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**CONVEX GAMES WITH COUNTABLE NUMBER OF PLAYERS  
AND SEQUENCING SITUATIONS**

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# Convex Games with Countable Number of Players and Sequencing Situations\*

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## Abstract

In this paper we study convex games with an infinite countable set of agents and provide characterizations of this class of games. Some difficulties arise when dealing with these infinite games, especially to tackle the vectors of marginal contributions. In order to solve these problems we use a continuity property. Infinite sequencing situations where the number of jobs is countable infinite and the related cooperative TU games are introduced. It is shown that these infinite games are convex and the marginals associated with some orders turn out to be extreme points of the core.

**Key words:** Cooperative games, countable number of players, convexity, infinite sequencing situations.

**JEL Classification:** C71

**AMS Classification:** 91A12

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# 1 Introduction

The class of convex games with transferable utility and with finite number of players was introduced by Shapley (1971). There are several equivalent ways to introduce this class of games. Supermodularity, increasing marginal contributions (groups and individuals), marginals in the core or the core is equal to the Weber set, are examples of these equivalent formulations.

In this paper we consider cooperative  $TU$  games where the set of players is countable infinite. The concept of convexity can be extended, in a natural way, to these games. However, we will show that many problems arise when we try to find characterizations for this class of infinite games. In particular, to assure that the vector of marginal contributions with respect to any order is in the core, we need to introduce an additional property. This is a continuity property and it guarantees, for instance, that we can reach the worth of an infinite coalition from monotonic sequences of finite coalitions.

A finite sequencing situation arises when a (finite) number of jobs has to be processed in a machine, according to a certain order and one tries to optimize a cost function. Given a sequencing situation we can associate to it the cost related to its initial order as the sum of the costs of the jobs, where the cost of job  $i$  is given by the product of its cost per unit of time,  $\alpha_i$ , and the time that it spends in the system, i.e. its service time,  $p_i$ , plus the waiting time for completing all the jobs that precede  $i$  in the queue. The problem related to a sequencing situation is to determine the optimal order of the jobs. Smith (1956) proved that the optimal order can be obtained reordering the jobs according to decreasing urgency indices, where the urgency index of job  $i$  is defined as  $u_i = \alpha_i/p_i$ .

If the jobs belong to the same agent he will agree to reorder optimally, according to the previous result. The situation is completely different when each job belongs to a different agent. In this case, we can tackle the situation involving two or more interacting agents through an Operations Research Game (see Borm *et al*, 2001), and a reordering requires that at least the agents involved agree on the new order. So we can say that a switch among two jobs is always possible if they are consecutive in the initial order or if all the agents that own one of the jobs in between the two that are switched agree. If all the agents agree, the optimal order can be obtained, generating a cost savings with respect to the initial order. The following question arises: Is it possible to share this cost savings among the agents in such a way that the new order results to be stable? In other words, we want to find a fair amount

to be given to the different agents, in such a way that all of them agree on the optimal order and have no incentive to recede from the agreement. This situation finds its natural habitat in cooperative Game Theory.

In 1989 Curiel, Pederzoli and Tijs introduced the sequencing games. An updated survey on these games can be found in Curiel *et al* (2002). A finite sequencing game is a pair  $(N, v)$  where  $N$  is the (finite) set of players, that coincides with the set of jobs, and the characteristic function  $v$  assigns to the players of a coalition  $S$  the maximal cost savings they can obtain by reordering only their jobs. Curiel *et al* (1989) show that finite sequencing games are convex games and, consequently, balanced. A balanced game is a cooperative  $TU$  game with a non-empty core.

In this paper, we deal with cooperative  $TU$  games where there is a countable infinite number of agents. In the next section we present three equivalent formulations of convexity for games with a countable number of players. Section 3 is devoted to the study of the marginals for these infinite games. In Section 4 we analyse infinite sequencing situations and show that, for some orderings, the marginals are extreme core elements of the corresponding cooperative  $TU$  games. Some comments about extensions of these infinite sequencing games and other remarks are given in Section 5.

