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Engwerda, J.C.; Douven, R.C.M.H.

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# Discussion Paper

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## **ON THE SENSITIVITY MATRIX OF THE NASH BARGAINING SOLUTION**

By Jacob Engwerda, Rudy C. Douven

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# On the sensitivity matrix of the Nash bargaining solution

Jacob C. Engwerda \*

and

Rudy C. Douven †

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## Abstract

In this note we derive the sensitivity matrix of the Nash bargaining solution w.r.t. the disagreement point  $d$ . This first order derivative is completely specified in terms of the Pareto frontier function. We show that whenever one player increases his threatpoint always at least one player will loose utility: i.e. the dual result of Pareto optimality. Furthermore, the  $d$ -monotonicity property is easily re-established from this matrix. This matrix also enables us to consider the concept of local strong  $d$ -monotonicity. That is, under which conditions on the Pareto frontier function  $\varphi$  an infinitesimal increase of  $d_i$ , while for each  $j \neq i$ ,  $d_j$  remains constant, it happens that agent  $i$  is the only one who's payoff increases. We show that for the Nash bargaining solution this question is closely related to non-negativity of the Hamiltonian matrix of  $\varphi$  at the solution.

**Keywords:** Nash bargaining solution,  $d$ -monotonicity, diagonally dominant Stieltjes matrix.

**Jel-codes:** C61, C62, C71, C78.

## 1 Introduction

In this note we investigate how the Nash bargaining solution [12],  $N$ , responds to changes in the disagreement point  $d$  for a fixed feasible set.

Following Thomson [16], we define an  $n$ -person bargaining problem to be a pair  $(S, d)$ , where  $S \subset \mathbb{R}^n$  is called the feasible set,  $\mathbb{R}^n$  the utility space and  $d$  the disagreement point. If the agents unanimously agree on a point  $x \in S$ , they obtain  $x$ . Otherwise, they obtain  $d$ . In this paper we are interested in the effect of changes in the disagreement point on the point of agreement for a fixed feasible set. Therefore, we will be considering not just one single bargaining problem but a whole class of

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\*Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

†CPB, Netherlands Bureau for Economic Policy Analysis, P.O.Box 80510, 2508 GM The Hague, The Netherlands.

bargaining problems obtained by varying the threatpoint. If one does not just view the outcome of the bargaining problem as a once and for all given result, but merely as the outcome of a time-dependent process where the disagreement point  $d$  of the players may change over time, the sensitivity matrix may indicate in what direction the bargaining solution will evolve or give information on coalition behaviour between various players.

Thomson considers two classes of bargaining problems: 1)  $\bar{\Sigma}^n$ , where the feasible set  $S$  is assumed to be convex, compact and such that there exists a  $x \in S$  with  $x > d$  (here we use the vector inequality notation); and 2)  $\Sigma^n$ , which is a subclass of  $\bar{\Sigma}^n$ , the so-called class of comprehensive bargaining problems. This subclass is obtained by considering just those elements in  $S$  satisfying the additional property that whenever  $x \in S$  and  $d < \bar{x} \leq x$ , then  $\bar{x} \in S$ .

We will consider in this paper a subclass  $\Sigma_P^n$  of  $\Sigma^n$ . We assume that the (fixed) feasible set in this subclass  $\Sigma_P^n$  satisfies the additional requirement that the set  $P$  of (weak) Pareto optimal solutions can be described by a smooth strictly concave function  $\varphi$ , that is  $\Sigma_P^n = \{(x_1, \dots, x_n)^T \in S | x_i \geq d_i, x_n \leq \varphi(x_1, \dots, x_{n-1})\}$ , and whenever  $x \in S$  and  $d \leq y \leq x$ , then  $y \in S$ . This class of problems (for larger classes of bargaining problems, see e.g., [13] or [16]) is particular popular in applied economic sciences (see e.g. the literature on policy coordination [8], [14], [17], [5]).

Given this class of  $n$ -person bargaining problems, a solution is a function  $F$  associating with every  $(S, d)$  in this class the point of agreement  $F(S, d) \in S$ . Note that, since we consider here a fixed feasible set, the dependence of  $F$  on  $S$  will be omitted.  $F$  is called the Nash solution,  $N$ , if for every fixed pair  $(S, d)$ ,  $F(S, d)$  is assigned the point where the product  $\Pi(x_i - d_i)$  is maximized for  $x \in S$  with  $x \geq d$ .

