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van Dam, E.R.; Haemers, W.H.; Koolen, J.H.

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**COSPECTRAL GRAPHS AND THE GENERALIZED
ADJACENCY MATRIX**

By E.R. van Dam, W.H. Haemers, J.H. Koolen

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Cospectral graphs and the generalized adjacency matrix

E.R. van Dam* & W.H. Haemers

Tilburg University, Dept. Econometrics and Operations Research
P.O. Box 90153, 5000 LE Tilburg, The Netherlands
e-mail: Edwin.vanDam@uvt.nl, Haemers@uvt.nl

J.H. Koolen[†]

POSTECH, Dept. Mathematics, Pohang 790-784, South Korea
e-mail: koolen@postech.ac.kr

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Abstract

Let J be the all-ones matrix, and let A denote the adjacency matrix of a graph. An old result of Johnson and Newman states that if two graphs are cospectral with respect to $yJ - A$ for two distinct values of y , then they are cospectral for all y . Here we will focus on graphs cospectral with respect to $yJ - A$ for exactly one value \hat{y} of y . We call such graphs \hat{y} -cospectral. It follows that \hat{y} is a rational number, and we prove existence of a pair of \hat{y} -cospectral graphs for every rational \hat{y} . In addition, we generate by computer all \hat{y} -cospectral pairs on most nine vertices. Recently, Chesnokov and the second author constructed pairs of \hat{y} -cospectral graphs for all rational $\hat{y} \in (0, 1)$, where one graph is regular and the other one is not. This phenomenon is only possible for the mentioned values of \hat{y} , and by computer we find all such pairs of \hat{y} -cospectral graphs on at most eleven vertices.

1 Introduction

For a graph Γ with adjacency matrix A , any matrix of the form $M = xI + yJ + zA$ with $x, y, z \in \mathbb{R}$, $z \neq 0$ is called a *generalized adjacency matrix* of Γ (As usual, J is the all-ones matrix and I the identity matrix). Since we are interested in the relation between Γ and the spectrum of M , we can restrict to generalized adjacency matrices of the form $yJ - A$ without loss of generality.

Let Γ and Γ' be graphs with adjacency matrices A and A' , respectively. Johnson and Newman [7] proved that if $yJ - A$ and $yJ - A'$ are cospectral for two distinct values of y , then they are cospectral for all y , and hence they are cospectral with respect to all generalized adjacency matrices. If this is the case we will call Γ and Γ'

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\mathbb{R} -cospectral. So if $yJ - A$ and $yJ - A'$ are cospectral for some but not all values of y , they are cospectral for exactly one value \hat{y} of y . Then we say that Γ and Γ' are \hat{y} -cospectral. Thus cospectral graphs (in the usual sense) are either 0-cospectral or \mathbb{R} -cospectral. For both types of cospectral graphs, many examples are known (see for example [5]). In Figure 1 we give an example of both. This figure also gives examples of \hat{y} -cospectral graphs for $\hat{y} = \frac{1}{3}$ and $\hat{y} = -1$. Note that Γ and Γ' are \hat{y} -cospectral if and only if their complements are $(1 - \hat{y})$ -cospectral. So we also have examples for $\hat{y} = 1, \frac{2}{3}$ and 2. If $\hat{y} = \frac{1}{2}$, one can construct a graph cospectral with a given graph Γ by multiplying some rows and the corresponding columns of $\frac{1}{2}J - A$ by -1 . The corresponding operation in Γ is called *Seidel switching*. This shows that every graph with at least two vertices has a $\frac{1}{2}$ -cospectral mate.