## 2 Countable Infinite Convex Games

A cooperative  $TU$  game can be represented by  $(N, v)$ , where  $N$  is the set of players and the characteristic function  $v$  assigns to each group (or coalition) of players  $S \subset N$  the value  $v(S)$ , which stands for the reward to the members in  $S$  when they cooperate. The core of a  $TU$  game  $(N, v)$  is subset of  $x \in \mathbb{R}^N$  satisfying

$$\begin{aligned} \text{(Efficiency)} \quad & \sum_{i \in N} x_i = v(N), \text{ and} \\ \text{(Coalitional rationality)} \quad & \sum_{i \in S} x_i \geq v(S), \text{ for all } S \subset N. \end{aligned}$$

We are interested in extending the concept of a finite convex game to the situation with a countable infinite player set. For this purpose we introduce a definition of convex games for those with  $N = \mathbb{N}$ . To avoid convergence problems we will restrict to nonnegative games with bounded value.

**Definition 1** *An infinite game  $(\mathbb{N}, v)$  is called convex if and only if*

$$v(S_1 \cup \{i\}) - v(S_1) \leq v(S_2 \cup \{i\}) - v(S_2),$$

for all  $S_1, S_2 \subset \mathbb{N}, i \in \mathbb{N}$  such that  $S_1 \subset S_2 \subset \mathbb{N} \setminus \{i\}$ .

An extra condition is needed in order to assure that if a game is convex, then

$$(i) \quad v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \text{ for all } S, T \subset N,$$

holds, i.e. the game is supermodular. This condition is usually known as the *Inner Continuity Property (ICP)*:

**Definition 2** A game  $(\mathbb{N}, v)$  is inner continuous at a coalition  $S \subset \mathbb{N}$ , if

$$\lim_{k \rightarrow \infty} v(S_k) = v(S)$$

where  $S_1 \subset S_2 \subset \dots$  and  $\bigcup_{k=1}^{\infty} S_k = S$ . An infinite game is inner continuous if it is inner continuous at every coalition  $S$ .

As the next example shows this is not a redundant condition if one seeks to reach the value of a coalition in this infinite context.

**Example 3** Consider the infinite unanimity game  $(\mathbb{N}, u_{\mathbb{N}})$  and its restrictions to the first  $n$  players present. In this case,  $u_{\mathbb{N}}(\mathbb{N}) = 1$  and it can not be reached from the limit of the values of the corresponding finite games, which are always 0.

**Proposition 4** If the game  $(\mathbb{N}, v)$  is convex and inner continuous, then it is supermodular.

**Proof.** Given  $S, T \subset \mathbb{N}$ , let  $S \setminus T = \{s_1, s_2, s_3, \dots\}$  with  $s_1 < s_2 < s_3, \dots$ . Note that this is possible because each subset of the natural numbers has always a smallest element, and  $S \setminus T$  can be finite or infinite. Then, taken into account the ICP, we have for  $r \geq 1$  and  $s_r \in S \setminus T$

$$v(S) - v(S \cap T) = \sum_{r \geq 1} (v((S \cap T) \cup \{s_1, \dots, s_r\}) - v((S \cap T) \cup \{s_1, \dots, s_{r-1}\})) \quad (1)$$

and

$$v(S \cup T) - v(T) = \sum_{r \geq 1} (v((T) \cup \{s_1, \dots, s_r\}) - v((T) \cup \{s_1, \dots, s_{r-1}\})) \quad (2)$$

where for convenience  $\{s_1, \dots, s_{r-1}\} = \emptyset$  for  $r = 1$ .

Since each term in the sum of (1) is by hypothesis not larger than the corresponding term in (2), assumption (i)

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \text{ for all } S, T \subset N,$$

holds. ■

In a similar way, it can be shown that another kind of continuity has to be considered when we try to achieve the value of a coalition from a monotonic non-increasing sequence of coalitions.

**Definition 5** A game  $(\mathbb{N}, v)$  is outer continuous at a coalition  $S \subset \mathbb{N}$ , if

$$\lim_{k \rightarrow \infty} v(S_k) = v(S)$$

where  $S_1 \supset S_2 \supset \dots$  and  $\bigcap_{k=1}^{\infty} S_k = S$ . An infinite game is outer continuous if it is outer continuous at all coalitions.

In order to analyse the convexity, in the next theorem we give equivalent formulations which lead to convex games when the infinite game is continuous. An infinite game is *continuous* if it is inner and outer continuous. The proof is left to the reader because the main steps resemble those in the finite case (cf Branzei *et al*, 2005).