The paper is organized as follows. In section two we present some preliminaries. Then, in section three we consider the sensitivity matrix of the  $N$ -solution. First, we derive the sensitivity matrix and then discuss some consequences and examples. In particular we consider monotonicity properties of the  $N$ -solution. We show that the  $d$ -monotonicity property follows directly from the sensitivity matrices. This property states that, given some agent  $i$ , if  $d_i$  increases while  $d_j$  remains constant for all  $j \neq i$  then agent  $i$ 's payoff increases (or at least not decreases) (see Thomson ([15])). In fact Thomson also considered the stronger requirement, that not only agent  $i$ 's payoff does not decrease but also the payoffs of none of the other agents increases, which is called strong  $d$ -monotonicity. Thomson shows by means of a counterexample that the  $N$ -solution does not satisfy this notion of strong  $d$ -monotonicity. This notion of  $d$ -monotonicity is a global property in the sense that this property should hold for every positive increment of  $d_i$  at every threatpoint  $d$ .

In fact it is also interesting to see under which conditions for a fixed  $d$  this property holds locally. That is, to see what the gains/losses will be for the other players if one (arbitrarily chosen) player unilaterally changes his threatpoint something. If this player is the only one who gains from such a small (positive) deviation and this property holds irrespective of which player alters his threatpoint we call the bargaining solution *local strong  $d$ -monotonic* at the threatpoint  $d$ . We derive a necessary and sufficient condition for local strong  $d$ -monotonicity. Furthermore, we present a sufficient condition in terms of the Pareto frontier function  $\varphi$  under which this property holds. Given the threatpoint  $d$  and the corresponding bargaining point, this notion tells us something about the stability of the realized bargaining point. This, in the following sense. Assume that the threatpoint can be controlled to some extent by an exogenous authority (e.g a European commission who might consider to change some directives which might favor some outside options of participating countries). If the bargaining

point is local strong  $d$ -monotonic at  $d$  then whenever this threatpoint is changed at one entry only, this action will be disapproved by all other players. This, in contrast to the case that such a change in the threatpoint is beneficial for some other player(s) too. In that case it is rational for that (those) other player(s), at least, to be not against such a change in the threatpoint. So, a less number of players will be against a reopening of the bargaining process in such a case. In this sense, the threshold to reopen the bargaining process will be lower, and the bargaining point might be called less stable. We will illustrate this point in an example in section four.

The paper ends with some concluding remarks.

## 2 Preliminaries

We assume that the number of players equals  $n$ . For notational convenience  $\mathbf{n}$  will be used to denote the set  $\{1, \dots, n\}$ . Furthermore,  $I$  will be used to denote the identity matrix,  $e_i$  to denote the  $i^{\text{th}}$  standard basis vector in  $\mathbb{R}^n$ ,  $v^T$  the transpose of a vector/matrix  $v$ ,  $e$  the vector  $(1, \dots, 1)^T$  and  $\mathbf{0}$  the zero vector  $(0, \dots, 0)^T$ . The dimension of these vectors will be clear from the context. Furthermore, the notation  $\text{diag}(a_i)$  is used to denote a diagonal matrix with as its  $i^{\text{th}}$  diagonal entry  $a_i$ ;  $(A|B)$  to denote the extended matrix of  $A$  and  $B$ ; and  $\text{sgn}(a)$  to denote the sign of the number  $a$ . If  $x := (x_1, \dots, x_n)$  is a vector,  $x_-$  will denote the truncated vector  $(x_1, \dots, x_{n-1})$ .  $\varphi'_i$  is the  $i$ -th partial derivative of  $\varphi$ .

The property of local strong  $d$ -monotonicity with respect to the disagreement point  $d$  is now formalized as follows:

Definition 1: A bargaining solution  $F$  on  $\Sigma_P^N$  is called *locally strong  $d$ -monotonic* at a problem  $(S, d) \in \Sigma_P^N$ , if  $F$  is differentiable in  $d$ , and for all  $i$  and  $j \neq i$ ,  $\frac{\partial F_j(S, d)}{\partial d_i} \leq 0$  and  $\frac{\partial F_i(S, d)}{\partial d_i} \geq 0$ .  $\square$

In the ensuing analysis we will see that the set of so-called  $M$ -matrices arise in a natural way. An  $M$ -matrix is an  $n \times n$  matrix with nonpositive off-diagonal entries whose inverse exists and is entry-wise nonnegative. Symmetric  $M$ -matrices are called *Stieltjes matrices*. From Berman et al. [2, pp.141] we recall the following result.

Theorem 1:

- 1) Symmetric  $M$ -matrices are positive definite.
- 2) Symmetric positive definite matrices with nonpositive off-diagonal entries are  $M$ -matrices.

