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THE NON- AND SEMIPARAMETRIC ANALYSIS OF MS MODELS: SOME APPLICATIONS*

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ABSTRACT. This paper illustrates how to compare different microscopic simulation (MS) models and how to compare a MS model with real data in case the parameters of interest are estimated non- or semiparametrically. As examples we investigate the marginal single-period probability density function of stock returns, and the corresponding spectral density function and memory parameters. We illustrate the methodology by the MS models developed by Levy, Levy, Solomon (2000) and the market fraction model developed by He and Li (2005a, b), and confront the resulting return data with the S&P 500 stock index data.

JEL classification: C14, G12

Keywords: Microscopic simulation models; Probability density function; Spectral density function; Memory parameters

1. INTRODUCTION

Many of the classical models in finance are based upon the assumptions of investor homogeneity and expected utility theory, including, for instance, the Capital Asset Pricing Model (CAPM), see, for example, Cochrane (2001). Although recent research reveals that these assumptions are hard to maintain (see, for instance, Barberis *et al.*, 2001, and Haliassos and Hassapis, 2001, for recent discussions), they are still used because of their analytical tractability. The developments of computational power provide the possibility to relax these assumptions through the use of Microscopic Simulation (MS) techniques. The idea is to study financial markets by representing each of the investors individually (on a computer) and by simulating the behavior of the entire market, keeping track of all of the investors and their interactions over time. Throughout the simulation, the variables that are of interest, including, for example, asset prices or asset returns, can be recorded, and their dynamics can be investigated.

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The MS models of financial markets that result from the interaction of heterogeneous agents, having different attitudes towards risk and different expectations about future asset returns, have been studied intensively in the literature, see, for example, Chiarella and He (2003), Hommes (2005), LeBaron (2000, 2005), Levy *et al.* (2000), and Lux (1998). So far, research has mainly focused on investigating whether a single or some 'representative' simulation runs of an MS model shares some important characteristics found in actual financial markets, the so-called stylized facts, such as short-term momentum, excess volatility, heavy trading volume, a positive correlation between volume and contemporaneous absolute returns, endogenous market crashes, etc., see Hommes (2005), for instance. Although these works provide various ways to explain the stylized facts, to our knowledge, systematic procedures to investigate the differences between two MS models or to judge whether a MS model is realistic or not have not yet been developed.

The work of Li *et al.* (2005) is a first attempt along this line. In Li *et al.* (2005), we study how to compare different microscopic simulation (MS) models and how to compare a MS model with actual data. Essentially, we investigate the comparison of some given features of the distribution of the outcome series of interest (like asset returns) by applying econometric techniques. In the current paper we apply this methodology to some particular relevant features of the distribution of asset returns, namely the marginal one-period probability density function, the spectral density function, and particular memory parameters. The marginal one-period probability density function gives insights into the distribution of the return process. This is linked directly to one of the stylized facts of high frequency financial data, namely leptokurtosis, in which case the return process exhibits high peaks around the mean and has fat tails. The spectral density function measures the global cyclical behavior of the return process, and as such contains much information about the dynamics of the return process of financial markets. In fact, in the macroeconomics literature (see, for instance, Diebold *et al.*, 1998), the comparison of the spectral density functions is a standard way to compare the dynamics of model generated data with actual data. Finally, as memory parameters we take the order of fractional integration in an ARFIMA process, which measures the speed of decay of autocorrelations. In this way we can investigate the presence of long range dependence in, for instance, squared or absolute returns.

We shall make the assumption of (strict) stationarity. Since we are dealing with return data, this seems to be an acceptable approximation. Standard estimators and tests are then available for the marginal one-period probability density function, the spectral density function, and the memory parameters that we consider, and can be applied straightforwardly. For instance, when making a comparison in terms of the marginal one-period probability density function or the spectral density function, nonparametric kernel estimators and tests based upon these can be applied. For the memory parameter we shall make use of available semiparametric techniques, including the estimators of the fractional differencing parameter proposed by Geweke and Poter-Hudak (1983) and Robinson and Henry (1999).

A special feature of MS models is that its outcome series can be observed along two dimensions, namely, we can observe the outcome series for each run of the simulation, and we can run the MS model independently many times. Consequently, the parameters of interest might be estimated in different ways. We shall exploit this possibility heavily in this paper when applying the non- and semiparametric techniques, particularly, when comparing actual data with outcomes of MS models.

The remainder of the paper is organized as follows. In Section 2, we introduce how to compare two different MS models, and how to compare a MS model with actual data in terms of the marginal probability density function, the spectral density function, and memory parameters. In Section 3 we illustrate the methods by the MS models developed by Levy, Levy, Solomon (see Levy *et al.*, 2000), and the market fraction model developed by He and Li (2005a, b), where we use in both cases the Standard & Poor 500 index (hereafter S&P 500) to represent the real world. We conclude in Section 4.

2. ECONOMETRIC METHODS

Given the outcomes of interest of MS models, for instance, the stocks returns, we are interested in how to assess the differences between two MS models and that between an MS model and actual data in terms of specific characteristics of the stock return distribution. As characteristics we shall consider in this section the marginal single-period probability density function, the spectral density function, and memory parameters. We shall make the assumption of (strict) stationarity, which seems to be an acceptable approximation, since we are dealing with return data. When dealing with asymptotic results, we shall also assume 'asymptotic independence' in the form of appropriate mixing conditions, cf., for example, Bierens (2004).

2.1. **The marginal one-period probability density function.**

With observations $\{x_t\}_{t=1}^T$, the marginal one-period probability density function f evaluated at $x \in (a,b)$ can be estimated consistently (under appropriate regularity conditions, see, for example, Pagan and Ullah, 1999) by the Nadaraya-Watson kernel estimator

$$\hat{f}(x) = \frac{1}{Th}\sum_t K(\frac{x - x_t}{h}) \tag{1}$$

for each x in the interval (a,b), where K is an rth order kernel function and h is the smoothing parameter.