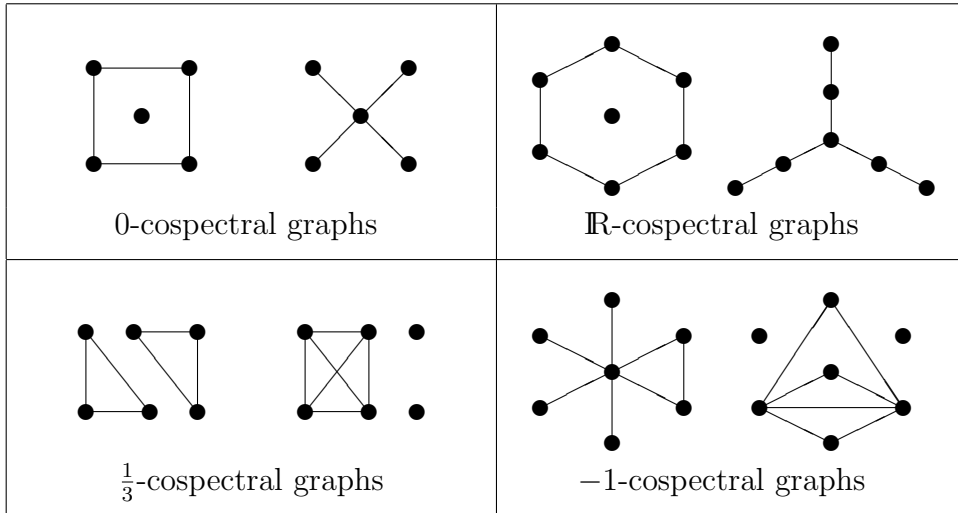


Figure 1: Some examples of (generalized) cospectral graphs

It is well-known that, with respect to the adjacency matrix, a regular graph cannot be cospectral with a nonregular one (see [2, p. 94]). In [5] this result is extended to generalized adjacency matrices $yJ - A$ with $y \notin (0, 1)$. In [1] a regular-nonregular pair of \hat{y} -cospectral graphs is constructed for all rational $\hat{y} \in (0, 1)$. In the next section we shall see that \hat{y} is rational for any pair of \hat{y} -cospectral graphs. Thus we have:

Theorem 1. *There exists a pair of \hat{y} -cospectral graphs, where one graph is regular and the other one is not, if and only if \hat{y} is a rational number satisfying $0 < \hat{y} < 1$.*

In the final section we will generate all regular-nonregular \hat{y} -cospectral pairs on at most eleven vertices. The smallest such pair has only six vertices; it is the $\frac{1}{3}$ -cospectral pair of Figure 1. In Section 3 we shall construct \hat{y} -cospectral graphs for every rational value of \hat{y} . Therefore:

Theorem 2. *There exists a pair of \hat{y} -cospectral graphs if and only if \hat{y} is a rational number.*

In the final section we also generate all pairs of \hat{y} -cospectral graphs on at most nine vertices.

2 The generalized characteristic polynomial

For a graph Γ with adjacency matrix A , the polynomial $p(x, y) = \det(xI + yJ - A)$ will be called the *generalized characteristic polynomial* of Γ . Thus $p(x, y)$ can be interpreted as the characteristic polynomial of $A - yJ$, and $p(x, 0) = p(x)$ is the characteristic polynomial of A . The generalized characteristic polynomial is closely related to the so-called *idiosyncratic polynomial*, which was introduced by Tutte [9] as the characteristic polynomial of $A + y(J - I - A)$. We prefer the polynomial $p(x, y)$, because it has the important property that the degree in y is only 1. Indeed, for an arbitrary square matrix M it is known that $\det(M + yJ) = \det M + y\Sigma \operatorname{adj} M$, where $\Sigma \operatorname{adj} M$ denotes the sum of the entries of the adjugate (adjoint) of M . It is also easily derived from the fact that by Gaussian elimination in $xI + yJ - A$ one can eliminate all y -s, except for those in the first row. In this way we will obtain more useful expressions for $p(x, y)$ as follows. Partition A according to a vertex v , the neighbors of v and the remaining vertices ($\mathbf{1}$ denotes an all-ones vector, and $\mathbf{0}$ an all-zeros vector):

$$A = \begin{bmatrix} 0 & \mathbf{1}^\top & \mathbf{0}^\top \\ \mathbf{1} & A_1 & B \\ \mathbf{0} & B^\top & A_0 \end{bmatrix}.$$