Unfortunately, the inverse of a nonsingular nonnegative matrix is not in general an  $M$ -matrix. In literature the problem has been addressed to characterize all matrices which do have this property. This turns out to be a difficult problem. A class of matrices that satisfy this property are e.g. the so-called strictly ultrametric matrices (see Nabben et al. [10] and [11]).

Finally, a square matrix  $A = (a_{ij})$  is called diagonally dominant if  $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ , for all  $i$ .

### 3 The Nash bargaining solution

By assumption, the Nash bargaining solution  $x^N := (x_1^N, \dots, x_n^N)$  is determined by the argument that solves the maximization problem

$$\max_{x_-} f(x_-) := \max_{x_-} \prod_{i \in \mathbf{n}-1} (x_i - d_i) (\varphi(x_-) - d_n).$$

This maximization problem has, according to Nash, exactly one solution. Obviously, this solution lies not on the edge of the Pareto frontier  $P$  of  $\Sigma_P^N$ , i.e., it is an interior point of  $P$ . Thus, the first order conditions yield that the Nash bargaining solution is uniquely determined by:

$$g_i(x_-^N, d) = 0, \forall i \in \mathbf{n}-1, \quad (1)$$

where  $g_i(x_-, d) := \varphi(x_-) - d_n + (x_i - d_i) \varphi'_i(x_-)$ ,  $i \in \mathbf{n}-1$ .

Note that all derivatives in these  $n-1$  equations are evaluated at the Nash solution. To simplify notation we will drop this argument whenever it is the Nash solution. So, unless stated differently, we assume from now on that the argument in the derivatives will always be the Nash solution throughout this section.

Moreover, since the solution is a maximum location we know that the second order derivative  $H$  of  $f$  evaluated at the Nash solution will be semi-negative definite. Simple calculations show that

$$H = Dg' \quad (2)$$

where the  $i^{\text{th}}$  entry,  $d_{ii}$ , of the diagonal matrix  $D$  is  $\prod_{j \neq i \in \mathbf{n}-1} (x_j^N - d_j)$  and

$$g'_{x_-}(x_-^N, d) = \frac{\partial g}{\partial x_-}(x_-^N, d) = (ee^T + I) \text{diag}(\varphi'_i) + \text{diag}(x_i^N - d_i) \varphi''. \quad (3)$$

We will assume throughout this note additionally that  $H$  is invertible. In particular it follows then from (2) that the inverse of  $g'$  exists and  $g'^{-1} = H^{-1}D$ . According the implicit function theorem

$$\frac{\partial x_-^N}{\partial d} = -\left\{ \frac{\partial g}{\partial x_-}(x_-^N, d) \right\}^{-1} \frac{\partial g}{\partial d}. \quad (4)$$

It is easily verified that

$$\frac{\partial g}{\partial d} = -(\text{diag}(\varphi'_i) \mid e). \quad (5)$$

To complete the picture of  $\frac{\partial x_i^N}{\partial d_j}$  we still have to consider  $\frac{\partial x_n^N}{\partial d_j}$ . To that end we recall that  $x_n^N = \varphi(x_-^N)$ . Consequently,

$$\frac{\partial x_n^N}{\partial d_j} = \varphi' \begin{pmatrix} \frac{\partial x_1^N}{\partial d_j} \\ \vdots \\ \frac{\partial x_{n-1}^N}{\partial d_j} \end{pmatrix},$$

where  $\varphi' := (\varphi'_1, \dots, \varphi'_{n-1})$ . So, with  $L := \begin{pmatrix} I \\ \varphi' \end{pmatrix}$ , we have that

$$\frac{\partial x^N}{\partial d} = -L \left\{ \frac{\partial g}{\partial x_-}(x_-^N, d) \right\}^{-1} \frac{\partial g}{\partial d}. \quad (6)$$

Before we present the sensitivity matrix we introduce for notational convenience

$$v_i^N := \frac{x_i^N - d_i}{\sqrt{\varphi(x_i^N) - d_n}} \text{ and } G := \left( -(ee^T + I) + (\varphi - d_n) \text{diag}\left(\frac{1}{\varphi_i}\right) \varphi'' \text{diag}\left(\frac{1}{\varphi_i}\right) \right)^{-1}.$$

Theorem 3:

Under the assumption that the hamiltonian of the Pareto frontier evaluated at the Nash bargaining solution is invertible, the sensitivity matrix of the Nash bargaining solution is given by

$$\frac{\partial x^N}{\partial d} = - \begin{pmatrix} \text{diag}(v_i^N) \\ -v_n^N e^T \end{pmatrix} G \left( \text{diag}\left(\frac{1}{v_i^N}\right) \mid \frac{-1}{v_n^N} e \right). \quad (7)$$

Proof:

Using (3,5) it is clear that

$$\frac{\partial x^N}{\partial d} = \begin{pmatrix} I \\ \varphi' \end{pmatrix} \left( (ee^T + I) \text{diag}(\varphi'_i) + \text{diag}(x_i^N - d_i) \varphi'' \right)^{-1} \left( \text{diag}(\varphi'_i) \mid e \right). \quad (8)$$

Some elementary rewriting of this equation (8) gives:

$$\frac{\partial x^N}{\partial d} = \begin{pmatrix} I \\ \varphi' \end{pmatrix} \left( (ee^T + I) \text{diag}\left(\frac{\varphi'_i(x_i^N - d_i)}{x_i^N - d_i}\right) + \text{diag}(x_i^N - d_i) \varphi'' \text{diag}(x_i^N - d_i) \text{diag}\left(\frac{1}{x_i^N - d_i}\right) \right)^{-1} \left( \text{diag}\left(\frac{\varphi'_i(x_i^N - d_i)}{x_i^N - d_i}\right) \mid e \right).$$

From (1) we have that at the N-solution

$$\varphi'_i(x_i^N - d_i) = \varphi'_j(x_j^N - d_j) = -(\varphi - d_n). \quad (9)$$

Using this, we can rewrite the above equation as follows

$$\begin{aligned} \frac{\partial x^N}{\partial d} &= \begin{pmatrix} I \\ \varphi' \end{pmatrix} \text{diag}(x_i^N - d_i) \left( -(\varphi - d_n)(ee^T + I) + \text{diag}(x_i^N - d_i) \varphi'' \text{diag}(x_i^N - d_i) \right)^{-1} \\ &\quad \left( -(\varphi - d_n) \text{diag}\left(\frac{1}{x_i^N - d_i}\right) \mid e \right) \\ &= \begin{pmatrix} \text{diag}(v_i^N) \\ -v_n^N e^T \end{pmatrix} \left( -(ee^T + I) + \text{diag}(v_i^N) \varphi'' \text{diag}(v_i^N) \right)^{-1} \left( -\text{diag}\left(\frac{1}{v_i^N}\right) \mid \frac{1}{v_n^N} e \right) \\ &= \begin{pmatrix} \text{diag}(v_i^N) \\ -v_n^N e^T \end{pmatrix} \left( -(ee^T + I) + \text{diag}\left(\frac{\varphi'_i v_i^N}{\varphi_i}\right) \varphi'' \text{diag}\left(\frac{\varphi'_i v_i^N}{\varphi_i}\right) \right)^{-1} \left( -\text{diag}\left(\frac{1}{v_i^N}\right) \mid \frac{1}{v_n^N} e \right). \end{aligned}$$

From this, using (9) and the above introduced notation, (7) is obtained.  $\square$

Elementary spelling out (7) shows that the sensitivity matrix can also be written as

$$\frac{\partial x^N}{\partial d} = - \begin{pmatrix} \text{diag}(v_i^N) G \text{diag}\left(\frac{1}{v_i^N}\right) & \frac{-1}{v_n^N} \text{diag}(v_i^N) G e \\ -v_n^N e^T G \text{diag}\left(\frac{1}{v_i^N}\right) & e^T G e \end{pmatrix}. \quad (10)$$

Since  $\varphi$  is by assumption strictly concave,  $\varphi''$  is negative definite. Consequently,  $G$  is negative definite too.

Using this, it follows immediately from (10) that all diagonal entries of the sensitivity matrix are always positive. Or stated differently,

Corollary 4:

The N-solution is  $d$ -monotonic. □

Another result which easily follows is:

Corollary 5:

If some player increases his threatpoint, always at least one other player will be worse of (in terms of utility).

Proof:

What we have to show is that in each column of the sensitivity matrix there is at least one entry which has a negative sign.

Consider the row vector  $p := (\frac{v_n^N}{v_1^N}, \dots, \frac{v_n^N}{v_{n-1}^N}, 1)$ . It is easily verified from (7) that  $p \frac{\partial x^N}{\partial d} = 0$ . Since all entries of  $p$  are positive, we conclude in particular that a positive combination of the entries of, e.g., the first column yields zero. Since the first entry of this column is positive at least one of the other entries must be negative. □

Next, we address the question under which conditions on  $\varphi$  the N-solution is locally strong  $d$ -monotonic. We have the following result:

Theorem 6:

The N-solution is locally strong  $d$ -monotonic if and only if  $-G$  is a diagonally dominant Stieltjes matrix.