Under some regularity conditions (see Pagan and Ullah, 1999, for instance), this nonparametric estimator has an asymptotic normal distribution, when $Th \to \infty$, $T \to \infty$, and $h \to 0$

$$\sqrt{Th}(\hat{f}(x) - s(x)) \to \mathcal{N}\left(0, f(x)\int K^2(u)du\right),\ a < x < b, \tag{2}$$

where K is an appropriate kernel function, and where the asymptotic bias vanishes when $\sqrt{Th}h^2 \to \infty$. On the basis of this limit distribution, one can easily construct

pointwise confidence intervals. However, rather than relying on the asymptotic limit distribution, we apply the bootstrap (see, for instance, Horowitz 2001, 2003, for general discussions) to construct such pointwise confidence intervals, where we follow Hall (1992). As discussed by Hall (1992), there are two ways to deal with the asymptotic bias. One is the method of explicit bias removal, the other method is undersmoothing, which relies on choosing the bandwidth h so small that the asymptotic bias becomes negligible. Hall (1992) compares these two methods in terms of the errors in the coverage probabilities of bootstrap confidence intervals. The conclusion seems to be that undersmoothing is the better method for handling the asymptotic bias, when the aim is to minimize the differences between the true and nominal rejection and coverage probabilities of the bootstrap-based confidence intervals. To get this result a fourth order kernel, such as $K(z) = (15/32)(7z^4 - 10z^2 + 3)$ if $|z| < 1$, and $K(z) = 0$ otherwise, needs to be used. Accordingly, this kernel and Hall's suggestion on undersmoothing will be used for the bias removal in our study.

Obviously, a set of $(1 - \alpha)$ pointwise confidence intervals constructed for a discretized finite interval will not achieve $(1 - \alpha)$ joint coverage probability. So, we also consider a uniform confidence band. Hall (1993) suggests a bootstrap procedure to construct a simultaneous confidence bands which we will use in our study. Hall (1993) suggests to remove the asymptotic bias by the so-called explicit method, because it is difficult to determine the appropriate amount of undersmoothing. Hall (1993) uses the standard normal kernel to estimate f, while the second order approximation of the bias is estimated by estimating the second order derivative of f, appearing in the bias approximation, by using the second derivative of the standard normal kernel. We will follow these suggestions in our study.

When the aim is to compare two MS models, we are dealing with two time series $\{x_t\}$ and $\{y_t\}$ associated with the marginal probability density function f of one MS model and g of an other MS model, respectively, and we are interested in testing the null hypothesis $H_0 : f(x) = g(x)$ against the alternative $H_1 : f(x) \neq g(x)$. We shall test these hypotheses by applying a test proposed by Li (1996), which is based on $\hat{I} = \int (\hat{f} - \hat{g})^2 dx$. After rewriting, $\hat{I}$ can be expressed as

$$\hat{I} = \hat{I}_1 + 2\frac{K(0)}{Th} + O_p(T^{-1}) \tag{3}$$

where

$$\hat{I}_1 = \frac{1}{T^2 h} \sum_{i=1}^{T} \sum_{j=i, j\neq i}^{T} \left[K(\frac{x_i - x_j}{h}) + K(\frac{y_i - y_j}{h}) - K(\frac{y_i - x_j}{h}) - K(\frac{x_i - y_j}{h}) \right].$$

Li (1996) shows that under H_0, when $h \to 0$, and $Th \to \infty$,

$$J_T := \frac{\sqrt{Th}\hat{I}_1}{\hat{\sigma}_1} \xrightarrow{d} \mathcal{N}(0, 1), \tag{4}$$

where

$$\hat{\sigma}_1^2 = \frac{2}{T^2 h} \sum_{i=1}^{T} \sum_{j=1}^{T} \left[K(\frac{x_i - x_j}{h}) + K(\frac{y_i - y_j}{h}) + 2K(\frac{x_i - y_j}{h}) \right] \int K^2(\psi) d\psi.$$

To use this result to compare the marginal probability density functions of two MS models, we define the averaged, over simulations, estimator of the probability density function $f(x)$ as

$$\hat{f}^{Si}(x) := \frac{1}{N} \sum_{n} \hat{f}^n(x), \tag{5}$$

where the superscript Si indicates that this estimator is obtained from the average over independent simulations; $\hat{f}^n$ is the kernel estimator of the probability density function f based upon the n-th realization of the MS model, so $\hat{f}^{Si}$ is the average of the kernel estimators over independent simulations. The estimator $\hat{f}^{Si}(x)$ is a consistent estimator of $f(x)$ for $T \to \infty$ with

$$\sqrt{Th}(\hat{f}^{Si}(x) - f(x)) \to \mathcal{N}\left(0, \frac{f(x)}{N} \int K^2(u) du\right), \; a < x < b, \tag{6}$$

when $Th \to \infty$, $h \to 0$, and $T \to \infty$. When T goes to infinity and, sequentially, also N goes to infinity, we have

$$\sqrt{TNh}(\hat{f}^{Si}(x) - f(x)) \xrightarrow{d} \mathcal{N}\left(0, f(x) \int K^2(v) dv\right), \tag{7}$$

when $TNh \to \infty$, $h \to 0$, $T \to \infty$ and $N \to \infty$ (and the asymptotic bias vanishes when $\sqrt{TNh}h^2 \to 0$). In other words, we can test $H_0 : f(x) = g(x)$ based upon the estimators $\hat{f}^{Si}$ and $\hat{g}^{Si}$ using all the data in one test. The corresponding $\hat{\sigma}_1^2$ appearing in Li's test can be computed by using all of the NT observations

Finally, we turn to the problem of comparing the probability density function of a MS model with that of the real world. So, let $f(x)$ and $g(x)$ be the probability density functions of the real world and a MS model, respectively, and we are again interested in testing the null hypothesis $H_0 : f(x) = g(x)$ against the alternative $H_1 : f(x) \neq g(x)$. $f(x)$ can be estimated by the kernel estimator, $\hat{f}(x)$, using the time series observations; $g(x)$ can be estimated by $\hat{g}^{Si}$ using the time series observations over many simulation runs. Obviously, $\hat{g}^{Si}$ converges much faster than $\hat{f}(x)$, if we also exploit the simulation limit $N \to \infty$ in addition (and sequential) to $T \to \infty$. This means, when testing equality of $f(x)$ and $g(x)$, that we can ignore the estimation/simulation inaccuracy in $\hat{g}^{Si}$. Testing equality of $f(x)$ and $g(x)$ can be done, for instance, by simply checking whether $\hat{g}^{Si}$ lies in the confidence bound around $\hat{f}(x)$, where the confidence bound is solely based on the estimation inaccuracy of $\hat{f}(x)$.