Then

$$\begin{aligned} p(x, y) &= \det \begin{bmatrix} x + y & (y - 1)\mathbf{1}^\top & y\mathbf{1}^\top \\ (y - 1)\mathbf{1} & xI + yJ - A_1 & yJ - B \\ y\mathbf{1} & yJ - B^\top & xI + yJ - A_0 \end{bmatrix} = \\ &= \det \begin{bmatrix} x + y & (y - 1)\mathbf{1}^\top & y\mathbf{1}^\top \\ (-1 - x)\mathbf{1} & xI + J - A_1 & -B \\ -x\mathbf{1} & J - B^\top & xI - A_0 \end{bmatrix} = \\ &= p(x) + y \det \begin{bmatrix} 1 & \mathbf{1}^\top & \mathbf{1}^\top \\ (-1 - x)\mathbf{1} & xI + J - A_1 & -B \\ -x\mathbf{1} & J - B^\top & xI - A_0 \end{bmatrix} = \\ &= p(x) + y \det \begin{bmatrix} 1 & 2\mathbf{1}^\top & \mathbf{0}^\top \\ -\mathbf{1} & xI - A_1 & J - B \\ \mathbf{0} & J - B^\top & xI - A_0 \end{bmatrix} - xy \det \begin{bmatrix} 0 & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{1} & xI - A_1 & -B \\ \mathbf{1} & -B^\top & xI - A_0 \end{bmatrix}. \end{aligned}$$

This expression provides the coefficients of the three highest powers of x in $p(x, y)$. A similar expression is used for the computations in Section 4.

Lemma 1. *Let Γ be a graph with n vertices, e edges, and generalized characteristic polynomial $p(x, y) = \sum_{i=0}^n (a_i + b_i y)x^i$. Then $a_n = 1$, $b_n = 0$, $a_{n-1} = 0$, $b_{n-1} = n$, $a_{n-2} = -e$ and $b_{n-2} = 2e$.*

Proof. By using the above expression for $p(x, y)$, and straightforward calculations. \square

Thus the coefficient of x^{n-2} in $p(x, y)$ equals $e(2y - 1)$. This implies the known fact that, for any $y \neq \frac{1}{2}$, the number of edges of a graph can be deduced from the spectrum of $yJ - A$. Note that a $\frac{1}{2}$ -cospectral pair with distinct numbers of edges can easily be made by Seidel switching.

Now let Γ and Γ' be graphs with generalized characteristic polynomials $p(x, y)$ and $p'(x, y)$, respectively. It is clear that $p(x, y) \equiv p'(x, y)$ if and only if Γ and Γ' are \mathbb{R} -cospectral, and Γ and Γ' are \widehat{y} -cospectral if and only if $p(x, \widehat{y}) = p'(x, \widehat{y})$ for all $x \in \mathbb{R}$, whilst $p(x, y) \not\equiv p'(x, y)$. If this is the case, then $a_i + \widehat{y}b_i = a'_i + \widehat{y}b'_i$ with $(a_i, b_i) \neq (a'_i, b'_i)$ for some i ($0 \leq i \leq n-3$). This implies $\widehat{y} = -(a_i - a'_i)/(b_i - b'_i)$. Thus we proved the mentioned result of Johnson and Newman, that there is only one possible value of \widehat{y} . Moreover, we see that \widehat{y} is a rational number, and that $|\widehat{y}|$ is bounded by $|a_i - a'_i|$ which, in turn, is at most $4(1 + \frac{1}{2}\sqrt{n+1})^{n+1}$. (Indeed, every coefficient a_i of the characteristic polynomial of a graph satisfies $|a_i| \leq \binom{n}{i}2^{i-n}(n-i+1)^{(n-i+1)/2} \leq \sum_{i=0}^n \binom{n}{i}2^{i-n}(n-i+1)^{(n-i+1)/2} = 2^{-n}(2 + \sqrt{n+1})^{n+1}$.)

The generalized characteristic polynomial $p(x, y)$ of a graph Γ is related to the set of main angles $\{\beta_1, \dots, \beta_\ell\}$ of Γ . Suppose the adjacency matrix A of Γ has ℓ distinct eigenvalues $\lambda_1 > \dots > \lambda_\ell$ with multiplicities m_1, \dots, m_ℓ , respectively, then the main angle β_i is defined as the cosine of the angle between the all-ones vector $\mathbf{1}$ and the eigenspace of λ_i . For $i = 1, \dots, \ell$, let V_i be an $n \times m_i$ matrix whose columns are an orthonormal basis for the eigenspace of λ_i . Then $\beta_i\sqrt{n} = \|V_i^\top \mathbf{1}\|$. Moreover, we can choose V_i such that $V_i^\top \mathbf{1} = \beta_i\sqrt{n}\mathbf{e}_1$ (where \mathbf{e}_1 is the unit vector in \mathbb{R}^{m_i}). Put $V = [V_1 \dots V_\ell]$, then $V^\top AV = \Lambda$, where Λ is the diagonal matrix with the spectrum of A , and $V^\top \mathbf{1} = \sqrt{n}[\beta_1\mathbf{e}_1^\top \dots \beta_\ell\mathbf{e}_1^\top]^\top$.