**Theorem 6** Given an infinite continuous game  $(\mathbb{N}, v)$ , the following assertions are equivalent:

- (i)  $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ , for all  $S, T \subset N$ ;
- (ii)  $v(S_1 \cup U) - v(S_1) \leq v(S_2 \cup U) - v(S_2)$ , for all  $S_1, S_2, U \subset N$  such that  $S_1 \subset S_2 \subset N \setminus \{U\}$ ;
- (iii)  $(\mathbb{N}, v)$  is convex.

In a similar way, the notion of concavity can be introduced by reversing the inequalities. As in the finite case, when we consider a countable infinite number of agents, we are interested in those types of games which turn out to be convex/concave. Finite bankruptcy and sequencing are two of the best known situations which give rise to convex games. In this countable infinite setting, it can be shown that infinite bankruptcy games with finite total claims are convex. The case of infinite sequencing situations is analysed in Section 4. Airport games with a countably infinite player set and bounded total cost can be seen as concave games.

### 3 The Marginals of Countable Infinite Games

There are many characterizations of finite convex games, for instance, there is a well-known with five criteria (see, e.g. Branzei *et al*, 2005). In the finite case Weber (1988) proved, by induction on the number of players, that the core is always included in the convex hull of the so-called Weber allocations. In his proof Derks (1992) uses a separation theorem. Shapley (1971) stated that if the game is supermodular, then the core and the Weber set coincide. Ichiishi (1981) showed that if the marginal vectors are core elements for all orders of the player set, then the game is convex. Thus, for finite games, the core is equal to the Weber set if and only if the game is convex.

In this infinite setting, before we can introduce a suitable Weber set we have to look at the orders in  $\mathbb{N}$ . Using the fact that each subset of  $\mathbb{N}$  always has a least element, all orders in  $\mathbb{N}$  can be written through the predecessors and/or successors of the members of a partition that each order  $\sigma$  induces in  $\mathbb{N}$ . For instance, partitions induced in  $\mathbb{N}$  by the natural order  $\sigma_1 = (1, 2, 3, \dots)$  and the reverse order  $\sigma_2 = (\dots, 3, 2, 1)$  consist in only one class: that of the smallest element 1. The elements in  $\sigma_1$  can be written by listing the successors of 1, while for  $\sigma_2$  we have to use its predecessors. On the other hand, if we consider  $\sigma_3 = (1, 3, 5, \dots, \dots, 6, 4, 2)$ , i.e. first the odd numbers in the natural order and, then, the even numbers in the reverse order, we can reconstruct the sequence using the successors of 1 inside its class and the predecessors of 2 which are in his own class.

Formally, each order  $\sigma$  in  $\mathbb{N}$  can be represented by  $(\sigma^k)_{k \in \Sigma}$  where  $\bigcup_{k \in \Sigma} \sigma^k = \mathbb{N}$ ,  $\sigma^{k_1} \cap \sigma^{k_2} = \emptyset$ , for all  $k_1 \neq k_2$  such that  $k_1, k_2 \in \Sigma$ , and  $\Sigma = \{k \in \mathbb{N} | k \text{ is the smallest element in } \sigma^k\}$ . Note that  $\sigma^k$  can be finite or infinite and  $\Sigma$  is always non-empty since  $1 \in \Sigma$ .

In a convex game with a countable number of players,  $(\mathbb{N}, v)$ , the marginal corresponding to each player  $i \in \mathbb{N}$  for an order  $\sigma$  can be described by

$$m_i^\sigma(v) = v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))$$

where  $P^\sigma(i)$  represents the set of predecessors of  $i$  in  $\sigma$ . Note that in this context it is not obvious that a marginal vector has to be in the core of a convex game. Example 3 illustrates this fact because the marginal contributions are 0.

**Theorem 7** *If the infinite game  $(\mathbb{N}, v)$  is convex and continuous, then  $m^\sigma(v)$  is in the core of  $(\mathbb{N}, v)$ , for all  $\sigma$ .*



**Proof.** For the sake of brevity, in the sequel we denote by  $m^\sigma$  the marginal vector for an order  $\sigma$ . First, we have

$$\begin{aligned} \sum_{i \in \mathbb{N}} m_i^\sigma &= \sum_{k \in \Sigma} \left( \sum_{i \in \sigma_k} m_i^\sigma \right) = \\ &= \sum_{k \in \Sigma} \left( \sum_{i \in \sigma_k} (v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))) \right) = v(\mathbb{N}), \end{aligned}$$

where the last equality holds because we are dealing with a generalized telescopic sum of a countable number of nonnegative terms and the game is continuous.