2.2. **The spectral density function.** A standard way of investigating the dynamics commonly used in the macroeconomics literature is based on the spectral density function (see, for instance, Diebold *et al.*, 1998). The spectral analysis treats the time

series globally and focuses on analyzing the series as a whole, decomposing it into its periodic components corresponding to different frequencies. This is in contrast to comparing only a limited number of variances and covariances of the time series. As a consequence, this method yields a complete second-order comparison of the dynamic properties of model and data, providing a complete summary of the time series dynamics.

We start with reviewing some basic concepts. For a time series $\{x_t\}$, let $\gamma(k)$ denote its autocovariance at lag k. Then its spectral density function is defined as

$$s(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} e^{-ik\omega}\gamma(k) = \frac{1}{\pi}\left[\gamma(0) + 2\sum_{k=1}^{+\infty}\gamma(k)\cos(\omega k)\right]. \tag{8}$$

Suppose we have observations $x_1, x_2, ..., x_T$. Let L denote the largest integer less than or equal to $T/2$, i.e., $L = [T/2]$, and let ω_k be the frequency given by $2\pi k/T$, $-L \leq k \leq L$. Let

$$I_T(\omega) = \frac{1}{T}\left|\sum_{k=1}^{T} x_k e^{ik\omega}\right|^2, \ -\pi \leq \omega \leq \pi, \tag{9}$$

be the periodogram of the sample, then the spectral density function $s(\omega)$ can be estimated by

$$\hat{s}(\omega, h) = \frac{1}{Th}\sum_{k=-L}^{L} K(\frac{\omega - \omega_k}{h}) I_T(\omega_k), \ -\pi \leq \omega \leq \pi. \tag{10}$$

Because the precision of the sample analogue of $\gamma(k)$ decreases as the lag k increases, this estimation consists of a weighting procedure, which gives less weight to the values of the autocovariance function at higher lags. Various weighting functions (kernel functions) can be used to estimate the spectrum, for instance, the Blackman-Tukey weight (see Chatfield, 2004, for more detailed discussion on these issues).

Under some regularity conditions (see Priestly, 1981, for instance), the above nonparametric estimator of the spectral density function has an asymptotic normal distribution. We have

$$\sqrt{Th}(\hat{s}(\omega) - s(\omega)) \to \mathcal{N}\left(0, s^2(\omega)\int K^2(u)du\right), \ 0 < \omega < \pi \tag{11}$$

when $Th \to \infty$, $T \to \infty$, and $h \to 0$, where K is a kernel function satisfying the assumptions in Priestly (1981). On the basis of this limit distribution, one can easily construct point wise confidence intervals. However, similar to the case of the marginal one-period probability density function, we shall construct point wise confidence intervals based on the procedure of bootstrapping kernel estimation of the spectral density function. A way to do this is proposed by Franke and Härdle (1992). We shall follow this approach in our application.

For the uniform confidence band, similar to that of Hall (1993) for the probability density function and Swanepoel and van Wyk (1986) for the spectral density function

but with parametric estimation, we discuss in the appendix how *potentially* such a uniform confidence band might be constructed. We shall use this potential uniform confidence band in addition to the point wise version. Notice, however, that the formal theory behind this construction still needs to be developed.

In our situation, we can run a MS model for many time periods and independently many times. In order to use all of the available information, we define, similar to the case of the marginal one-period probability density function, the estimator

$$\hat{s}^{Si}(\omega) := \frac{1}{N}\sum_{n}\hat{s}^{n}(\omega), \tag{12}$$

where $\hat{s}^n$ is the estimated spectral density based upon the nth realization of the MS model, and then $\hat{s}^{Si}$ is the average of estimated spectral density over independent simulation. So, $\hat{s}^{Si}$ is a consistent estimator of s for $T \to \infty$ with

$$\sqrt{Th}(\hat{s}^{Si}(\omega) - s(\omega)) \to \mathcal{N}\left(0, \frac{s^2(\omega)}{N}\int K^2(u)du\right),\ 0 < \omega < \pi. \tag{13}$$

When, sequentially, also N goes to infinity, we get[1]

$$\sqrt{TNh}(\hat{s}^{Si}(\omega) - s(\omega)) \to \mathcal{N}\left(0, s^2(\omega)\int K^2(u)du\right),\ 0 < \omega < \pi. \tag{14}$$

So, similar to the case of the probability density function, the spectral density function estimator in case of the actual data will have rate of convergence $\sqrt{Th}$, while in case of the simulation based case, it will have rate of convergence $\sqrt{TNh}$. So, again, when sequentially to $T \to \infty$ also $N \to \infty$, the simulation based estimator will have a much higher rate of convergence, and may be considered (at least, asymptotically) to be known, when compared with the actual data based estimator.

2.3. **Semiparametric estimation of memory parameters.** There are several possible definitions of the property of "long memory". Following Baillie (1996), given a series x_t, $t = 0, \pm 1$..., with autocorrelation function ρ_j at lag j, we say that the process possesses long memory if ρ_j decays "slowly", i.e., if the quantity

$$\lim_{n\to\infty}\sum_{j=-n}^{j=n}|\rho_j| \tag{15}$$

is nonfinite. Equivalently, the spectral density $s(\omega)$ will be unbounded at low frequencies. We semiparametrically model long memory in a covariance stationary series x_t, $t = 0, \pm 1, ...$, by

$$s(\omega) \approx G\omega^{-2d},\ \omega \to 0^{+}, \tag{16}$$

where $0 < G < \infty$, and $s(\omega)$ is the spectral density of x_t. Under (16), $s(\omega)$ has a pole at $\omega = 0$ for $0 < d < 1/2$ (when there is long memory in x_t), while $d \geq 1/2$

[1]Notice that if T is fixed and only N goes to infinity, in general, it is impossible to get a consistent estimator for the spectral density function unless we have specific restrictions on $\gamma(k)$, for instance, $\gamma(k) = 0$ for $k > T$.

implies the process is not covariance stationary; $s(\omega)$ is positive and finite for $d = 0$; for $-1/2 < d < 0$, we have short memory, negative dependence, or antipersistence.

