A new solution property in optimal control

Feichtinger, G.; Hartl, R.F.; Kort, P.M.; Yegorov, Y.A.

Published in:
Automatica

Publication date:
2007

A New Solution Property in Optimal Control: the Lens
suggested running title: The Lens

Gustav Feichtinger* Richard F. Hartl†
Peter M. Kort‡§
Tilburg University, Department of Econometrics and Operations Research,
P.O. Box 90153, 5000 LE Tilburg, The Netherlands
Yuri A. Yegorov¶

August 16, 2006

Abstract

We consider a new property of an optimal control problem called a lens.

A *lens* is an interior point in the state-control phase plane where – given the value of the state variable – there is only one control value satisfying the necessary optimality conditions and – given the value of the control variable – there is only one state value satisfying the necessary optimality conditions.

We build a simple model that generates a lens and give necessary and sufficient conditions under which a lens occurs.

Keywords: optimal control, maximum principle, phase plane, economic dynamics

*Vienna University of Technology, Department of Operations Research and Nonlinear Dynamical Systems, Vienna, Austria

†University of Vienna, Department of Business Studies, Vienna, Austria

‡Tilburg University, Department of Econometrics and Operations Research & CentER, Tilburg, The Netherlands, and University of Antwerp, Department of Economics, Antwerp, Belgium

§Corresponding author. Tel.:+31 13 4662062; fax: 31 13 4663280. E-mail address: kort@uvt.nl.

¶Institute for Advanced Studies, Vienna, Austria

1 Introduction

We restrict ourselves to autonomous optimal control problems with one state and one control variable. We identify a class of optimal control models, the solution of which contains a point, called a lens, in the state-control phase plane with the following properties. In the lens the value of the state variable, say $\hat{x}$, is such that either the control does not influence the contribution to the objective at this point (Case 1), or the control has no effect on the change of the state variable at this point (Case 2). The implication for Case 1 is that in the lens the optimal control, say $\hat{u}$, is set such that its contribution to the change of the state variable is optimized. This implies that, in case of a maximization problem, the increase of the state variable is maximized (minimized) when the co-state variable is positive (negative). Consequently, (i) for a given sign of the co-state variable the value of the co-state as such does not influence the control, and (ii) for the value of the state being equal to $\hat{x}$, only the control being equal to $\hat{u}$ satisfies the necessary optimality conditions.

In Case 2 something similar is going on. Only here $\hat{x}$ is such that the control does not influence the change of the state variable in the lens. This implies that the optimal control solely optimizes the contribution to the objective at this particular point in time. This "lens property" was detected for the first time in Caulkins et al. (2005), in which a problem of migration is studied. Just a little later this same phenomenon appeared in Kort et al. (2006), while analyzing the problem of a fashion designer managing the value of its brand. The meaning of a lens is that in that particular point the optimal control can just be found by taking a myopic approach, i.e. the optimal control can be determined by maximizing the immediate contribution to the objective (Case 2) or the state variable (Case 1). Although we operate in a dynamic optimization framework, at the lens we can thus forget about the long term.

The contents of the paper is as follows. Section 2 starts out by presenting a simple model in which this lens property is present. Then, in Section 3 we proceed by giving a formal definition of the lens property, followed by a proposition in which necessary and sufficient conditions are given under which this lens will occur. Finally, a corollary is presented, which contains a set of sufficiency conditions for a lens in linear-quadratic optimal control problems.

2 Introductory Example

Consider an autonomous optimal control model with one state variable, x, one control variable, u, and r, δ, b being parameters having a positive value. The model is:

$$\max_u \int_0^\infty e^{-rt}(xu - u)dt$$

subject to

$$\dot{x} = u(1 - bu) - \delta x, \quad x(0) = x_0 \geq 0, \tag{1}$$

and

$$u \geq 0.$$

The Hamiltonian H for this problem is given by

$$H = xu - u + \lambda (u(1 - bu) - \delta x).$$

Consequently, the adjoint equation is

$$\dot{\lambda} = r\lambda - H_x = (r + \delta)\lambda - u, \tag{2}$$

while an interior optimal control must satisfy

$$H_u = x - 1 + \lambda [1 - 2bu] = 0. \tag{3}$$

Provided that λ is nonzero (in fact, it will turn out that $\lambda > 0$ for $x > 0$), for optimal trajectories it holds that

