\bar{J}(0, T, v, 0) \geq 0$  for all  $v$ , where  $\bar{J}$  is given by (48). □*

**Theorem 4.6** *Assume either  $T$  is finite or  $(A, B)$  is stabilizable. Then  $J(0, T, u, x_0)$  is convex for all/an  $x_0$  if and only if  $\inf \bar{J}(0, T, v, 0)$  exists.*

**Proof:**

"  $\Rightarrow$  " From Corollary 4.5 it follows that if  $J(0, T, u, x_0)$  is convex  $\bar{J}(0, T, v, 0) \geq 0$  for all  $v$ . In case  $T$  is finite obviously  $\inf \bar{J}(0, T, v, 0) \leq J(0, T, 0, 0)$  whereas in case  $(A, B)$  is stabilizable,  $\inf \bar{J}(0, \infty, v, 0)$  is bounded from above too. Consequently,  $\inf \bar{J}(0, T, v, 0)$  exists.

"  $\Leftarrow$  " Assume  $\inf \bar{J}(0, T, v, 0) = m$ . Then  $m \geq 0$ . For if  $m < 0$  there would exist a  $\bar{v}$  such that  $\bar{J}(0, T, \bar{v}, 0) < 0$ . But from this it follows directly from the linearity of the system that  $\bar{J}(0, T, \lambda \bar{v}, 0) = \lambda^2 \bar{J}(0, T, \bar{v}, 0)$ . From which it is clear that  $\inf \bar{J}(0, T, v, 0)$  would not exist. So  $\bar{J}(0, T, v, 0) \geq 0$  for all  $v$ . This implies, see Corollary 4.5 again, that  $J(0, T, u, x_0)$  is convex for all/an  $x_0$ .  $\square$

Next we consider the case  $R > 0$  in  $M$ . It is well-known that then the next Riccati equations play an important role

$$\dot{K}(t) = A^T K(t) + K(t)A - (K(t)B + V)R^{-1}(B^T K(t) + V^T) + Q, \quad K(T) = Q_T; \quad (\text{DRE})$$

$$0 = A^T K + KA - (KB + V)R^{-1}(B^T K + V^T) + Q. \quad (\text{ARE})$$

Let  $\Gamma$  denote the set of all symmetric solutions of (ARE). From, e.g., [5] and [27] (see also [18], [11], [30], [14] [15] and [28]) we recall the next results. Notice that these results are formulated in terms of existence of the performance function for an arbitrary initial state. In the above cited references one can also find conditions for the existence of  $\inf \bar{J}(0, T, v, 0)$ . Since the formulation of these conditions is more involved and is somewhat outside the main scope of this note they are not presented here.

**Theorem 4.7** *Consider the linear quadratic cost function*

$$J := \int_0^T [x^T(t) \ u^T(t)] M \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + x^T(T) Q_T x(T) \text{ with } \dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0. \quad (52)$$

Then the following holds:

1. Let  $T < \infty$  and  $\mathcal{U} = L_2[0, T]$ . Then,  $\inf_u J$  exists for all  $x_0$  if and only if (DRE) has a solution on  $[0, T]$ . Moreover, if this infimum exists it is in fact a minimum. This  $\min_u J(u) = x_0^T K(0) x_0$  is attained uniquely by  $u^*(t) = -R^{-1}(B^T K(t) + V)x^*(t)$ , where  $K(t)$  is the unique symmetric solution of (DRE) and  $x^*(\cdot)$  solves  $\dot{x}(t) = (A - BR^{-1}(B^T K(t) + V))x(t)$ ,  $x^*(0) = x_0$ .
2. Let  $T = \infty$ ,  $\mathcal{U} = L_{2,e,s}^+$  and  $(A, B)$  be stabilizable. Then,  $\inf_u J$  exists for all  $x_0$  if and only if  $\Gamma \neq \emptyset$ . Moreover, there exists a  $u^*$  attaining this infimum if and only if (ARE) has a stabilizing solution  $K^+$ . Under this condition  $\min_u J(u) = x_0^T K^+ x_0$  is attained uniquely by  $u^*(t) = -R^{-1}(B^T K^+ + V)x^*(t)$ , where  $x^*(\cdot)$  solves  $\dot{x}(t) = (A - BR^{-1}(B^T K^+ + V))x(t)$ ,  $x^*(0) = x_0$ .
3. Let  $Q_T = 0$ ,  $T = \infty$ ,  $\mathcal{U} = L_{2,e}^+$  and  $(A, B)$  be controllable. Then,  $\inf_u J$  exists for all  $x_0$  if (ARE) has a symmetric solution  $K \leq 0$ . Moreover, if this condition is satisfied there exists a  $u^*$  attaining this infimum if and only if  $\Delta \subset K^-$ . Here  $K^-$  is the smallest solution of  $\Gamma$  and  $\Delta := K^+ - K^-$ , where  $K^+$  is the largest solution of  $\Gamma$ . In that case  $\min_u J(u) = x_0^T K_f x_0$  is attained uniquely by  $u^*(t) = -R^{-1}(B^T K_f + V)x^*(t)$ , where  $x^*(\cdot)$  solves  $\dot{x}(t) = (A - BR^{-1}(B^T K_f + V))x(t)$ ,  $x^*(0) = x_0$  and  $K_f = K^- P_N + K^+(I - P_N)$ . Here  $P_N$  is the