Assume that Γ and Γ' are cospectral graphs with the same angles. Then there exist matrices V and V' such that $V^\top AV = V'^\top A'V' = \Lambda$ and $V^\top \mathbf{1} = V'^\top \mathbf{1}$. Define $Q = VV'^\top$, then $Q^\top AQ = A'$ and $Q\mathbf{1} = Q^\top \mathbf{1} = \mathbf{1}$. This implies that $Q^\top(yJ - A)Q = yJ - A'$, so $yJ - A$ and $yJ - A'$ are cospectral for every $y \in \mathbb{R}$, hence Γ and Γ' have the same generalized characteristic polynomial.

Cvetković and Rowlinson [3] (see also [4, p. 100]) proved the following expression for $p(x, y)$ in terms of the spectrum and the main angles of Γ :

$$p(x, y) = p(x)\left(1 + yn \sum_{i=1}^{\ell} \beta_i^2 / (x - \lambda_i)\right).$$

This formula also shows that the main angles can be obtained from $p(x, y)$, as can be seen as follows. Suppose $q(x) = \prod_{i=1}^{\ell} (x - \lambda_i)$ is the minimal polynomial of A , put $r(x) = p(x)/q(x)$ and $q(x, y) = p(x, y)/r(x)$. Then $q(x, y)$ is a polynomial satisfying $q(\lambda_j, 1) = n\beta_j^2 \prod_{i \neq j} (\lambda_j - \lambda_i)$, which proves the claim. As a consequence, we also proved (a result due to Johnson and Newman [7]) that Γ and Γ' are \mathbb{R} -cospectral if and only if there exist an orthogonal matrix Q such that $Q^\top AQ = A'$ and $Q\mathbf{1} = \mathbf{1}$. The next theorem recapitulates the conditions we have seen for graphs being \mathbb{R} -cospectral.

Theorem 3. *Let Γ and Γ' be graphs with adjacency matrices A and A' . Then the following are equivalent:*

- Γ and Γ' have identical generalized characteristic polynomials,
- Γ and Γ' are cospectral with respect to all generalized adjacency matrices,
- Γ and Γ' are cospectral, and so are their complements,
- Γ and Γ' are cospectral, and have the same main angles,
- $yJ - A$ and $yJ - A'$ are cospectral for two distinct values of y ,
- $yJ - A$ and $yJ - A'$ are cospectral for any irrational value of y ,

- $yJ - A$ and $yJ - A'$ are cospectral for any y with $|y| > 4(1 + \frac{1}{2}\sqrt{n+1})^{n+1}$,
- There exist an orthogonal matrix Q , such that $Q^\top A Q = A'$ and $Q\mathbf{1} = \mathbf{1}$.

3 A construction

We construct pairs of graphs Γ and Γ' on n vertices. For each pair the vertex set is partitioned into three parts with sizes a , b , and c for Γ , and a' , b' , and $c' = c$ for Γ' . Thus $a + b = a' + b' = n - c$. With these partitions Γ and Γ' are defined via their adjacency matrices A and A' as follows (O denotes the all-zeros matrix):

$$A = \begin{bmatrix} OO & O \\ OO & J \\ OJ & J - I \end{bmatrix}, \quad A' = \begin{bmatrix} OJ & O \\ JO & J \\ OJ & J - I \end{bmatrix}.$$

So for the matrices $M = yJ - A$ and $M' = yJ - A'$ we get:

$$M = \begin{bmatrix} yJ & yJ & yJ \\ yJ & yJ & (y-1)J \\ yJ & (y-1)J & (y-1)J + I \end{bmatrix}, \quad M' = \begin{bmatrix} yJ & (y-1)J & yJ \\ (y-1)J & yJ & (y-1)J \\ yJ & (y-1)J & (y-1)J + I \end{bmatrix}.$$