$$u = \hat{u} = \frac{1}{2b} \Longleftrightarrow x = \hat{x} = 1. \tag{4}$$

Hence, every optimal trajectory passes through one unique point (*critical point, lens*) when $x = 1$. This is similar to Caulkins et al. (2005) and Kort et al. (2006). In the lens, $\hat{x} = 1$ implies that the contribution to the objective is independent of the value of u.Therefore, in this point the control u is chosen such that the contribution to the state equation is optimized. Since the state variable x is a "good stock", optimization implies maximization of $u(1 - bu)$ with respect to u,which in turn implies that $u = \hat{u} = \frac{1}{2b}$. An important observation is that in the lens the choice of u is independent of the co-state variable λ.

Next, we set up the state-control phase diagram. To determine the $\dot{u}-$equation, we first observe that from (3) it is obtained that

$$\lambda = \frac{1-x}{1-2bu}. \tag{5}$$

Now, differentiating (3) with respect to time eventually gives

$$\dot{u} = \frac{1-2bu}{2b\,(1-x)}\left((r+\delta) - x\,(r+2\delta) + bu^2\right). \tag{6}$$

In order to develop the phase plane, we start by determining the isoclines. From (1), we obtain that the $\dot{x}=0$ isocline becomes

$$\begin{aligned} x &= \frac{1}{\delta}u\,(1-bu) \text{ i.e.} \\ u &= \frac{1}{2b}\left(1 \pm \sqrt{1-4bx\delta}\right). \end{aligned}$$

From (6) it is obtained that $\dot{u}$ is infinite (undefined) for

$$x = 1. \tag{7}$$

Furthermore, $\dot{u}=0$ holds for

$$\begin{aligned} u &= \frac{1}{2b} \text{ or} \\ x &= \frac{(r+\delta)+bu^2}{r+2\delta} \text{ which is equivalent to} \\ u &= \frac{1}{\sqrt{b}}\sqrt{x\,(r+2\delta)-(r+\delta)}. \end{aligned}$$

This leads to the phase plane depicted in Figure 1.
The "top" (maximum x value) of the $\dot{x}=0$ isocline is $x=1/4\delta b$. For a lens to occur, the intersection of the two $\dot{u}=0$ isoclines, $x=\frac{r+\delta+\frac{1}{4b}}{r+2\delta}$, must lie in between $x=1$ and $x=1/4\delta b$. It can be easily derived that this is the case iff

$$b < \frac{1}{4\delta}.$$

In Figure 1 the isoclines are the dash-dotted curves. The saddle point path converging to the (saddle point) equilibrium $(1.5,\ 0.4)$ is the solid curve. In order to see the lens property more clearly, two other trajectories are depicted as dashed curves which also pass through the lens $(1,\ 0.32)$. It is also seen that for small values of x the saddle point path does not exist, so that optimal trajectories corresponding to initial values of the state variable being sufficiently low, will converge to the origin. Hence, the solution is history dependent, meaning that it depends on the initial state where the