On the other hand,

$$\begin{aligned} \sum_{i \in S} m_i^\sigma &= \sum_{i \in S} (v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))) \geq \\ &= \sum_{i \in S} (v(P^{\sigma|S}(i) \cup \{i\}) - v(P^{\sigma|S}(i))) = \sum_{i \in S} m_i^{\sigma|S} = \\ &= \sum_{k \in \Sigma \cap S} \left( \sum_{i \in \sigma_k} (v(P^{\sigma|S}(i) \cup \{i\}) - v(P^{\sigma|S}(i))) \right) = v(S) \end{aligned}$$

where  $P^{\sigma|S}(i) = P^\sigma(i) \cap S$ . The inequality holds because the game is nonnegative and convex, and the last equality follows from the fact that there is a countable nonnegative telescopic sum and the game is continuous. Thus, the marginal vector  $m^\sigma$ , for all order  $\sigma$ , is in the core of the infinite game  $(\mathbb{N}, v)$ . ■

The marginal vectors are not only in the core, but they are also extreme points as the next result shows.

**Theorem 8** *Let  $(\mathbb{N}, v)$  a convex and continuous infinite game. For all order  $\sigma$ ,  $m^\sigma$  is an extreme point of the core.*

**Proof.** (i) To prove that  $m^\sigma$  is an extreme point of the core,  $C(v)$ , we need to show that for pairs  $x, y \in C(v)$  with  $m^\sigma = \frac{1}{2}(x + y)$  we have  $x = y = m^\sigma$ .

(ii) We claim that  $\sum_{i \in S} m_i^\sigma = v(S)$ , for  $S$  of the form  $P^\sigma(\sigma(k))$  and also of the form  $P^\sigma(\sigma(k)) \cup \{\sigma(k)\}$  with  $k \in \mathbb{N}$ . Using continuity the first claim follows, similarly as in the proof of the efficiency of  $m^\sigma$  of theorem 7, by decomposing  $\sum_{i \in P^\sigma(\sigma(k))} m_i^\sigma$  in generalized telescopic sums. The second claim follows by noting that

$$\sum_{i \in P^\sigma(\sigma(k)) \cup \{\sigma(k)\}} m_i^\sigma = v(P^\sigma(\sigma(k))) + m_{\sigma(k)}^\sigma = v(P^\sigma(\sigma(k)) \cup \{\sigma(k)\}).$$

(iii) Consider a pair  $x, y \in C(v)$  with  $m^\sigma = \frac{1}{2}(x + y)$ . Note that for such pairs we have

$$\sum_{i \in S} x_i = \sum_{i \in S} y_i = v(S), \quad (3)$$

for each  $S \in 2^N$  with  $\sum_{i \in S} m_i = v(S)$ . Taking into account (ii) and (3) we find, for pairs  $x, y$  and  $k \in \mathbb{N}$ :

$$\begin{aligned} x_{\sigma(k)} &= \sum_{i \in P^\sigma(\sigma(k)) \cup \{\sigma(k)\}} x_i - \sum_{i \in P^\sigma(\sigma(k))} x_i = \\ &v(P^\sigma(\sigma(k)) \cup \{\sigma(k)\}) - v(P^\sigma(\sigma(k))) = m_{\sigma(k)}^\sigma \end{aligned}$$

and it also holds that  $y_{\sigma(k)} = m_{\sigma(k)}^\sigma$ . Thus, we can conclude that  $x = y = m^\sigma$ . ■

Therefore, if we consider the Weber set as the closure (in the weak topology) of the convex hull of all marginal vectors,  $cl(\text{conv}\{m^\sigma \mid \sigma \in \Pi(\mathbb{N})\})$ , we have shown that it is contained in the core of an infinite convex game, assuming that it is continuous.