projector onto the subspace  $N$  along  $\Delta^{-1}N^\perp$  with  $N$  the undetectable (w.r.t.  $\mathbb{C}^- \cup \mathbb{C}^0$ ) subspace of  $(K^-, (A - BR^{-1}(B^TK^- + V)))$ .  $\square$

**Remark 4.8** If (ARE) has a solution  $K \geq 0$  then  $K_f$  in Theorem 4.7.3) is the smallest positive semi-definite solution of (ARE).

If one merely assumes  $(A, B)$  to be stabilizable instead of controllable in Theorem 4.7.3) (ARE) has in general not a smallest solution anymore. It can still be shown under the same condition as in 3) (that (ARE) has a solution  $K \leq 0$ ) that the infimum exists and equals  $x_0^TK_fx_0$  for some  $K_f \in \Gamma$ . However there does not exist a nice characterization like in 3) of  $K_f$  (see [10]).  $\square$

Next introduce  $B := [B_1 \ B_2]$ . Combining the results of Lemmas 2.1, 2.12 and Theorems 4.6, 4.7 one can derive now straightforwardly existence and computational algorithms for both the finite and infinite horizon game problems. We will just state the result for the infinite planning horizon case where the state converges to zero. The formulation of the other results is left to the reader.

**Corollary 4.9** Consider the cooperative game (29,45) with  $T = \infty$ ,  $\mathcal{U} = L_{2,e,s}^+$  and  $(A, B)$  stabilizable.

Assume (ARE), with  $V := [V_i \ W_i]$  and  $R := \begin{bmatrix} R_{1i} & N_i \\ N_i^T & R_{2i} \end{bmatrix}$  has a stabilizing<sup>5</sup> solution  $K_i$ , for  $i = 1, 2$ , respectively.

Then all Pareto efficient solutions are obtained by determining for  $\alpha \in \mathcal{A}$

$$u^*(\alpha) := \arg \min_{u \in \mathcal{U}} \alpha_1 J_1 + \alpha_2 J_2, \text{ subject to (45)}. \quad (53)$$

The corresponding Pareto solutions are

$$(J_1(u^*(\alpha)), J_2(u^*(\alpha))). \quad \square$$

Notice that, since by assumption both cost functions are strict convex, the minimization problem (53) has for all  $\alpha \in \mathcal{A}$  a unique solution. From this result one obtains then the next procedure to calculate all Pareto efficient outcomes for this game.

#### Theorem 4.10 (Solution Cooperative Game)

Consider the cooperative game (29,45) with  $T = \infty$ ,  $\mathcal{U} = L_{2,e,s}^+$  and  $(A, B)$  stabilizable.

For  $\alpha \in \mathcal{A}$  let

$$M(\alpha) := \alpha_1 M_1 + \alpha_2 M_2 =: \begin{bmatrix} \tilde{Q} & \tilde{V} \\ \tilde{V}^T & \tilde{R} \end{bmatrix},$$

where

$$\tilde{Q} = \alpha_1 Q_1 + \alpha_2 Q_2, \quad \tilde{V} = [\alpha_1 V_1 + \alpha_2 V_2, \alpha_1 W_1 + \alpha_2 W_2], \quad \text{and} \quad \tilde{R} = \alpha_1 \begin{bmatrix} R_{11} & N_1 \\ N_1^T & R_{21} \end{bmatrix} + \alpha_2 \begin{bmatrix} R_{12} & N_2 \\ N_2^T & R_{22} \end{bmatrix}.$$

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<sup>5</sup>That is,  $\sigma(A - BR^{-1}V^T - SK_i) \subset \mathbb{C}^-$ .