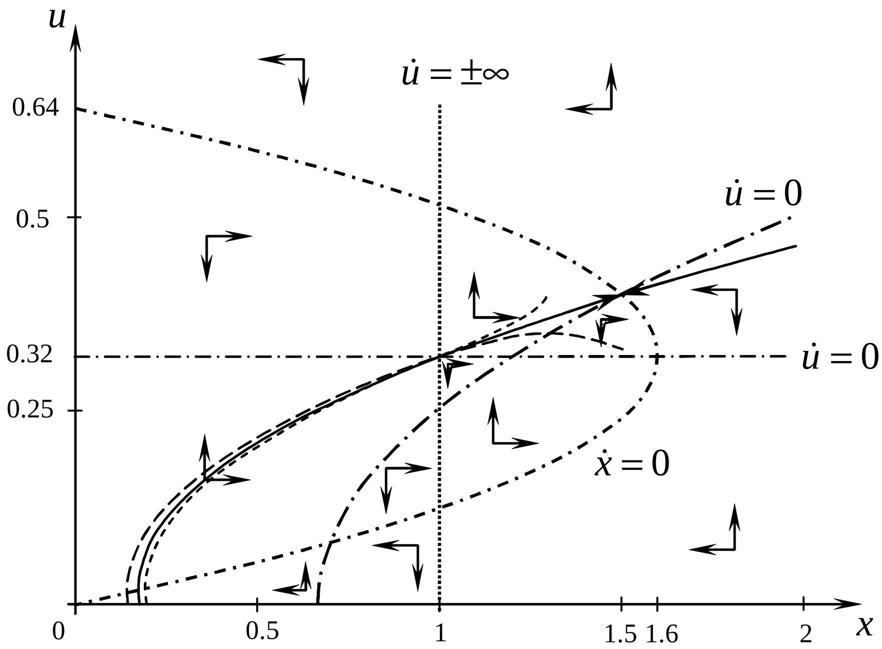

Figure 1: The phase plane for $r = 0.1,\ \delta = 0.1,\ b = \frac{25}{16}$. The lens is $(1,\ 0.32)$ while the saddle point is (1.5, 0.4).

optimal trajectory eventually will converge to. The level of the initial value of the state variable below which optimal trajectories converge to the origin and above which we have convergence to the saddle point is called a Skiba point (see, e.g., Dechert and Nishimura (1983)). The Skiba point is located somewhere at the right of the intersection between the saddle point path and the x-axis.

3 General Characteristics of a model with a lens

3.1 The lens property

Definition 1 *A* lens $(\hat{x}, \hat{u})$ *is an interior point in the state-control phase plane where:*
1. given $\hat{x}$, $\hat{u}$ *is the only control level satisfying the necessary optimality conditions;*
2. given $\hat{u}$, $\hat{x}$ *is the only value for the state variable satisfying the necessary optimality conditions.*

Let us consider the general autonomous infinite horizon optimal control problem with one state and one control:

$$\max_u \int_0^\infty e^{-rt} F(x,u)dt$$

subject to

$$\dot{x} = f(x,u), \quad x(0) = x_0 \geq 0.$$

The Hamiltonian is

$$H = F(x,u) + \lambda f(x,u),$$

so that the maximization condition for an interior control is

$$H_u = F_u(x,u) + \lambda f_u(x,u) = 0$$

A lens property apparently needs $\lambda \neq 0$. Then there can be two situations w.r.t. occurrence of a lens:
Case 1:

$$[F_u(x,u) = 0 \Longleftrightarrow x = \hat{x}] \quad \Longleftrightarrow \quad [f_u(x,u) = 0 \Longleftrightarrow u = \hat{u}], \tag{8}$$

Case 2:

$$[f_u(x,u) = 0 \Longleftrightarrow x = \hat{x}] \quad \Longleftrightarrow \quad [F_u(x,u) = 0 \Longleftrightarrow u = \hat{u}]. \tag{9}$$

This directly leads to the following result:

Proposition 2 ***Necessary and sufficient conditions***
Necessary conditions for a lens to occur are:

- *Case 1:* $F_u(x,u)$ *has no root in* u *for* $x \neq \hat{x}$. *Furthermore,* $x = \hat{x}$ *is the unique root in* x *of* $F_u(x,u)$ *for all possible values of* u, *and*
  $f_u(x,u)$ *has no root in* x *for* $u \neq \hat{u}$, *while* $u = \hat{u}$ *is the unique root in* u *of* $f_u(x,u)$ *for all possible values of* x.

- *Case 2:* $f_u(x,u)$ *has no root in* u *for* $x \neq \hat{x}$. *Furthermore,* $x = \hat{x}$ *is the unique root in* x *of* $f_u(x,u)$ *for all possible values of* u, *and*
  $F_u(x,u)$ *has no root in* x *for* $u \neq \hat{u}$, *while* $u = \hat{u}$ *is the unique root in* u *of* $F_u(x,u)$ *for all possible values of* x.

If $\lambda \neq 0$ in $(\hat{x}, \hat{u})$ then these necessary conditions are also sufficient for $(\hat{x}, \hat{u})$ to be a lens.