Geweke and Poter-Hudak (1983), henceforth GPH, suggested a semiparametric estimator of the fractional differencing parameter, d, that is based on a regression of the ordinates of the log spectral density. The estimator exploits the theory of linear filters to write the process $(1 - L)^d y_t = u_t$, where $u_t \sim I(0)$, i.e., the process $\{u_t\}$ is stationary. Let $s_y(\omega)$ and $s_u(\omega)$ be the spectral densities of y_t and u_t, respectively. Then

$$s_y(\omega) = \left|1 - e^{-i\omega}\right|^{-2d} s_u(\omega). \tag{17}$$

This can be expressed as

$$\log s_y(\omega) = \log(4\sin^2(\omega/2))^{-d} + \log s_u(\omega). \tag{18}$$

Given spectral ordinates $\omega_1, \omega_2, ..., \omega_m$, this becomes

$$\log s_y(\omega_j) = \log s_u(0) - d\log(4\sin^2(\omega_j/2)) + \log(s_u(\omega_j)/s_u(0)). \tag{19}$$

GPH suggest estimating d from a regression of the ordinates from the periodogram of y_t, that is $I_y(\omega_j)$. Hence, for $j = 1, 2, ..., m$,

$$\log I_y(\omega_j) = c - d\log(4\sin^2(\omega_j/2)) + v_j, \tag{20}$$

where

$$v_j = \log(s_u(\omega_j)/s_u(0)) \tag{21}$$

and v_j is assumed to be i.i.d. with zero mean and variance $\pi^2/6$. When u_t is white noise, then the above regression should provide a good estimate of d. When u_t is autocorrelated, GPH show that the above regression holds approximately for frequencies in the neighborhood of zero. If this neighborhood shrinks at an appropriate rate with the sample size, then the GPH procedure should realize a consistent estimator of d. If the number of ordinates m is chosen such that $m = g(T)$, where $g(T)$ is such that $\lim_{T\to\infty} g(T) = \infty$, $\lim_{T\to\infty} g(T)/T = 0$, $\lim_{T\to\infty}(\log(T)^2)/g(T) = 0$, then the OLS estimator of d based on (20) will have the limiting distribution

$$\sqrt{m}(\hat{d}_{GPH} - d) \xrightarrow{d} \mathcal{N}(0, \frac{\pi^2}{24}) \tag{22}$$

Robinson (1995) provided the formal proof for $-1/2 < d < 1/2$, Velasco (1998) proved the consistency of $\hat{d}_{GPH}$ in the case $1/2 \leq d < 1$ and its asymptotic normality in the case $1/2 \leq d < 3/4$. The variance of this estimator can be obtained from the usual OLS regression formula. It is clear from this result that the GPH estimator is not $T^{1/2}$ consistent and will converge at a slower rate.

Another most often used estimator of d is developed by Robinson and Henry (1999), henceforth RH. They suggest a semiparametric Gaussian estimate of the memory parameter d, by considering

$$\hat{d}_{RH} = \arg\min_d R(d), \tag{23}$$

where $R(d)$ is

$$R(d) = \log\left\{\frac{1}{m}\sum_{j=1}^{m}\omega_j^{2d}I(\omega_j)\right\} - 2\frac{d}{m}\sum_{j=1}^{m}\log\omega_j, \tag{24}$$

in which $m \in (0, [T/2])$ and $\omega_j = 2\pi j/T$. They proved that when $m < [T/2]$ such that, as $T \to \infty$,

$$\frac{1}{m} + \frac{m}{T} \to 0, \tag{25}$$

and under some further conditions (see Robinson and Henry, 1999), we have

$$\sqrt{m}(\hat{d}_{RH} - d) \to \mathcal{N}(0, \frac{1}{4}) \tag{26}$$

as $T \to \infty$.

In case of the simulation models we can construct $\hat{d}_b^{Si} = \frac{1}{N}\sum_{n=1}^{N}\hat{d}_b^n$, with $\hat{d}_b^n$ the estimator of d in simulation run n, and $b \in \{GPH, RH\}$. We have

$$\sqrt{Nm}(\hat{d}_{GPH}^{Si} - d) \xrightarrow{d} \mathcal{N}(0, \frac{\pi^2}{24}),$$

and

$$\sqrt{Nm}(\hat{d}_{RH}^{Si} - d) \xrightarrow{d} \mathcal{N}(0, \frac{1}{4}).$$

So, in case of the actual data the estimators will have rate of convergence $\sqrt{m}$, while in case of the simulation based case, it will have rate of convergence $\sqrt{Nm}$. So, again as in the previous cases, when sequentially also $N \to \infty$, the simulation based estimator will have a much higher rate of convergence, and may be considered (at least, asymptotically) to be known, when comparing it with the actual data based estimator.

3. APPLICATIONS

The MS literature attempts to explain various types of market behavior, and to replicate the well documented empirical findings of actual financial markets, the stylized facts. The recent literature has demonstrated the ability to explain various stylized facts, see, for instance the recent survey papers Hommes (2005) and LeBaron (2005). But to our knowledge, most of the MS models match some stylized facts only to a limited extent; on the other hand, systematic procedures to investigate the differences between two MS models or to judge whether a MS model is realistic or not have not yet been developed. Hence, our applications here are made for illustrative purposes, in the hope that they will gain a better understanding of the MS models (under consideration) and that they might be of help in developing even more realistic MS models.

In this section, we will study two MS models. Firstly, we investigate the model by Levy *et al.* (2000), to which we shall refer to as the LLS model. Secondly, we study the market fraction (MF) model proposed by He and Li (2005a, b).

3.1. **Investigating the LLS model.** We use the LLS model as a representation of MS models. Because the LLS model is calibrated to quarterly frequency, we use the quarterly data of the S&P 500 from Datastream as representation of the real life situation, starting in 1965 and running to the first quarter of 2003.