Clearly $\text{rank}(M) \leq c + 2$, so the characteristic polynomial $p(x, y)$ of M has a factor x^{n-c-2} . Moreover $\text{rank}(M - I) \leq a + b + 1$, so $p(x, y)$ has a factor $(x - 1)^{c-1}$. In a similar way we find that the characteristic polynomial $p'(x, y)$ of M' also has a factor $x^{n-c-2}(x - 1)^{c-1}$. Define

$$r(x, y) = \frac{p(x, y)}{x^{n-c-2}(x - 1)^{c-1}} \quad \text{and} \quad r'(x, y) = \frac{p'(x, y)}{x^{n-c-2}(x - 1)^{c-1}}.$$

Then $r(x, y)$ and $r'(x, y)$ are polynomials of degree 3 in x and degree 1 in y . Clearly M and M' are cospectral if $r(x, y) = r'(x, y)$ for all $x \in \mathbb{R}$. Write

$$r(x, y) = t_0 + t_1x + t_2x^2 + t_3x^3, \quad \text{and} \quad r'(x, y) = t'_0 + t'_1x + t'_2x^2 + t'_3x^3,$$

where t_i and t'_i are linear functions in y . Then $t_3 = t'_3 = 1$, and $t_2 = t'_2 = -ny + c - 1$, because $-ny = -\text{trace}(M) = -\text{trace}(M')$, which equals the coefficient of x^{n-1} in $p(x, y)$ and $p'(x, y)$. We shall require that

$$b'(a' + c) = bc.$$

This means that Γ and Γ' have the same number of edges. In the previous section we saw that the number of edges determines the coefficient of x^{n-2} in the generalized characteristic polynomial. Therefore the above requirement gives $t_1 = t'_1$. Finally we shall use the fact that $r(x, y)$ and $r'(x, y)$ are the characteristic polynomials of the quotient matrices R and R' of M and M' , respectively (the quotient matrices are the 3×3 matrices consisting of the row sums of the blocks). So if we choose $y = \hat{y}$ such that these quotient matrices have the same determinant we have $t_0 = t'_0$, and therefore M and M' have the same spectrum. We find

$$\det R = \det \begin{bmatrix} ya & yb & yc \\ ya & yb & yc - c \\ ya & yb - b & yc - c + 1 \end{bmatrix} = -yabc,$$

and

$$\det R' = \det \begin{bmatrix} ya' & yb'-b' & yc \\ ya'-a' & yb' & yc-c \\ ya' & yb'-b' & yc-c+1 \end{bmatrix} = (1-2y)a'b'(c-1).$$

Using $bc = b'(a' + c)$, this leads to $\hat{y} = a'(c-1)/(2a'c - 2a' - ac - aa')$. So any choice of positive integers a, a', b, b' , and c that satisfy $a + b = a' + b'$, $bc = b'(a' + c)$, and $2a'c - 2a' - ac - aa' \neq 0$ leads to a pair of \hat{y} -cospectral graphs with the above \hat{y} (indeed, \hat{y} is uniquely determined, hence Γ and Γ' are not \mathbb{R} -cospectral). For example $(a, a', b, b', c) = (2, 4, 3, 1, 2)$ leads to the two -1 -cospectral graph of Figure 1. Moreover, by a suitable choice of these numbers we can get every rational value of $\hat{y} > 1/2$. Indeed, write $\hat{y} = p/q$ with $2p - q \geq 2$, then

$$a = 2p - q - 1, \quad a' = a(p+1), \quad b = p(a+1), \quad b' = p, \quad \text{and} \quad c = p+1$$

satisfy the required conditions and gives $\hat{y} = p/q$. As remarked in the introduction, $\frac{1}{2}$ -cospectral graphs are easily made by use of Seidel switching, and we also saw that two graphs are \hat{y} -cospectral if and only if their complements are $(1 - \hat{y})$ -cospectral. Thus we have proved that a pair of \hat{y} -cospectral graphs exists for every rational \hat{y} .

Variations on the above construction are possible. The \hat{y} -cospectral pairs, with $0 < \hat{y} < 1$, constructed in [1] (where one graph is regular and the other one not) are of a completely different nature.