Furthermore, let  $\tilde{S} := B\tilde{R}^{-1}B^T$ .

Assume that (54) below has a stabilizing solution  $X_i$  for  $\alpha_i = 1$ ,  $i = 1, 2$ , respectively.

$$A^T X + X A - (X B + \tilde{V})\tilde{R}^{-1}(B^T X + \tilde{V}^T) + \tilde{Q} = 0. \quad (54)$$

Then the set of all cooperative Pareto solutions is given by

$$\{(J_1(u^*(\alpha)), J_2(u^*(\alpha))) \mid \alpha \in \mathcal{A}\}.$$

Here

$$u^*(t) = -\tilde{R}^{-1}(B^T X_s + \tilde{V}^T)x(t) - \tilde{R}^{-1}B^T \int_t^\infty e^{-A_{cl}^T(t-s)} X_s c(s) ds$$

where  $X_s$  is the stabilizing solution of (54) and, with  $A_{cl} := A - B\tilde{R}^{-1}\tilde{V}^T - \tilde{S}X_s$ , the closed-loop system is  $\dot{x}(t) = A_{cl}x(t) - Sm(t) + c(t)$ ,  $x(0) = x_0$ . In case  $c(\cdot) = 0$  the with these actions corresponding cost are  $J_i(x_0, u^*) = x_0^T M_i x_0$ , where  $M_i$  is the unique solution of the Lyapunov equation

$$A_{cl}^T M_i + M_i A_{cl} = -[I, -(X_s B + \tilde{V})\tilde{R}^{-1}] M_i \begin{bmatrix} I \\ -\tilde{R}^{-1}(B^T X_s + \tilde{V}^T) \end{bmatrix}. \quad \square$$

**Example 4.11** To illustrate the above result consider the next simplistic environmental example. Consider a fishery management game, where two industries pollute a lake and their income depends on the fish-stock of the lake. Assume that both the revenues and cost depend quadratically on the fish-stock and the produced pollution. Within this context we consider the next model:

$$\dot{x}(t) = 2x(t) - u_1(t) - u_2(t), \quad x(0) = 1,$$

with cost functions

$$J_1 = \int_0^\infty -x^2(t) + 2u_1^2(t) + u_2^2(t) dt \quad \text{and} \quad J_2 = \int_0^\infty -2x^2(t) + u_1^2(t) + 2u_2^2(t) dt.$$

Here  $x$  describes the fish-stock and  $u_i$  is the pollution produced by industry  $i$ . The fact that industry 1 gives more weight to the pollution it produces than to the quantity produced by the other industry can be interpreted as that industry 1 is really concerned about the pollution it produces itself (and which it is able to control in contrast to the pollution produced by the other industry). In figure 1 we plotted the with this problem corresponding Pareto frontier if both players would cooperate in this game. □

## 5 The Cooperative Game

As we motivated in the introduction the set of Pareto solution(s) constitute a natural set of outcomes to consider when players cooperate. As we illustrated in the previous sections in a number of examples in a two-player game usually the Pareto frontier will be a one-dimensional surface in  $\mathbb{R}^2$ . However, this does not always have to be the case. This can, e.g., already be illustrated by a simple two-player

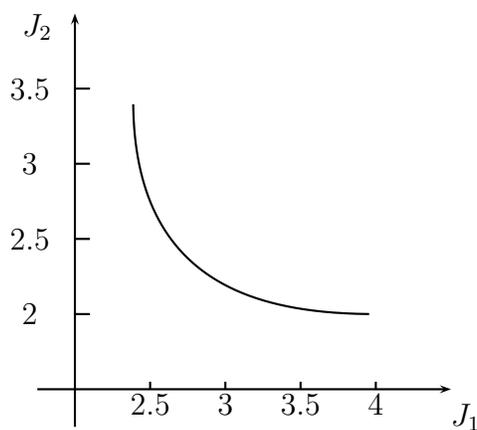


Figure 1: Pareto frontier Example 4.11.

scalar linear quadratic game. Assuming both players have the same cost function (with positive weights) it is easily verified that in that case the Pareto frontier reduces to a single point in  $\mathbb{R}^2$ .