For the method of comparison of two different MS models, we perform a sensitivity analysis of the initial conditions and parameter settings in the LLS-economy to illustrate the method to compare two different MS models. We illustrate the sensitivity analysis in terms of the initial price. The subsequent sensitivity analysis in terms of initial dividend, the risk aversion parameter, the average dividend growth rate, and initial wealth is performed in a similar way. To analyze the sensitivity to the initial price, we first simulate the benchmark model. In the benchmark LLS model, we choose, following Levy *et al.* (2000), the initial price $IP = 20.94$, the initial dividend $ID = 0.5$, the risk aversion parameter $RA = 1.5$, the maximal one period dividend decreases $MDD = -0.07$, and the initial wealth $IW = 1000$. Next, we keep all the conditions and parameters the same, except for the initial price. Two additional simulations are performed, one with an initial price $IP = 26$ that is higher than the benchmark price, and an other one with a lower initial price, namely $IP = 16$. Then we look at the impact of these different initial prices on the output parameters, in particular, the probability density function, and see whether these parameters significantly deviate from those of the benchmark model. Next, we do the same exercises in terms of the initial dividend (with $ID \in \{0.4, 0.6\}$), the risk aversion parameter (with $RA \in \{1.45, 1.55\}$), the maximal one period dividend decreases (with $MDD \in \{-0.08, -0.06\}$), and the initial wealth. In the latter case we consider two variations: In the first case (unif.), the initial wealth is uniformly distributed over [500, 1500], while in the second case (50%) half of the investors have an initial wealth of 500 and the other half have of 1500. In this study, for each set of parameter, we ran 5,000 independent simulations over 1,000 time periods, and for each run of the model we use the last 152 observations to match the sample size of the S&P 500 that we use.

Probability density function: In Table 1, we report the results of the sensitivity analysis in terms of Li (1996)'s test (J_T). We find that the changes considered do not have a serious impact, with one exception: MDD has a significant impact. The reason for this seems to be that the dividend process is the driving force in the LLS-economy, and changing MDD means changing the dividend distribution. These test results are quite similar to those in Li *et al.* (2005), where a sensitivity analysis is performed in terms of parametrically estimated output parameters.

TABLE 1. Sensitivity analysis in terms of probability density function

	IP		ID		RA		MDD		IW	
	16	26	0.4	0.6	1.45	1.55	-0.08	-0.06	Unif.	50%
J_T	1.7237	1.8158	1.7436	1.8845	1.8150	1.7028	17.0767	10.9783	1.7699	1.8100

Besides the comparison of two different economies, we next compare the benchmark LLS model with the real world, which is represented by the S&P 500. We plot

the probability density functions of the LLS model, the S&P 500, and that of a normal distribution with mean and variance equal to that of S&P 500 in Figure 1[2]. In addition, we also plot a 95%-point wise confidence interval in Figure 1(a) and a 95% uniform confidence band in Figure 1(b) around the S&P 500 density estimate. The normal density fits in both the point wise and uniform confidence intervals, but the LLS-model based average density does not so fully in the point wise confidence interval, while it almost fits in the uniform confidence band. Thus, the actual return distribution as a whole can be reproduced by the LLS-model according to the confidence band, but at some particular return outputs the LLS model has difficulty in fitting the S&P 500 distribution. This applies particularly to returns close to zero.

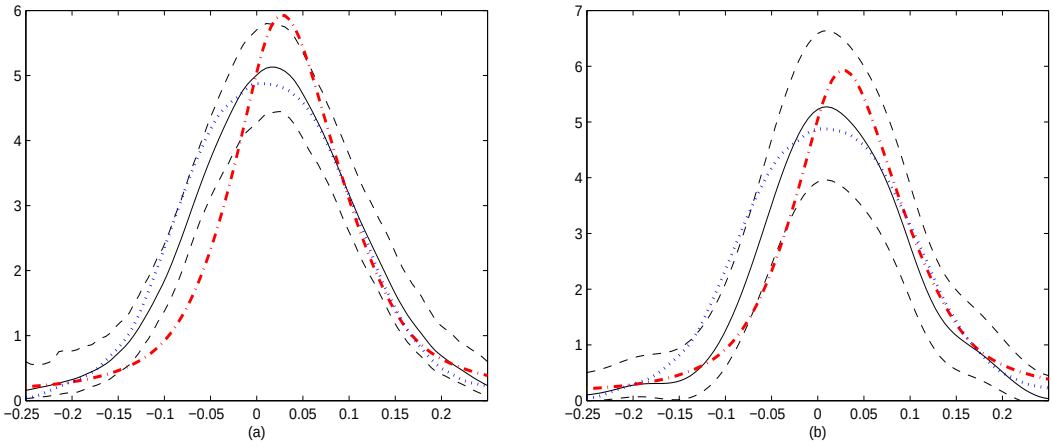

FIGURE 1. The estimated probability density functions of the LLS model (dash-dot line), normal distribution (dot line), and of the S&P 500 with confidence intervals (a) and confidence band (b)

Spectral density function: First, we implement the sensitivity analysis in terms of the spectral density function. Although we do not apply a formal test, we base our sensitivity analysis on a comparison of the confidence intervals, following here Diebold *et al.* (1998). We present the estimated spectral density functions for the benchmark LLS model, for the LLS model with price $IP = 16$, and that with price $IP = 26$ and their corresponding 95% pointwise confidence intervals in Figure 2. Similarly, we do the same exercise in terms of initial dividend, the risk aversion parameter, the maximal one period dividend decrease, and initial wealth; these results are presented in Figure 3. We find that the LLS model is quite robust in terms of the spectral density function with respect to the chosen initial parameters, except, again, for the maximal one period dividend decrease, MDD, indicating that the dividend process also seriously influences the shape of the spectral density function.

[2]The estimated density functions of the S&P 500 are slightly different in (a) and (b), this is because we follow Hall (1992, 1993) and use different kernels.

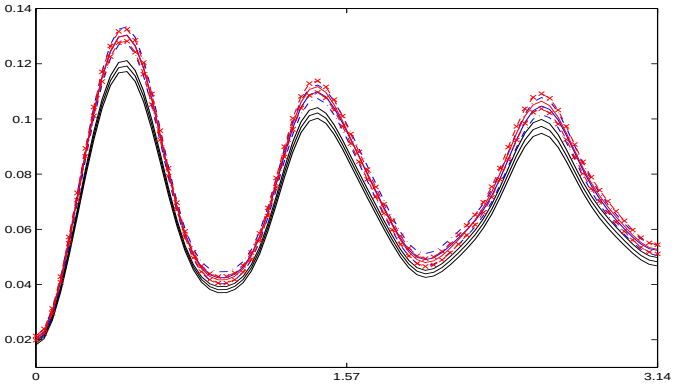

FIGURE 2. The spectral density functions with its 95% pointwise confidence intervals of benchmark LLS model (solid line), and those of LLS model with $IP = 16$ (dash-dot line), and of LLS model with $IP = 26$ (line with cross).