Proof:

Consider (10). Since  $v_i^N > 0$  it follows that  $\text{sgn}((\frac{\partial x^N}{\partial d})_{ij}) = \text{sgn}(-G_{ij})$ ,  $i, j \in \mathbf{n} - \mathbf{1}$ . As already noted before,  $-G$  is positive definite. So, by Theorem 1.2),  $-G$  is a Stieltjes matrix. Moreover it follows from (10) that  $\text{sgn}((\frac{\partial x^N}{\partial d})_{in}) = \frac{v_i^N}{v_n^N} G e$ . So,  $(\frac{\partial x^N}{\partial d})_{in} \leq 0$  if and only if entry  $i$  of  $G e \leq 0$ ,  $i \in \mathbf{n} - \mathbf{1}$ . Or, stated differently,  $-G$  is diagonally dominant. □

Lemma 7:

Assume  $S$  is an invertible matrix and  $D$  is a positive diagonal matrix. Consider  $P := (S + D)^{-1}$ .

- 1) If  $S^{-1}$  is diagonally dominant, then  $P$  is diagonally dominant.
- 2) If  $S^{-1}$  is a Stieltjes matrix, then  $P$  is a Stieltjes matrix.

Proof:

- 1) First notice that

$$(S + D)^{-1} = D^{-1} - D^{-1}(D^{-1} + S^{-1})^{-1}D^{-1}. \quad (11)$$



Next consider

$$H := \begin{pmatrix} D^{-1} + S^{-1} & D^{-1} \\ D^{-1} & D^{-1} \end{pmatrix}$$

Due to our assumptions, it is easily verified that  $H$  is diagonally dominant. From e.g. Lei et al. [9] (see also Carlson et al. [3]) we conclude then that the Schur complement of  $H$ , which equals (11), is also diagonally dominant.

2) Since by assumption  $S^{-1}$  is a Stieltjes matrix, by Theorem 1.1),  $S^{-1}$  is a positive definite matrix. From this it is obvious that  $P$  will be positive definite too. So, the diagonal entries of  $P$  are positive. Furthermore since, by assumption, both  $S$  and  $D$  are a positive matrix also  $S + D$  is a positive matrix. Next we consider the off-diagonal entries of  $P$ . Since both  $D^{-1}$  and  $S^{-1}$  are Stieltjes matrices, also  $D^{-1} + S^{-1}$  is a Stieltjes matrix. So, in particular, all entries of  $(D^{-1} + S^{-1})^{-1}$  are positive. From (11) it is obvious then that all off-diagonal entries of  $P$  are negative. Since we already argued above that  $P$  is positive definite, Theorem 1.2) shows that  $P$  is a Stieltjes matrix.  $\square$

Theorem 8:

Assume that

$$\Phi^{-1} := -[\text{diag}(\frac{1}{\varphi_i})\varphi''\text{diag}(\frac{1}{\varphi_i})]^{-1} \quad (12)$$

is a diagonally dominant Stieltjes matrix. Then, the N-solution is strong d-monotonic.

Proof:

What has to be shown is that irrespective of the choice of the threatpoint  $d$  the N-solution will be locally strong d-monotonic. Or, equivalently (see Theorem 6), that matrix  $-G$  is irrespective of the choice of the threatpoint  $d$  a diagonally dominant Stieltjes matrix.

To that end first note that, since  $\Phi^{-1}$  is a diagonally dominant Stieltjes matrix, also  $(\varphi - d_n)\Phi^{-1}$  is a diagonally dominant Stieltjes matrix. So, by Lemma 7,

$$P := (I + (\varphi - d_n)\Phi)^{-1} \quad (13)$$

is a diagonally dominant Stieltjes matrix. Next consider  $-G$ . We have

$$\begin{aligned} -G &= \left( (ee^T + I) + (\varphi - d_n)\Phi \right)^{-1} \\ &= \left( ee^T + P^{-1} \right)^{-1} \\ &= P - Pe(e^T Pe + 1)^{-1}e^T P. \end{aligned} \quad (14)$$

Since  $P$  is diagonally dominant  $Pe \geq 0$ . Consequently,  $Pe(e^T Pe + 1)^{-1}e^T P \geq 0$ . So, all off-diagonal entries of  $-G$  are nonpositive. Obviously,  $-G$  is a positive definite matrix and all entries of  $-G^{-1}$  are nonnegative. So, by Theorem 1.2),  $-G$  is a Stieltjes matrix.