Assuming that in a cooperative setting more than one solution exists, a natural question that arises is which solution of all these possibilities will be chosen by the players. Since we assume an open-loop information structure of the game we can use concepts developed within the area of static game theory to motivate the choice for a certain solution. Within the area of static cooperative game theory one usually distinguishes two types of games: the transferable utility (TU) games and the nontransferable utility (NTU) games<sup>6</sup>.

By working together in coalitions, players can generate benefits. A typical example is that of a number of firms cooperating in order to save costs. Not only is it interesting to know how players can cooperate in an optimal way, but also the problem of allocation arises. TU-games were introduced by [29] and can be seen as an allocation problem in which an amount of money is to be divided and can be freely transferred between the players. Here one abstracts from the fact that the involved players may value the obtained payoffs differently.

Generally, the analysis of a TU-game focuses on how to allocate the joint benefits/costs of the grand coalition of all players, where the values of subcoalitions serve as a benchmark. In order to allocate the value of the grand coalition several one-point solutions, each with its own appealing properties, have been proposed in the literature. As examples we mention the Shapley value [23], the compromise value [26] and the nucleolus [19]. An important concept in the allocation of the value of a grand coalition is core stability. The core of a (N)TU-game consists of those allocation vectors such that no subcoalition has an incentive to splitt off. Hence, if the TU-game measures the benefits of each coalition, an allocation is an element of the core if for each coalition its members receive together at least as much as they could achieve when they would act as a separate coalition.

NTU games, introduced by [2], on the other hand assume that there is no common measure of utility. That is, the objects to be divided are not valued in the same way by all the players. Consequently, no side-payments take place. Within the NTU-games one discerns the so-called bargaining problems. For these games one assumes that only the set of feasible outcomes for the individual players and a so-called threatpoint is specified. This threatpoint (or disagreement point) is a value every individual player can obtain by playing noncooperatively. Below we will elaborate three bargaining solutions which have been proposed in literature.

Bargaining theory has its origin in two papers by Nash [16] and [17]. In these papers a bargaining

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<sup>6</sup>Note that from a formal point of view a TU-game can be viewed as a special case of a NTU-game.

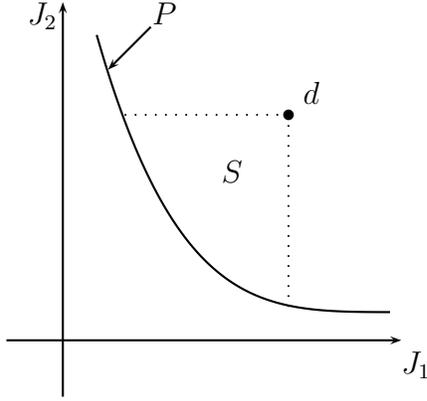


Figure 2: The bargaining game.

problem is defined as a situation in which two (or more) individuals or organizations have to agree on the choice of one specific alternative from a set of alternatives available to them, while having conflicting interests over this set of alternatives. Nash proposes in [17] two different approaches to the bargaining problem, namely the *axiomatic* and the *strategic* approach. The axiomatic approach lists a number of desirable properties the solution must have, called the *axioms*. The strategic approach on the other hand, sets out a particular bargaining procedure and asks what outcomes would result from rational behavior by the individual players.

So, bargaining theory deals with the situation in which players can realize -through cooperation- other (and better) outcomes than the one which becomes effective when they do not cooperate. This non-cooperative outcome is called the *threatpoint*. The question is to which outcome the players may possibly agree.

In Figure 2 a typical bargaining game is sketched (see also Figure 1). The inner part of the ellipsoid marks out the set of possible outcomes, the *feasible set*  $S$ , of the game. The point  $d$  is the threatpoint. The edge  $P$  is the set of individually rational Pareto-optimal outcomes.

We assume that if the agents unanimously agree on a point  $x = (J_1, J_2) \in S$ , they obtain  $x$ . Otherwise they obtain  $d$ . This presupposes that each player can enforce the threatpoint, when he does not agree with a proposal. The outcome  $x$  the players will finally agree on is called the solution of the bargaining problem. Since the solution depends on as well the feasible set  $S$  as the threatpoint  $d$ , it will be written as  $F(S, d)$ . Notice that the difference for player  $i$  between the solution and the threatpoint,  $J_i - d_i$ , is the reduction in cost player  $i$  incurs by accepting the solution. In the sequel we will call this difference the utility gain for player  $i$ . We will use the notation  $J := (J_1, J_2)$  to denote a point in  $S$  and  $x \succ y$  ( $x \prec y$ ) to denote the *vector inequality*, i.e.  $x_i > y_i$  ( $x_i < y_i$ ),  $i = 1, 2$ . In axiomatic bargaining theory a number of solutions have been proposed. In Thomson [25] a survey is given on this theory. We will present here the three most commonly used solutions: the Nash bargaining solution, the Kalai-Smorodinsky solution and the Egalitarian solution.