4 Computer enumeration

By computer we enumerated all graphs with a \hat{y} -cospectral ($\hat{y} \neq \frac{1}{2}$) mate on at most nine vertices. For fixed numbers of vertices (n) and edges (e) we generated all graphs with these numbers using *nauty* [8], and for each graph we computed $p(x, y)$ for $x = 0, \dots, n$. Note that these $n + 1$ linear functions in y uniquely determine the polynomial $p(x, y)$. For each pair of graphs we compared the corresponding linear functions, giving a system of $n + 1$ linear equations in y . If the system had infinitely many solutions, then we concluded that the pair was \mathbb{R} -cospectral; and if it had a unique solution \hat{y} , then the pair was \hat{y} -cospectral. The results of these computations are given in Table 1. Note that we only considered the cases where $2e \leq \binom{n}{2}$ since, as mentioned before, the complement of a pair of \hat{y} -cospectral graphs is a pair of $(1 - \hat{y})$ -cospectral graphs. In the table, the columns with e give the numbers of edges and the columns with $\#$ give the numbers of graphs with e edges. The columns with header \mathbb{R} contain the number of graphs that have an \mathbb{R} -cospectral mate. The columns with a number \hat{y} in the header contain the numbers of graphs that have a \hat{y} -cospectral mate. Note that this does not mean that a graph cannot be counted in more than one column; for example, of the triple of 0-cospectral graphs with seven vertices and five edges, one (the union of $K_{1,4}$ and an edge) also has a 1-cospectral mate, and another (the union of $K_{2,2}$, an isolated vertex, and an edge) also has a $\frac{1}{4}$ -cospectral mate.

We may conclude that the 0-cospectral pair of Figure 1 is the smallest pair of \hat{y} -cospectral graphs. The smallest pair of \hat{y} -cospectral graphs for $\hat{y} \neq 0$ is the pair of $\frac{1}{3}$ -cospectral graphs in Figure 1. This is also the smallest example where one graph is regular, and the other one not. The smallest \mathbb{R} -cospectral pair of graphs, and the smallest \hat{y} -cospectral pair of graphs for a negative \hat{y} are also given in Figure 1.

We remark that also in [6] all graphs with a \mathbb{R} -cospectral mate, as well as all graphs with a (usual) cospectral mate were enumerated (up to eleven vertices). The latter

Graphs on 5 vertices Graphs on 6 vertices Graphs on 7 vertices Graphs on 8 vertices

e	#	\mathbb{R}	0	$\frac{1}{3}$
2	2	.	.	.
3	4	.	.	.
4	6	.	2	.
5	6	.	.	.

e	#	\mathbb{R}	0	$\frac{1}{3}$
2	2	.	.	.
3	5	.	.	.
4	9	.	4	.
5	15	.	2	.
6	21	.	2	2
7	24	.	2	.

e	#	\mathbb{R}	-1	0	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{4}$	1
2	2
3	5
4	10	.	.	4
5	21	.	.	3	.	4	2
6	41	2	.	12	.	6
7	65	2	2	6	2	8	4
8	97	4	.	14	.	18	2
9	131	6	.	6	.	2	11	2	2	.	2
10	148	6	.	10	.	4	2	2	2	.	9

e	#	\mathbb{R}	-1	0	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	1	2
2	2
3	5
4	11	.	.	4
5	24	.	.	5
6	56	2	.	21	.	2	2
7	115	4	.	22	.	6	2
8	221	10	.	58	.	15	8	.	2
9	402	30	.	44	.	17	6
10	663	55	.	117	.	44	20	.	2	2	.	.	11	.
11	980	87	.	67	.	55	31	.	2	.	.	.	2	4
12	1312	133	2	90	2	36	30	2	6
13	1557	173	.	58	.	21	22	7	2
14	1646	178	2	47	.	2	31	.	4	.	4	31	2	47

Graphs on 9 vertices

e	#	\mathbb{R}	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{3}{14}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{16}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{7}{8}$	1	$\frac{5}{4}$	$\frac{3}{2}$		
2	2
3	5
4	11	4
5	25	5	.	2
6	63	2	24
7	148	6	2	47	2
8	345	18	130	.	4
9	771	61	165	2	21	4
10	1637	158	2	399	.	6	2
11	3252	383	2	2	403	.	8	2
12	5995	764	3	762	6	20	2
13	10120	1469	724	2	16	6
14	15615	2342	2	1133	.	14	11
15	21933	3557	2	915	15	16	2
16	27987	4624	2	5	15
17	32403	5619	4	9	.	858	2	6	5	
18	34040	5805	6	10	.	736	2	10	

Table 1: Numbers of graphs with a \hat{y} -cospectral mate

enumeration is different from our enumeration of graphs with a 0-cospectral mate since it also counts graphs with a \mathbb{R} -cospectral mate (whereas these are excluded in our enumeration).