## 4 Sequencing Situations with Countable Number of Players

In this section we consider that there are countably many jobs and, equivalently, a countably infinite number of agents. The difference between these situations and the semi-infinite ones, introduced in Fragnelli (2001), is that in the latter paper the number of jobs is infinite but the number of agents is finite. Therefore, in the so-called infinite sequencing situations the set of agents can be represented by  $\mathbb{N}$  and we suppose that they are numbered as they are in the queue in front a counter (machine). Thus, we are assuming that the initial order coincides with the natural order of  $\mathbb{N}$ . As in the finite case, for infinite sequencing situations if two consecutive jobs have increasing urgencies it is possible to reduce the cost with a switch. A set of jobs  $T$  is *connected according to an order*  $\sigma$  if

$$\sigma(i) < \sigma(k) < \sigma(j) \Rightarrow k \in T,$$

for all  $i, j \in T$  and  $k \in N$ . Switching two connected jobs  $i, j$  the change in cost is given by  $\alpha_j p_i - \alpha_i p_j$  (note that the variation is positive if and only

if the urgency indices verify  $u_i < u_j$ ); if the variation is negative the switch does not take place. We denote the gain of the switch by:

$$g_{ij} = (\alpha_j p_i - \alpha_i p_j)_+ = \max\{0, \alpha_j p_i - \alpha_i p_j\}$$

and, consequently, the gain of a connected coalition  $T$  according to an order  $\sigma$  as:

$$v(T) = \sum_{j \in T} \sum_{i \in P^\sigma(j) \cap T} g_{ij}.$$

In the sequel, we will focus on infinite sequencing situations where

$$\sum_{j \in \mathbb{N}} \sum_{i \in P^\sigma(j)} g_{ij} < +\infty,$$

i.e. the total gain that can be obtained reordering the jobs is bounded.

**Definition 9** *Given an infinite sequencing situation, the corresponding infinite sequencing game  $(\mathbb{N}, v)$  is a cooperative TU game with*

$$v(S) = \sum_{T \in S/\sigma} v(T) \quad \forall S \subset \mathbb{N}.$$

where  $S/\sigma$  is the partition in connected coalitions induced by the order  $\sigma$ .

**Remark 10** *We are interested in infinite sequencing games such that if we consider the finite sequencing situations in which keep only the first  $n$  jobs and the corresponding sequence of finite games  $([1, n], v_n)$ , where  $[1, n]$  denotes the set  $\{1, 2, \dots, n\}$ , we can obtain the characteristic function of this infinite sequencing game  $(\mathbb{N}, v)$  as:*

$$\lim_{n \rightarrow +\infty} v_n(S \cap [1, n]) = v(S),$$

for all  $S \subset \mathbb{N}$ . This means that, in the sequel, we will focus on infinite sequencing games which are inner continuous.

Since finite sequencing games turn out to be convex games, a natural question is to look for a similar property for infinite sequencing games.

**Proposition 11** *Let  $(\mathbb{N}, v)$  be the inner continuous sequencing game corresponding to an infinite sequencing situation. Then  $(\mathbb{N}, v)$  is a convex game.*

**Proof.** Since  $([1, n], v_n)$  is convex because it corresponds to a finite sequencing situation, we have  $v_n(S \cap [1, n]) + v_n(T \cap [1, n]) \leq v_n((S \cup T) \cap [1, n]) + v_n(S \cap T \cap [1, n])$ , for all  $S, T \subset \mathbb{N}$ . Taking the limit for  $n \rightarrow +\infty$ , by the ICP we obtain (i) of Theorem 6. So  $(\mathbb{N}, v)$  is a convex game. ■

For finite sequencing games interesting core elements are known. Curiel *et al* (1989) show that it is possible to determine a core allocation without computing the characteristic function of the finite sequencing game. They propose to share equally between the players  $i, j$  the gain  $g_{ij}$  produced by the switch and call this rule to obtain an allocation the *Equal Gain Splitting Rule* (*EGS*). It can be computed by:

$$EGS_i = \frac{1}{2} \sum_{k \in P^\sigma(i)} g_{ki} + \frac{1}{2} \sum_{j: i \in P^\sigma(j)} g_{ij} \quad \forall i \in N.$$

There exist two other simple allocation rules, denoted respectively by  $\mathcal{P}$  and  $\mathcal{S}$ . According to the first the gain of each switch is assigned to the predecessor in the initial order, while the second assigns the gain to the successor. We can write:

$$\begin{aligned} \mathcal{P}_i &= \sum_{j: i \in P^\sigma(j)} g_{ij} \quad \forall i \in N, \\ \mathcal{S}_i &= \sum_{j \in P^\sigma(i)} g_{ji} \quad \forall i \in N \end{aligned}$$

and it is easy to see that  $EGS = \frac{1}{2}(\mathcal{P} + \mathcal{S})$ .