The *Nash bargaining solution*,  $N(S, d)$ , selects the point of  $S$  at which the product of utility gains from  $d$  is maximal. That is,

$$N(S, d) = \arg \max_{J \in S} \prod_{i=1}^N (J_i - d_i), \text{ for } J \in S \text{ with } J \preceq d.$$

In Figure 3 we sketched the  $N$  solution. Geometrically, the Nash Bargaining solution is the point on

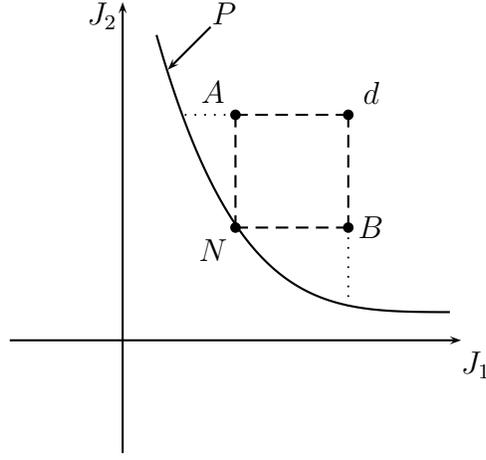


Figure 3: The Nash Bargaining solution  $N(S, d)$ .

the edge of  $S$  (that is a part of the Pareto frontier) which yields the largest rectangle  $(N, A, B, d)$ .

The Kalai-Smorodinsky solution,  $K(S, d)$ , sets utility gains from the threatpoint proportional to the player's most optimistic expectations. For each agent, the most optimistic expectation is defined as the lowest cost he can attain in the feasible set subject to the constraint that no agent incurs a cost higher than his coordinate of the threatpoint. Defining the *ideal point* as

$$I(S, d) := \max\{J_i \mid J \in S, J \preceq d\},$$

the *Kalai-Smorodinsky solution* is then

$$K(S, d) := \text{maximal point of } S \text{ on the segment connecting } d \text{ to } I(S, d).$$

In Figure 4 the Kalai-Smorodinsky solution is sketched for the two-player case. Geometrically, it is the intersection of the Pareto frontier  $P$  with the line which connects the threatpoint and the ideal point. The components of the ideal point are the minima each player can reach when the other player is fully altruistic under cooperation.

Finally, the *Egalitarian solution*,  $E(S, d)$ , represents the idea that gains should be equal divided between the players. Formal,

$$E(S, d) := \text{maximal point in } S \text{ for which } E_i(S, d) - d_i = E_j(S, d) - d_j, \quad i, j = 1, \dots, N.$$

Again, we sketched this solution for the two-player case. In Figure 5 we observe that geometrically this Egalitarian solution is obtained as the intersection point of the  $45^\circ$ -line through the threatpoint  $d$  with the Pareto frontier  $P$ .

Notice that in particular in contexts where interpersonal comparisons of utility is inappropriate or impossible, the first two bargaining solutions still make sense.

As already mentioned above these bargaining solutions can be motivated using an "axiomatic approach". In this case some people prefer to speak of an arbitration scheme instead of a bargaining game. An arbiter draws up the reasonable axioms and depending on these axioms, a solution results.

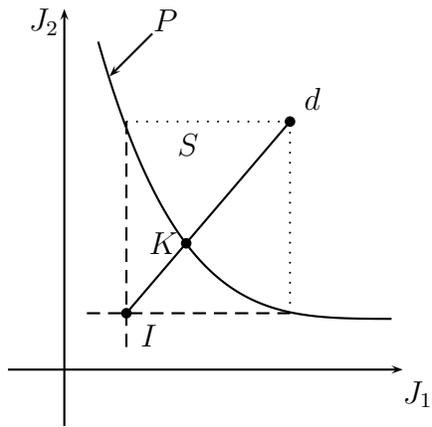


Figure 4: The Kalai-Smorodinsky solution  $K(S, d)$ .