Furthermore it follows from (14) that

$$\begin{aligned} -Ge &= (P - Pe(e^T Pe + 1)^{-1}e^T P)e \\ &= \left( 1 - \frac{e^T Pe}{1 + e^T Pe} \right) Pe \\ &\geq 0. \end{aligned}$$

That is,  $-G$  is diagonally dominant. □

Example 9:

1) Assume that  $\varphi''$  is a negative diagonal matrix (so,  $\varphi(x)$  is e.g. a plane or  $\varphi(x) = r + b^T x + \frac{1}{2} x^T A x$ , where  $b, x$  are  $n$ -dimensional vectors with  $b \leq 0$  and  $d \geq 0$  and  $A$  a negative diagonal matrix). Then,  $\Phi^{-1}$  is a positive diagonal matrix and thus in particular a diagonally dominant Stieltjes matrix. So, by Theorem 8, the  $N$ -solution is strong  $d$ -monotonic.

2) Assume that the Pareto frontier has a constant curvature, that is

$$\varphi(x) = \sqrt{r^2 - x_1^2 - \dots - x_n^2}.$$

Then  $\varphi'_i = \frac{-x_i}{\varphi(x)}$ ,  $\varphi''_{ij} = \frac{-x_i x_j}{\varphi^3(x)}$  if  $i \neq j$  and  $\varphi''_{ii} = \frac{-(\varphi^2(x) + x_i^2)}{\varphi^3(x)}$ . Consequently,

$$\Phi = \frac{1}{\varphi(x)} \begin{pmatrix} \frac{\varphi^2(x) + x_1^2}{x_1^2} & 1 & \dots & \dots & 1 \\ 1 & \frac{\varphi^2(x) + x_2^2}{x_2^2} & 1 & \dots & 1 \\ \vdots & \ddots & & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \dots & \dots & 1 & \frac{\varphi^2(x) + x_n^2}{x_n^2} \end{pmatrix}$$

and

$$\Phi^{-1} = \frac{1}{r^2 \varphi(x)} \begin{pmatrix} (r^2 - x_1^2)x_1^2 & -x_1^2 x_2^2 & \dots & \dots & -x_1^2 x_n^2 \\ -x_2^2 x_1^2 & (r^2 - x_2^2)x_2^2 & -x_2^2 x_3^2 & \dots & x_2^2 x_n^2 \\ \vdots & \ddots & & & \vdots \\ \vdots & & & \ddots & -x_{n-1}^2 x_n^2 \\ -x_n^2 x_1^2 & \dots & \dots & -x_n^2 x_{n-1}^2 & (r^2 - x_n^2)x_n^2 \end{pmatrix}.$$

Obviously  $\Phi^{-1}$  is a Stieltjes matrix. Furthermore  $\Phi^{-1}e = \frac{\varphi}{r^2} [x_1^2, \dots, x_n^2]^T$ . So  $\Phi^{-1}$  is diagonally dominant too. Therefore, by Theorem 8 again, the  $N$ -solution is strong  $d$ -monotonic.

3) Assume that the Pareto frontier is described by  $\varphi(x) = \prod_{i=1}^n (b_i - x_i)^{\alpha_i}$ , where  $0 < \alpha_i < 1$  and  $x_i \leq b_i$ ,  $i \in \mathbf{n}$ . Note that this type of functions includes e.g. the Cobb-Douglas function which often occurs in economics. Then,

$$\varphi'_i = \frac{-\alpha_i}{b_i - x_i} \varphi; \quad \varphi''_{ij} = \frac{\alpha_i \alpha_j}{(b_i - x_i)(b_j - x_j)} \varphi, \quad i \neq j; \quad \text{and} \quad \varphi''_{ii} = \frac{\alpha_i(-1 + \alpha_i)}{(b_i - x_i)^2} \varphi.$$

Elementary calculations show that then

$$-\Phi = \frac{1}{\phi} \left( ee^T + \text{diag}\left(\frac{-1}{\alpha_i}\right) \right).$$