In a similar way, we can define the  $EGS^\varepsilon$  solution for each  $\varepsilon \in [0, 1]$  as:

$$EGS^\varepsilon = \varepsilon \mathcal{P} + (1 - \varepsilon) \mathcal{S},$$

where for  $\varepsilon = 0$  we obtain  $\mathcal{S}$ , for  $\varepsilon = \frac{1}{2}$  we get  $EGS$ , and for  $\varepsilon = 1$  we have  $\mathcal{P}$ .

In the case of infinite sequencing games and referring to the gain splitting rules, it is easy to prove that  $\mathcal{P}$  and  $\mathcal{S}$  belong to the core. In fact, efficiency clearly holds and, for a connected coalition  $T$ , we have:

$$\sum_{i \in T} \mathcal{P}_i = \sum_{i \in T} \sum_{j: i \in P^\sigma(j)} g_{ij} \geq \sum_{i \in T} \sum_{j \in T: i \in P^\sigma(j)} g_{ij} = v(T)$$

and

$$\sum_{i \in T} \mathcal{S}_i = \sum_{i \in T} \sum_{j \in P^\sigma(i)} g_{ji} \geq \sum_{i \in T} \sum_{j \in P^\sigma(i) \cap T} g_{ji} = v(T).$$

Consequently, by the convexity of the core, we have that also  $EGS^\varepsilon$  belongs to the core for all  $\varepsilon \in [0, 1]$ .

Since we are assuming that the initial order coincides with the natural order of  $\mathbb{N}$ , then  $v(S) = \sum_{i \in S} \sum_{j \in S, j > i} g_{ij}$ , and  $\mathcal{P}$  and  $\mathcal{S}$  can be written as:

$$\mathcal{P} = \left( \sum_{j \geq 2} g_{1j}, \sum_{j \geq 3} g_{2j}, \dots \right) \text{ and } \mathcal{S} = \left( 0, g_{12}, \sum_{j \leq 2} g_{j3}, \sum_{j \leq 3} g_{j4}, \dots \right).$$

The following result shows that these allocations are not only in the core, but they are extreme points of the core when the game is continuous.

**Proposition 12**  *$\mathcal{P}$  and  $\mathcal{S}$  are extreme points of the core of the corresponding infinite continuous sequencing game.*

**Proof.** To prove that  $\mathcal{P} = \left( \sum_{j \geq 2} g_{1j}, \sum_{j \geq 3} g_{2j}, \dots \right)$  is an extreme point, let  $x, y$  be two core allocations such that  $\mathcal{P} = \frac{1}{2}(x + y)$ . We will show that  $x = y = \mathcal{P}$  through an induction procedure.

By efficiency, we know that  $\sum_{i \geq 1} x_i = \sum_{i \geq 1} \mathcal{P}_i$ , and using coalition rationality we have

$$\sum_{i \geq 2} x_i \geq v(\mathbb{N} \setminus \{1\}) = \sum_{i \leq 2} \sum_{j > i} g_{ij} = \sum_{i \geq 2} \mathcal{P}_i.$$

Thus, consequently,  $x_1 \leq \mathcal{P}_1$ .

Since the inequality  $y_1 \leq \mathcal{P}_1$  can be derived similarly and  $\mathcal{P}_1 = \frac{1}{2}(x_1 + y_1)$ , we obtain  $x_1 = y_1 = \mathcal{P}_1$ .

Now, we suppose that  $x_j = y_j = \mathcal{P}_j$ , for all  $j = 1, \dots, k - 1$ , and we are going to prove that  $x_k = y_k = \mathcal{P}_k$ .

Using the efficiency condition and the induction hypothesis, we have  $\sum_{i \geq k} x_i = \sum_{i \geq k} \mathcal{P}_i$ . By coalition rationality we can obtain

$$\sum_{i \geq k+1} x_i \geq v(\mathbb{N} \setminus \{1, \dots, k\}) = \sum_{i \leq k+1} \sum_{j > i} g_{ij} = \sum_{i \geq k+1} \mathcal{P}_i,$$

and then  $x_k \leq \mathcal{P}_k$ .

Similarly, we obtain  $y_k \leq \mathcal{P}_k$  and taking into account that  $\mathcal{P}_k = \frac{1}{2}(x_k + y_k)$ ,

it holds  $x_k = y_k = \mathcal{P}_k$ .