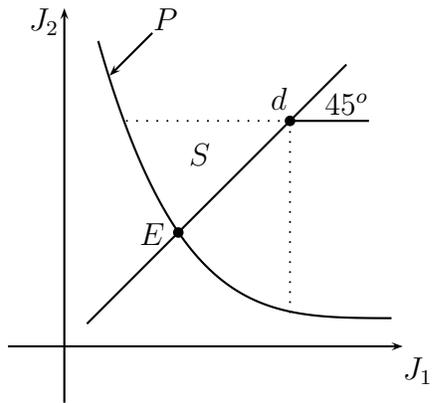


Figure 5: The Egalitarian solution  $E(S, d)$ .

Algorithms to calculate the first two solutions numerically are outlined in [5]. The calculation of the Egalitarian solution requires the solution of one non-linear constrained equations problem. The involved computer time to calculate this  $E$ -solution approximately equals that of calculating the  $N$ -solution.

**Example 5.1** Consider the following differential game on government debt stabilization (see van Aarle et al. [1]). Assume that government debt accumulation,  $\dot{d}(t)$ , is the sum of interest payments on government debt,  $rd(t)$ , and primary fiscal deficits,  $f(t)$ , minus the seignorage (i.e. the issue of base money)  $m(t)$ . So,

$$\dot{d}(t) = rd(t) + f(t) - m(t), d(0) = d_0.$$

Here  $d(t)$ ,  $f(t)$  and  $m(t)$  are expressed as fractions of GDP and  $r$  represents the rate of interest on outstanding government debt minus the growth rate of output. The interest rate  $r > 0$  is assumed to be exogenous. Assume that fiscal and monetary policies are controlled by different institutions, the fiscal authority and the monetary authority, respectively, which have different objectives. The objective of the fiscal authority is to minimize a sum of time profiles of the primary fiscal deficit, base-money growth and government debt

$$J_1 = \int_0^\infty e^{-\delta t} \{f^2(t) + \eta m^2(t) + \lambda d^2(t)\} dt.$$

The parameters,  $\eta$  and  $\lambda$  express the relative priority attached to base-money growth and government debt by the fiscal authority. The monetary authorities are assumed to choose the growth of base money such that a sum of time profiles of base-money growth and government debt is minimized. It is assumed that they (almost) do not care about the fiscal deficit. That is

$$J_2 = \int_0^\infty e^{-\delta t} \{\epsilon f^2(t) + m^2(t) + \kappa d^2(t)\} dt.$$

Here  $1/\kappa$  can be interpreted as a measure for the conservatism of the central bank w.r.t. the money growth and  $\epsilon$  is almost zero. Furthermore all variables are normalized such that their targets are zero, and all parameters are positive.

Introducing  $\tilde{d}(t) := e^{-\frac{1}{2}\delta t} d(t)$ ,  $\tilde{m} := e^{-\frac{1}{2}\delta t} m(t)$  and  $\tilde{f} := e^{-\frac{1}{2}\delta t} f(t)$  the above model can be rewritten as

$$\dot{\tilde{d}}(t) = (r - \frac{1}{2}\delta)\tilde{d}(t) + \tilde{f}(t) - \tilde{m}(t), \tilde{d}(0) = d_0.$$

Where the cost functions of both players are

$$J_1 = \int_0^\infty \{\tilde{f}^2(t) + \eta \tilde{m}^2(t) + \lambda \tilde{d}^2(t)\} dt$$

and

$$J_2 = \int_0^\infty \{\epsilon \tilde{f}^2(t) + \tilde{m}^2(t) + \kappa \tilde{d}^2(t)\} dt.$$

If both the monetary and fiscal authority agree to cooperate in order to reach their goals, by Theorem 2.12 the set of all Pareto solutions is obtained by considering the simultaneous minimization of

$$\begin{aligned} J_c(\alpha) &:= \alpha J_1 + (1 - \alpha) J_2 \\ &= \int_0^\infty \{ \alpha_1 \tilde{f}^2(t) + \beta_1 \tilde{m}^2(t) + \beta_2 \tilde{d}^2(t) \} dt, \end{aligned}$$

where  $\alpha_1 = \epsilon + \alpha(1 - \epsilon)$ ,  $\beta_1 = 1 + \alpha(-1 + \eta)$  and  $\beta_2 = \kappa + \alpha(\lambda - \kappa)$ . This cooperative game problem can be reformulated as the minimization of

$$J_c(\alpha) = \int_0^\infty \{ \beta_2 \tilde{d}^2(t) + [\tilde{f} \ \tilde{m}] \begin{bmatrix} \alpha & 0 \\ 0 & \beta_1 \end{bmatrix} \begin{bmatrix} \tilde{f} \\ \tilde{m} \end{bmatrix} \} dt,$$

subject to

$$\dot{\tilde{d}}(t) = (r - \frac{1}{2}\delta)\tilde{d}(t) + [1 \ -1] \begin{bmatrix} \tilde{f} \\ \tilde{m} \end{bmatrix}, \quad \tilde{d}(0) = d_0.$$