Consequently,

$$\begin{aligned} -G &= \left( ee^T + I + (\varphi - d_n)\Phi \right)^{-1} \\ &= \left( ee^T + I - (\varphi - d_n) \frac{1}{\phi} \left( ee^T + \text{diag}\left(\frac{-1}{\alpha_i}\right) \right) \right)^{-1} \end{aligned}$$

$$\begin{aligned}
&= \phi \left( d_n e e^T + \text{diag} \left( \phi + \frac{\phi - d_n}{\alpha_i} \right) \right)^{-1} \\
&= \phi \left( D^{-1} - D^{-1} e \left( e^T D^{-1} e + \frac{1}{d_n} \right)^{-1} e^T D^{-1} \right), \tag{15}
\end{aligned}$$

where  $D := \text{diag}(\phi + \frac{\phi - d_n}{\alpha_i})$ . From (15) it is easily verified that  $-G$  is a Stieltjes matrix. Furthermore  $-Ge = (1 - \frac{e^T D^{-1} e}{e^T D^{-1} e + \frac{1}{d_n}}) D^{-1} e$ . Clearly this vector is positive, so  $-G$  is diagonally dominant too. Since, irrespective of the location of the threatpoint  $d$ ,  $-G$  is a diagonally dominant Stieltjes matrix we conclude that the  $N$ -solution is strong  $d$ -monotonic.  $\square$

## 4 A policy game

In this section we elaborate an example on a policy game in the Economic and Monetary Union (EMU) considered by van Aarle et al. in [1]. In this paper, the EMU economy is represented by a dynamic two-country EMU framework. The model consists of the following equations:

$$y_1(t) = \delta_1 s(t) - \gamma_1 r_1(t) + \rho_1 y_2(t) + \eta_1 f_1(t) \tag{16}$$

$$\dot{p}_1(t) = \xi_1 y_1(t) + \mu_1 \dot{p}_2(t) \tag{17}$$

$$y_2(t) = -\delta_2 s(t) - \gamma_2 r_2(t) + \rho_2 y_1(t) + \eta_2 f_2(t) \tag{18}$$

$$\dot{p}_2(t) = \xi_2 y_2(t) + \mu_2 \dot{p}_1(t) \tag{19}$$

$$s(t) = p_2(t) - p_1(t) \tag{20}$$

$$f_1(t) = g_1(t) - z(t) \tag{21}$$

$$f_2(t) = g_2(t) + z(t) \tag{22}$$

where  $y_j$  denotes real output in country  $j$  (defined in terms of deviations from the equilibrium output, i.e. the output gap),  $s$  competitiveness of country 2 vis-à-vis country 1,  $r_j := i_E(t) - \dot{p}_j(t)$  the real interest rate,  $p_j$  the general price level,  $g_j$  the real fiscal deficit,  $f_j$  the net government expenditures in country  $j \in \{1, 2\}$ , and  $i_E$  the common nominal interest rate. The variable  $z(t) = \epsilon(y_1(t) - y_2(t))$  is an automatic fiscal transfer from the country that has a higher output to the country that has a lower output<sup>1</sup>. All variables are in logarithms, except for the interest rate that is in percentages. Variables are expressed in deviations from the long term equilibrium (balanced growth path) that has been normalized to zero, for simplicity. A dot above a variable denotes its time derivative. (16) and (18) represent aggregate demand as a function of competitiveness, the real interest rate, output in the other country and net government spending. (17) and (19) point to Phillips-curve (or short-run aggregate supply) relations that relate domestic inflation to output and foreign inflation. The first variable measures the effect of demand-pull inflation, the second variable the pass-through of foreign inflation through imported goods. (20) defines competitiveness as the intra-EMU price differential. Net government spending is defined in (21) and (22) as the gross fiscal deficit minus/plus the fiscal transfer paid to/received from the other country.

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<sup>1</sup>See also Engwerda *et al.* [6].

The fiscal policymakers are assumed to have intertemporal objective functions:

$$L^i(t_0) = \frac{1}{2} \int_{t_0}^{\infty} \{ \alpha_i \dot{p}_i^2(t) + \beta_i y_i^2(t) + \chi_i g_i^2(t) \pm \kappa z^2(t) \} e^{-\theta(t-t_0)} dt \quad (23)$$

for  $i \in \{1, 2\}$ . The fiscal authorities control their fiscal policy instrument  $g_i(t)$  such as to minimize a quadratic loss function which features domestic inflation, output, fiscal deficits, and the transfers that increase losses of one country and reduce the losses of the other one. The term  $\pm \kappa z^2$  in the intertemporal objective function means that fiscal transfer increases the loss of the country with higher output (contributor) and decreases the loss of the country of the lower output (recipient). Therefore, the sign of this expression depends on the circumstances.