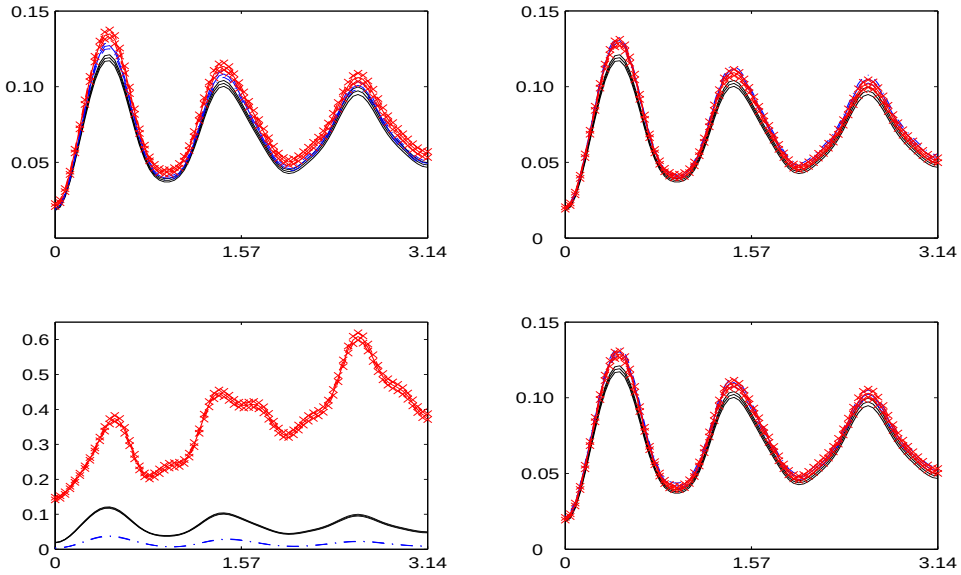

FIGURE 3. The spectral density functions with its 95% pointwise confidence intervals of the benchmark LLS model, and those of the LLS model in terms of different initial dividend (a), different risk aversion parameter (b), different MDD (c), and different initial wealth (d)

Next, we present the estimated spectral density function for the quarterly returns of the S&P 500, and its corresponding 95% confidence intervals in Figure 4. We also estimate the spectral density function for each simulation of the benchmark LLS model. Again, for the purpose of comparison, we only use the last 152 periods of the simulated time series for estimation, which matches the length of the S&P 500 data

that we use. In Figure 4(a) we plot the averaged spectral density function over 5,000 simulations, and in Figure 4(b) we do the same for the uniform confidence band. We find that, except for a few rather small frequencies, the LLS-based spectral densities lies outside of the 95% confidence bands of the spectral density function of the S&P 500. When we compare the frequencies corresponding to the peaks of the spectra, which describe the cycles that dominate the cyclical behavior of the dynamics, we see that they are not at the same pace. Thus, there seems to be a large difference when we assess the second order moments between the LLS generated data and the real life data.

An obvious approach to obtain a better fit of the actual data seems to be calibration. Taking into account that changes in the dividend process influence both the marginal one-period probability density function and the spectral density function, it seems to make sense to perform a calibration in terms of the parameters of the current dividend process of the LLS model, or, alternatively, in terms of the parameters of some adapted, more flexibly specified dividend process.

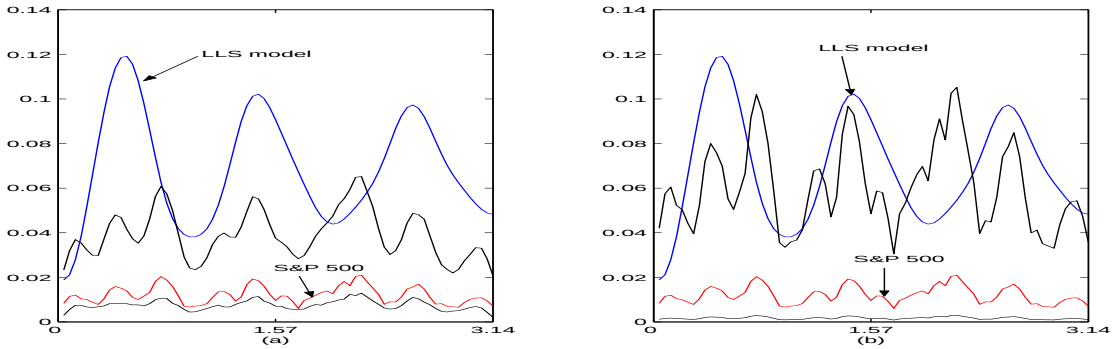

FIGURE 4. The spectral density functions with its 95% confidence intervals (a) and uniform confidence band (b) of the S&P 500 and the LLS model

3.2. **Investigation the Market Fraction model.** As another example, this section provides an analysis on the market fraction model proposed by He and Li (2005a, b), and a calibrated version, calibrated to actual data as described in Li *et al.* (2005), where the calibration is based on minimizing a distance between actual and model based autocorrelations of various orders of both the returns, the squared returns, and the absolute returns. The real world is represented by the S&P 500 stock market index. The market fraction model is calibrated to daily frequencies, using the daily closing price index of the S&P 500. There are altogether 5,306 observations from Oct 20, 1982 to Oct 27, 2003. Denote p_t as the price index for S&P 500 at time t ($t = 0, ..., 5305$) and log returns r_t are defined as $r_t = \ln p_t - \ln p_{t-1}$. In case of the market fraction model we are particularly interested in the study of the long memory properties of the market fraction model, since this model aims at giving an explanation of long range dependence in stock returns.

In this study, for each set of parameters, we ran 1,000 independent simulations over 6,306 time periods and discard the first 1,000 time periods to wash out the initial noise effect. Thus, for each run of the model we have 5,306 observations, which matches the sample size of the S&P 500 that we use.