In Figure 5 we plotted the set of Pareto solutions in case  $\eta = 0.1$ ;  $\lambda = 0.6$ ;  $\epsilon = 0.00001$ ;  $\kappa = 0.5$ ;  $r = 0.06$ ,  $\delta = 0.04$  and  $d(0) = 1$ . Furthermore we plotted the corresponding  $N$ ,  $K$  and  $E$  solution if the threatpoint is  $(2, 1)$ . For these parameters we also calculated the corresponding non-cooperative open-loop Nash equilibrium (see e.g. [5, Section 7.7]). This solution is given by  $(J_1, J_2) = (0.4658, 0.7056)$ . Taking this as the threatpoint, we illustrated in Figure 5 the new (more realistic) bargaining solutions. Furthermore we tabulated in Table 1 below the percentage gains for the fiscal and monetary authority if they would cooperate instead of playing non-cooperative. Assuming that real society's cost are measured by the average of the cost of the fiscal and monetary authority we observe from Table 1 that cooperation leads to a reduction of society's cost with approximately 30%.

| Solution | Coordinates      | % gain $J_1$ | % gain $J_2$ |
|----------|------------------|--------------|--------------|
| $N$      | (0.3383, 0.471)  | 27.4         | 33.2         |
| $K$      | (0.3445, 0.4582) | 26.0         | 35.1         |
| $E$      | (0.3755, 0.42)   | 19.4         | 40.5         |

Table 1: Percentage gains of cooperation

□

## 6 Concluding Remarks

In this paper we considered both necessary and sufficient conditions for the existence of a Pareto solution in a cooperative dynamic differential game with an open-loop information structure. This problem setting naturally occurs if either one person has more than one objectives he likes to optimize, or, the dynamics of the considered system is affected by more than one person and these persons agree to cooperate in order to optimize their individual objective.