Preference for a low fiscal deficit reflects the costs of excessive deficits. In this way, the fiscal stringency requirements of the Stability and Growth Pact can be included into the analysis. In particular, a high value of  $\chi_i$  can be interpreted as a strict implementation of the Stability and Growth Pact where countries perceive high costs to incurring (higher) deficits and, therefore, prefer fiscal deficit smoothing. In both cases the total cost to be minimized is a discounted sum of the costs incurred at each period, with  $\theta$  denoting the discount rate.

The ECB cares about aggregate inflation, aggregate output and smoothing of interest rates:

$$L^E(t_0) = \frac{1}{2} \int_{t_0}^{\infty} \{ \dot{P}^2(t) + Y^2(t) + \chi_E i_E^2(t) \} e^{-\theta(t-t_0)} dt \quad (24)$$

where  $\dot{P}(t) := \sum_{i=1}^2 \alpha_{iE} \dot{p}_i(t)$ ,  $Y(t) := \sum_{i=1}^2 \beta_{iE} y_i(t)$ .

Differences in the transmission of monetary policy and fiscal policies are likely to prevail under EMU. For that reason the case of asymmetric fiscal policy transmission was considered in Section 5.4 of [1]. That is, the case that  $\eta_1 < \eta_2$ . So, country two is more effective in its stabilization policy. In this section, using the corresponding parameters from [1], we will study the cooperative Nash bargaining solution for this case. In particular we study its local strong  $d$ -monotonicity. The threatpoint is in this case chosen to be the non-cooperative Nash equilibrium for this game. According Table 1 in [1] the non-cooperative (open-loop) Nash equilibrium  $d := (J_1, J_2, J_E) = (0.7421, 0.5104, 0.0145)$ . Using e.g. the numerical algorithm outlined in Douven [4, Section 3.3.2] (see also Engwerda [7, Section 6.4]), the with this threatpoint corresponding  $N$ -solution is  $(J_1^N, J_2^N, J_E^N) = (0.6296, 0.4753, 0.0039)$ . Numerical calculation shows that the derivatives of the Pareto frontier at the  $N$ -solution are approximately

$$\varphi' = (-0.0945, -0.303) \text{ and } \varphi'' = \begin{pmatrix} 50 & 46 \\ 46 & 56 \end{pmatrix}.$$

Since we are dealing in this example with costs, we first have to transform the problem into the framework we used in the previous sections. Simple calculations show that after this transformation,

$$\begin{aligned} G &= \left( -(ee^T + I) - (J_E - J_E^N) \text{diag}\left(\frac{1}{\varphi_i}\right) \varphi'' \text{diag}\left(\frac{1}{\varphi_i}\right) \right)^{-1} \\ &= \left( - \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - (0.0145 - 0.0039) \begin{pmatrix} \frac{-1}{0.0945} & 0 \\ 0 & \frac{-1}{0.303} \end{pmatrix} \begin{pmatrix} 50 & 46 \\ 46 & 56 \end{pmatrix} \begin{pmatrix} \frac{-1}{0.0945} & 0 \\ 0 & \frac{-1}{0.303} \end{pmatrix} \right)^{-1} \\ &= \begin{pmatrix} -0.0435 & 0.0927 \\ 0.0927 & -0.3155 \end{pmatrix}. \end{aligned}$$

From this it is clear that  $-G$  is a Stieltjes matrix. But, obviously,  $-G$  is not diagonally dominant. So, we conclude from Theorem 6 that the  $N$ -solution is not locally strong  $d$ -monotonic. From, e.g. (7), it follows that the sensitivity matrix of the  $N$ -solution is

$$\frac{\partial x^N}{\partial d} = \begin{pmatrix} 0.0435 & -0.2973 & 0.5205 \\ -0.0289 & 0.3155 & -0.7352 \\ 0.0046 & -0.0675 & 0.1736 \end{pmatrix}.$$

From this matrix we observe in particular that the ECB and the first fiscal player profit both from an increase of eachother's threatpoint, whereas the second fiscal player is the one who gets worse off.

## 5 Concluding remarks

In this note we derived, under some technical conditions, the sensitivity matrix of the Nash bargaining matrix w.r.t. the disagreement point  $d$ . In particular, this makes it possible to analyze the local strong  $d$ -monotonicity of the  $N$ -solution. We showed that the  $N$ -solution satisfies this property if and only if a certain matrix,  $-G$ , evaluated at the Nash bargaining solution is a diagonally dominant Stieltjes matrix. Using this result, a class of bargaining problems was characterized for which the  $N$ -solution satisfies the strong  $d$ -monotonicity property. The results were illustrated in a number of examples.

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