We also enumerated all regular graphs with a nonregular \hat{y} -cospectral mate ($\hat{y} \neq \frac{1}{2}$) on at most eleven vertices; see Table 2. The columns with a number \hat{y} in the header contain the numbers of graphs that have a \hat{y} -cospectral mate. The column with n gives the number of vertices, the column with k gives the valency and the column with $\#$ gives the number of k -regular graphs with n vertices. The computations were restricted to $v \geq 2k + 1$ for a similar reason as before. We remark further that for missing pairs (v, k) in the considered range, such as $(11, 2)$, there are no regular graphs with a \hat{y} -cospectral mate ($\hat{y} \neq \frac{1}{2}$). In Figure 2 we give all regular-nonregular \hat{y} -cospectral pairs on eight vertices (up to complements).

n	k	$\#$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{4}{11}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{5}{12}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{4}{7}$	$\frac{7}{12}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$
6	2	2	.	.	.	1
8	2	3	.	.	.	1	.	.	1
8	3	6	1	.	.	3
9	2	4	.	1	.	1
9	4	16	.	1	.	1	.	.	2	1	2	.	2	1	2	1	1
10	2	5	2	1	.	1
10	3	21	1	2	1	5	.	.	1	.	1
10	4	60	.	4	.	3	.	.	4	.	.	1	1	.	1	1	.
11	4	266	.	45	.	22	1	2	5	1	.

Table 2: Numbers of regular graphs \hat{y} -cospectral with nonregular graphs

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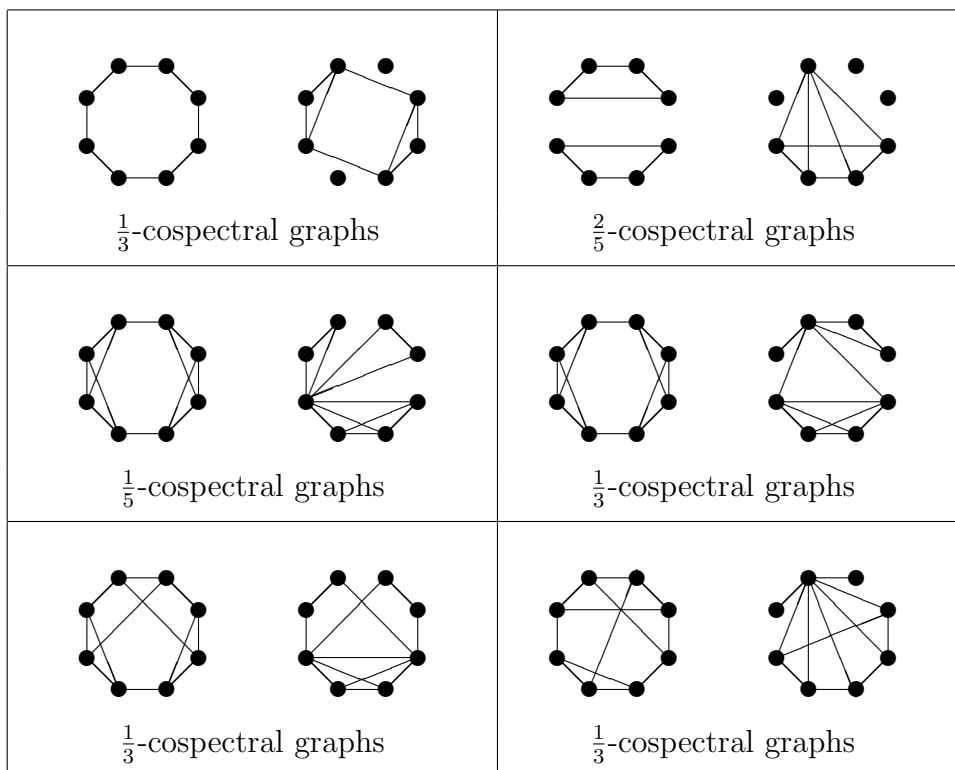


Figure 2: Regular-nonregular $\hat{\gamma}$ -cospectral pairs on eight vertices