Memory parameters: A well known empirical feature of high frequency (daily) financial time series is that the returns themselves contain little serial correlation, but the squared returns and absolute returns have a significant positive serial correlation over long lags, see, for instance, Ding *et al.* (1993). We first report the estimation results for the S&P 500, and then we analyze the market fraction model. We notice that a major issue in the application of the GPH and the RH estimators is the choice of m, due to the fact that some limited knowledge is now available concerning this issue (see, Geweke, 1998, for instance), it is a wise precaution to report the estimated results for a range of bandwidths. For instance, Lobato and Savin (1998) use $m = 30$, 40, 50, 60, 70, and 80 for S&P 500 data from 1973 to 1994, in the comment by Geweke (1998), the experiments results seem to suggest larger m, and Robinson (1998) suggests to report the results for larger values of m given the sample size and the fact that $\hat{d}$ is only $m^{1/2}$-consistent. So in our study, for both the GPH and the RH estimation of d, we report the corresponding estimates for $m = 50$, 100,150,200, and 250, respectively.

TABLE 2. The GPH and RH estimation of d for the S&P 500 with $m = 50, 100, 150, 200, 250$

	$\hat{d}_{GPH}$	t	p-value	95% CI	$\hat{d}_{RH}$	t	p-value	95% CI
r_t	0.0819	0.795	0.427	[-0.1201, 0.2838]	0.0602	0.852	0.394	[-0.0784, 0.1988]
	0.0907	1.307	0.191	[-0.0454, 0.2269]	0.0358	0.715	0.475	[-0.0622, 0.1338]
	0.0532	0.957	0.338	[-0.0558, 0.1622]	0.0167	0.408	0.683	[-0.0634, 0.0967]
	0.0365	0.766	0.444	[-0.0569, 0.1298]	0.0070	0.199	0.842	[-0.0623, 0.0763]
	0.0102	0.242	0.809	[-0.0727, 0.0932]	0.0001	0.003	0.997	[-0.0619, 0.0621]
r_t^2	0.2386	2.316	0.021	[0.0367, 0.4406]	0.2553	3.610	0.000	[0.1167, 0.3939]
	0.2175	3.132	0.002	[0.0814, 0.3537]	0.2452	4.904	0.000	[0.1472, 0.3432]
	0.1651	2.969	0.003	[0.0561, 0.2741]	0.1921	4.705	0.000	[0.1121, 0.2721]
	0.1452	3.048	0.002	[0.0518, 0.2386]	0.1737	4.913	0.000	[0.1044, 0.2430]
	0.1387	3.277	0.001	[0.0557, 0.2216]	0.1715	5.422	0.000	[0.1095, 0.2334]
$\|r_t\|$	0.6492	6.301	0.000	[0.4473, 0.8512]	0.6249	8.837	0.000	[0.4863, 0.7635]
	0.6366	9.166	0.000	[0.5005, 0.7727]	0.6241	12.48	0.000	[0.5261, 0.7221]
	0.5606	10.08	0.000	[0.4516, 0.6696]	0.5323	13.04	0.000	[0.4523, 0.6124]
	0.4940	10.35	0.000	[0.4004, 0.5876]	0.5021	14.20	0.000	[0.4328, 0.5714]
	0.4564	10.78	0.000	[0.3734, 0.5394]	0.4843	15.31	0.000	[0.4223, 0.5463]

Table 2 report the GPH and the RH estimation of d for returns, squared returns, and absolute returns, respectively. For instance, in the panel of r_t in Table 2, the first row reports the results from the GPH and the RH estimation with $m = 50$, the second row reports the results of the GPH and the RH estimation with $m = 100$, and so on. This also holds for the panels of r_t^2 and $|r_t|$, and for other tables in this section. The reported

t statistic is that compared with 0. We see that all of the estimated d for the returns are not significant at all conventional significance levels while those for the squared returns, and the absolute returns are significant. Thus, for the S&P 500, there is clear evidence of long memory for the squared and the absolute returns, where d is positive.

Next, we examine the market fraction model. Table 3 and Table 4 report the GPH and RH estimation of the memory parameter d, with corresponding t-value and p-value, averaged over the 1000 simulations. The Tables also contain the 95% confidence interval, representing the simulation inaccuracy, the number of times the t-tests indicates a significant value of d, as well as the test results for testing equality between the d parameter according to the market fraction model, and the real data (column 'Wald'), using here the Wald test taking into account only the estimation inaccuracy in the estimated d according to the real data.

TABLE 3. GPH estimation of d for the market fraction model with $m = 50, 100, 150, 200, 250$

	$\hat{d}$	t	p-value	95% CI	Sig%	Wald
r_t	-0.0079	-0.0768	0.4731	[-0.0143, -0.0015]	9	0.760
	-0.0115	-0.1651	0.4626	[-0.0158, -0.0072]	9	2.162
	-0.0175	-0.3140	0.4406	[-0.0209, -0.0140]	11	1.617
	-0.0252	-0.5287	0.4098	[-0.0281, -0.0222]	15	1.680
	-0.0344	-0.8128	0.3751	[-0.0370, -0.0318]	21	1.112
r_t^2	0.7502	7.2816	0.0000	[0.7439, 0.7566]	100	24.67
	0.5816	8.3744	0.0000	[0.5773, 0.5859]	100	27.45
	0.4791	8.6157	0.0000	[0.4757, 0.4826]	100	31.89
	0.4100	8.6065	0.0000	[0.4071, 0.4130]	100	30.95
	0.3631	8.5821	0.0000	[0.3605, 0.3657]	100	28.14
$\|r_t\|$	0.8738	8.4810	0.0000	[0.8674, 0.8802]	100	4.755
	0.7060	10.165	0.0000	[0.7017, 0.7103]	100	0.997
	0.5942	10.685	0.0000	[0.5908, 0.5977]	100	0.365
	0.5138	10.785	0.0000	[0.5108, 0.5168]	100	0.173
	0.4572	10.805	0.0000	[0.4546, 0.4598]	100	0.000

We see from Table 3 and Table 4 that both of the GPH and the RH estimation of d for the returns are in most if not all of the simulations not significant while those for the squared returns, and the absolute returns are all significant. Thus, the estimated ds for the absolute returns and the squared returns seem to provide clear evidence of long memory in the market fraction model for both the squared and absolute returns. Moreover, the Wald tests (notice that the critical values with significance levels 5% and 1% are 3.842 and 6.635, respectively) indicate that the estimated ds for the returns following from the market fraction model match with those of the S&P 500, the estimated ds for the squared returns are difficult to match with each other, and most of the

TABLE 4. RH estimation of d for the market fraction model with $m = 50, 100, 150, 200, 250$