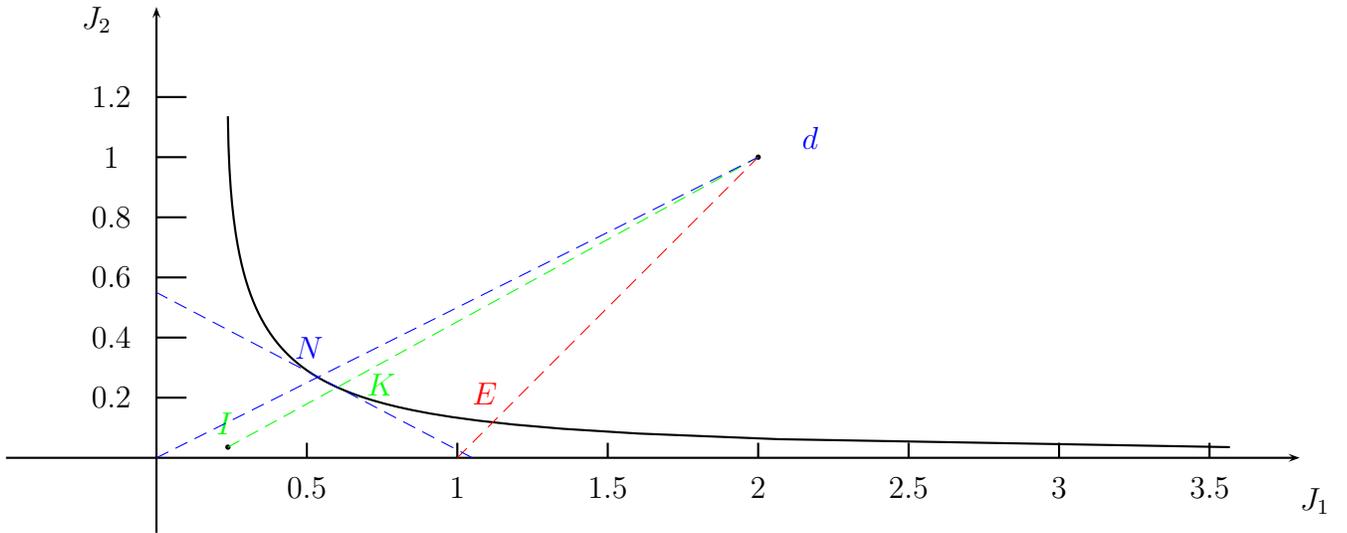


Figure 6: Pareto frontier Example 5.1 if  $\eta = 0.1$ ;  $\lambda = 0.6$ ;  $\kappa = 0.5$ ;  $r = 0.06$  and  $\delta = 0.04$ .

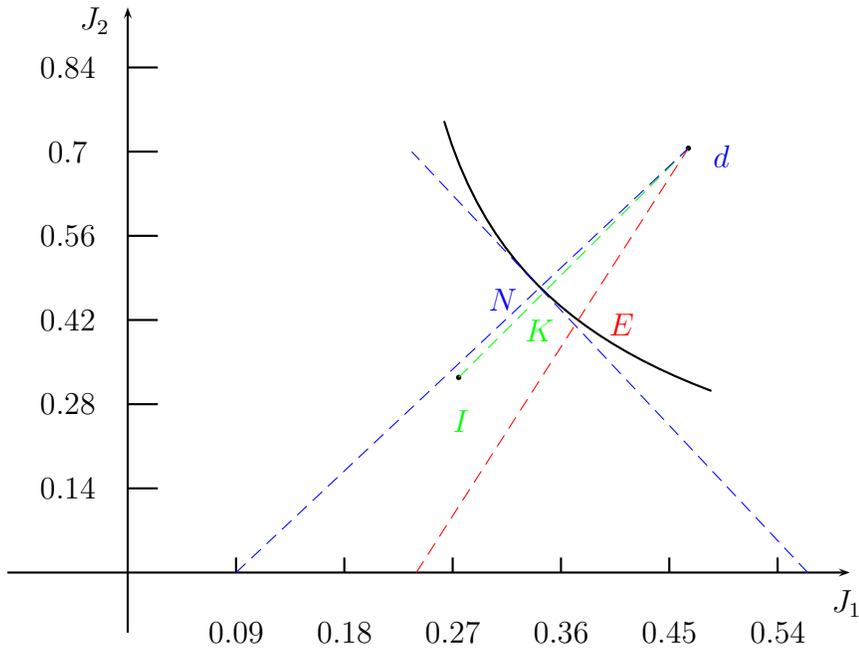


Figure 7: Bargaining solutions Example 5.1 if threatpoint is  $(0.4658, 0.7056)$ .

The presented necessary condition is a mixture of the maximum principle conditions and the necessary conditions for existence of a Pareto solution in a static game. In line with Arrow's conditions, we also gave conditions under which the necessary conditions are sufficient too. Finally we indicated the effects on the necessary conditions if the problem setting includes inequality constraints.

In the second part of this paper we considered the special case of regular indefinite linear quadratic control problems. First we elaborated the necessary conditions for the general case and next we considered the special case when the game is convex.

For the general case we showed how one can obtain all Pareto efficient outcomes for scalar systems if the game has Pareto solutions for every initial state. Unfortunately, the generalization of these results to non-scalar systems remains an open problem. We illustrated in an example some peculiarities that may occur.

For the convex game we showed how one can obtain all Pareto efficient outcomes. Using the theory of convex analysis we were able to treat both the finite and infinite planning horizon problems in a uniform way. For the infinite planning horizon we considered both the fixed endpoint problem and the free endpoint problem.

We believe that without too many complications the presented theory on the general case can be extended for an infinite planning horizon. The elaboration of the herewith involved technical problems remains however a point for future research. Another open problem that remains to be solved within this area is the case that the performance criteria of both players are not directly affected by the control efforts used by the other player (giving rise to singular control problems). Open problems that remain to be solved for the convex case are, e.g., the free endpoint infinite planning horizon problem under the assumption that the system is merely stabilizable.

Finally, in the third part of this paper, we considered the question which cooperative solution will result in a multi-player context. Given our open-loop information structure we argued that it makes sense to use solution concepts developed in static game theory to answer this question. In particular we reviewed some axiomatic bargaining solutions which have been developed for non-transferable utility games. We illustrated them in an example.

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