To prove that  $\mathcal{S} = \left(0, g_{12}, \sum_{j \leq 2} g_{j3}, \sum_{j \leq 3} g_{j4}, \dots\right)$  is an extreme point, let  $x, y$  be two core allocations with  $\mathcal{S} = \frac{1}{2}(x + y)$ . We will show that  $x = y = \mathcal{S}$ . By coalition rationality, we have  $x_1 \geq v(1) = 0 = \mathcal{S}_1$  and  $y_1 \geq \mathcal{S}_1$  can be derived in the same way. Since  $\mathcal{S}_1 = \frac{1}{2}(x_1 + y_1)$ , then  $x_1 = y_1 = \mathcal{S}_1$ .

Assume that  $x_j = y_j = \mathcal{S}_j$ , for all  $j = 1, \dots, k-1$ . We will prove that  $x_k = y_k = \mathcal{S}_k$ .

Using coalition rationality we have

$$\sum_{i=1, \dots, k} x_i \geq v(\{1, \dots, k\}) = \sum_{i=1, \dots, k} \sum_{j=i+1, \dots, k} g_{ij} = \sum_{i=1, \dots, k} \mathcal{S}_i$$

and, applying the hypothesis,  $x_k \geq \mathcal{S}_k$ . The related inequality,  $y_k \geq \mathcal{S}_k$ , is obtained using a similar procedure. As  $\mathcal{S}_k = \frac{1}{2}(x_k + y_k)$  we can conclude that  $x_k = y_k = \mathcal{S}_k$ . ■

## 5 Concluding Remarks

We have proved that the closure of the convex hull of all marginals is contained in the core, when the infinite game is convex and continuous. It could be interesting to know under which conditions the core of an infinite convex game is included in this Weber set.

In Curiel *et al* (1994) it is proved that finite sequencing situations with regular and additive cost functions are balanced. This result can be extended to infinite sequencing games. In fact,  $\mathcal{P}$  and  $\mathcal{S}$  are core elements. Similarly, as in the finite case, one can prove that  $\mathcal{S}$  is the unique drop out monotonic solution (cf Fernández *et al*, 2005) and  $\mathcal{P}, \mathcal{S}$  and  $EGS^\varepsilon$  are population monotonic allocation schemes.

Since we know from Remark 10 that  $v(S) = \lim_{n \rightarrow \infty} v(S \cap [1, n])$ , we may approximate  $\mathcal{P}, \mathcal{S}$  and  $EGS^\varepsilon$  with the corresponding solutions in a suitable finite game  $([1, n], v_n)$ .

More precisely for each  $\delta > 0$  we can find  $n_\delta \in \mathbb{N}$  such that

$$\sum_{j \in \mathbb{N}} \sum_{i \in P^\sigma(j)} g_{ij} - \sum_{j \in [1, n_\delta]} \sum_{i \in P^\sigma(j)} g_{ij} < \delta$$

and the allocations

$$\mathcal{P}(v_{n_\delta}) = \left( \sum_{j \in [2, n_\delta]} g_{1j}, \sum_{j \in [3, n_\delta]} g_{2j}, \dots, g_{n_\delta-1, n_\delta}, 0, 0, \dots \right)$$

$$\mathcal{S}(v_{n_\delta}) = \left( 0, g_{12}, \sum_{j \leq 2} g_{j3}, \dots, \sum_{j \leq n_\delta-1} g_{j, n_\delta}, 0, 0, \dots \right)$$

satisfy  $\mathcal{P}_i - \mathcal{P}_i(v_{n_\delta}) < \delta$  and  $\mathcal{S}_i - \mathcal{S}_i(v_{n_\delta}) < \delta$ , for all  $i \in \mathbb{N}$ . As a consequence also  $EGS^\varepsilon$  can be approximated in the same way.

If we allow an unbounded total gain  $\sum_{j \in \mathbb{N}} \sum_{i \in P^\sigma(j)} g_{ij} = +\infty$ , we can find core elements in an easy way like the so-called utopia payoff in Llorca *et al* (2004) for infinite assignment games. In this case, our proposal would be  $u_i = \mathcal{P}_i + \mathcal{S}_i$  for each agent  $i \in \mathbb{N}$ , to add the gains with the successors and the predecessors.

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