	$\hat{d}$	t	p-value	95% CI	Sig%	Wald
r_t	-0.0188	-0.2659	0.3971	[-0.0232, -0.0144]	17	1.249
	-0.0157	-0.3139	0.3924	[-0.0188, -0.0126]	15	1.061
	-0.0206	-0.5047	0.3728	[-0.0231, -0.0181]	18	0.836
	-0.0279	-0.7890	0.3478	[-0.0301, -0.0257]	23	0.972
	-0.0363	-1.1485	0.2963	[-0.0383, -0.0344]	31	1.327
r_t^2	0.7386	10.445	0.0000	[0.7342, 0.7430]	100	46.73
	0.5948	11.897	0.0000	[0.5917, 0.5979]	100	48.89
	0.5149	12.613	0.0000	[0.5124, 0.5174]	100	62.60
	0.4621	13.070	0.0000	[0.4599, 0.4643]	100	66.37
	0.4256	13.459	0.0000	[0.4236, 0.4276]	100	64.66
$\|r_t\|$	0.8557	12.101	0.0000	[0.8513, 0.8601]	100	10.66
	0.7133	14.265	0.0000	[0.7102, 0.7164]	100	3.183
	0.6267	15.350	0.0000	[0.6241, 0.6292]	100	5.353
	0.5678	16.060	0.0000	[0.5656, 0.5700]	100	3.445
	0.5262	16.641	0.0000	[0.5243, 0.5282]	100	1.758

estimated ds for the absolute returns match with each other at least at 1% significance level with only one significant difference for the RH estimate.

So, the conclusion seems to be that the calibrated MF model seems to be able to reproduce the presence of long memory patterns in the S&P 500, although not at the right magnitudes for the squared returns. Since the model is already calibrated to actual data, the lesson seems to be that the model needs to be improved, when the aim is to describe actual data.

4. CONCLUSION

Microscopic Simulation (MS-)models are a promising way to study financial markets, since they allow for the possibility to include all kinds of realistic and complex behavior of interacting economic agents, without having to worry about analytical tractability. However, in many cases judgements of the outcomes of MS models seem to be based solely on visual inference.

Following the methodology in Li *et al.* (2005), we investigate in this paper the time series characteristics of Microscopic Simulation (MS) models using the probability density function, the spectral density function, and particular memory parameters. Econometric techniques can be used to study the impact of changes in the parameters and the initial conditions on the simulated time series behavior of the relevant quantities of interest. We also present the method to compare real life data with data generated by MS economies. Here, we exploit our control over the number of simulations, creating one additional dimension for the asymptotic properties of the test

statistics. This allows us to ignore the estimation uncertainty present in the simulations. Hence, we only need to account for the estimation uncertainty of the features of the actual data.

We illustrate the use of the statistical and econometric techniques by studying one of the earlier MS models, the economy considered by Levy, Levy, and Solomon (LLS), and the market fraction model developed in He and Li (2005a, b).

For the LLS model, we perform a sensitivity analysis in terms of the marginal one-period probability density function and the spectral density, and we find that the LLS model is robust with respect to the changes we investigated in the initial prices, the initial dividend, the risk aversion parameter, and the initial wealth. However, a change in the one period maximal decrease of dividend has a serious impact, likely, because this changes the whole distribution of the dividend process, and the dividend process is the driving force in the LLS-model. We also illustrate how to compare the LLS generated data with the actual data. We find that both the marginal single-period probability density function and the second order characteristics of the LLS model are not fully in line with real data. Taking into account the sensitivity to the dividend process, a natural way to proceed seems to be to calibrate the LLS-model, focussing, in particular, on a parametrization of the dividend process.

For the market fraction model, we find that the model is able to mimic the long memory phenomena of real financial market, although not yet at the right magnitude for the squared returns. This suggests that the model needs improvement in case the goal is being able to describe actual data.

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APPENDIX A. 'UNIFORM CONFIDENCE BANDS' FOR THE SPECTRAL DENSITY FUNCTIONS

To construct the *potential* confidence band, we need to find c_U and c_L such that

$$Prob\left\{\sup_{-\pi\leq\omega\leq\pi}\left\{\sqrt{Th}\frac{(\hat{s}(\omega,h)-s(\omega))}{s(\omega)}\right\}\leq c_U\right\}=1-\frac{\alpha}{2}, \tag{27}$$

and

$$Prob\left\{\inf_{-\pi\leq\omega\leq\pi}\left\{\sqrt{Th}\frac{(\hat{s}(\omega,h)-s(\omega))}{s(\omega)}\right\}\geq c_L\right\}=1-\frac{\alpha}{2}, \tag{28}$$

then a $(1-\alpha)$ uniform confidence band for $\hat{s}$ is

$$\left[\hat{s}(\cdot,h)(1+c_L/\sqrt{Th}),\ \hat{s}(\cdot,h)(1+c_U/\sqrt{Th})\right]. \tag{29}$$

A bootstrap method to estimate them is as follows. First, define

$$c_U^{*,k}=\sup_{-\pi\leq\omega\leq\pi}\left\{\sqrt{Th}\frac{(\hat{s}^*(\omega,h,g)-\hat{s}(\omega,g))}{\hat{s}(\omega,g)}\right\},\ k=1,...,K. \tag{30}$$

Then let say c_U^*, be the $K(1-\alpha)$th order statistics of $\left\{c_U^{*,k}\right\}_{k=1}^{K}$. The upper band of the $(1-\alpha)$ uniform confidence band is then

$$(1+c_U^*/\sqrt{Th})\hat{s}(\omega,g). \tag{31}$$

Similarly, we can find the lower band as

$$(1+c_L^*/\sqrt{Th})\hat{s}(\omega,g). \tag{32}$$

In practice, there are many options on the choice of the kernel function, here we use the Bartlett-Priestley kernel

$$K(\theta)=\begin{cases}\frac{3M}{2}\left[1-\left(\frac{M\theta}{\pi}\right)^2\right] & |\theta|\leq\pi/M\\ 0 & |\theta|>\pi/M,\end{cases}$$

where M is the truncation point. This kernel satisfies the assumptions in Franke and Härdle (1992). We also note that Chatfield (2004) gives the relationship between the bandwidth and truncation point equal to $h=8\pi/3M$, the choice of bandwidths is based on the suggestion of Franke and Härdle.