An interaction term is clearly needed. This is because in a purely separable model where the objective function is $F(x, u) = K(x) + G(u)$ and the r.h.s. of the state equation is $f(x, u) = k(x) + g(u)$, the Hamiltonian maximization condition for the control does not contain the state variable x. Since control-state interaction is also one of the mechanisms that could generate history dependence (see, e.g., Hartl et al. (2004)), this explains the simultaneous occurrence of Skiba behavior in the example we presented in the previous section as well as in the models of Caulkins et al. (2005) and Kort et al. (2006).

3.2 Sufficient conditions for a lens

In Case 1 above we need $F_u(x, u)$ to have no root in u for $x \neq \hat{x}$. Furthermore, $x = \hat{x}$ is the unique root of $F_u(x, u)$ for all possible values of u. The simplest case is when F is linear in u, (i.e. F_u does not depend on u), and F_u is simply $F_u(x, u) = \gamma(x - \hat{x})$, i.e., $F(x, u) = \gamma(x - \hat{x})u$.

In Case 1 we further need $f_u(x, u)$ to have no root in x for $u \neq \hat{u}$, while $u = \hat{u}$ is the unique root of $f_u(x, u)$ for all possible values of x. The simplest case is when f does not contain x at all and $f_u = \alpha(u - \hat{u})$, i.e., $f(x, u) = \alpha\left(\frac{u}{2} - \hat{u}\right)u$.

In Case 2, simply the functions f and F need to be interchanged in the above arguments. The above is summarized in the following corollary.

Corollary 3 *A set of sufficient conditions for a lens to occur is*

- *one interaction term being linear in u, e.g. xu*
  one quadratic term in u
  two linear terms in u
- *interaction term and one linear term with opposite signs in objective or in state equation*
- *quadratic term and the other linear term with opposite signs in:*
  1. *the state equation when the interaction term occurs in the objective;*
  2. *the objective when the interaction term occurs in the state equation.*

Examples of Case 1, i.e., where the quadratic term occurs in the state equation while the interaction term occurs in the objective, are the fashion model by Kort et al. (2006) and the simple model presented in the previous section. The migration model by Caulkins et al. (2005) is an example of Case 2.

4 Epilogue

In this final section we like to discuss the generalization of the lens property to optimal control problems with more than one state or control, and the simultaneous occurrence of lens and Skiba point. Concerning the possibility for generalization to optimal control problems with more than one state or control, it is important to observe that the lens property results from a special property of the optimality condition with respect to the control. Then it follows that generalization is straightforward as long as this condition contains only one state and one control. This is the case for a particular control if it interacts only with one state and there are nonlinear terms associated with this control. This could also quite easily occur in models with more than one state or control.

Our final comment is about joint occurrence of Skiba and lens. In the three known models (the simple example in this paper, Caulkins et al. (2005), and Kort et al. (2006)), where the lens property occurs, there was also a Skiba point. In the main text we argued that control-state interaction is necessary for existence of a lens, while it is at the same time a mechanism that could generate a Skiba point. Therefore, it seems safe to say that occurrence of a lens raises the probability of existence of a Skiba point. However, in principle it must be possible to have lens without Skiba, although until now it was not found yet.

References

[1] Caulkins, J.P., Feichtinger, G., Johnson, M., Tragler, G., & Yegorov, Y. (2005). Skiba thresholds in a model of controlled migration, *Journal of Economic Behavior and Organization*, 57, 490-508.

[2] Dechert, D.W., & Nishimura, K. (1983). Complete characterization of optimal growth paths in an aggregated model with a nonconcave production function, *Journal of Economic Theory*, 31, 332-354.

[3] Hartl, R.F., Kort, P.M., Feichtinger, G., & Wirl, F. (2004). Multiple equilibria and thresholds due to relative investment costs, *Journal of Optimization Theory and Applications*, 123, 49-82.

[4] Kort, P.M., Caulkins, J.P., Hartl, R.F., & Feichtinger, G. (2006). Brand integrity and brand dilution in the fashion industry, *Automatica*, 42, 1363-1370.