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Publication date:
2005

Citation for published version (APA):
Navas, J., & Kort, P. M. (2005). *Time to Complete and Research Joint Ventures: A Differential Game Approach*. (CentER Discussion Paper; Vol. 2005-29). Microeconomics.

TIME TO COMPLETE AND RESEARCH JOINT VENTURES: A DIFFERENTIAL GAME APPROACH

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February 2005

Time to complete and Research Joint Ventures: a Differential Game Approach

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Abstract

In this paper we analyze cooperation in R&D in the form of RJVs. We show that the optimal size of an RJV does not only depend on the degree of spillovers, as literature suggests, but also on the cost function of R&D activities. Moreover, the explicit consideration of the fact that R&D projects take time to complete shows that benefits from cooperation in R&D not only allow RJVs to carry out larger R&D projects, but also to reduce the time to completion for projects with a given size and, consequently, to accelerate the acquisition of the benefits associated with the innovation.

JEL classification: C73; L13; O31

Keywords: Differential games; Research Joint Ventures; Time to complete

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1

1   Introduction

The study of innovations has become an important area of research in the economic literature due to its contribution to economic growth. However, there exists a series of market failures related to the research and development (R&D) activity including for instance the problem of appropriability or the duplication of effort[1] that could undermine incentives of innovators to undertake research investments. Therefore, governments apply a set of policy instruments to correct these market imperfections. A policy that has received attention in the recent past has been the promotion of cooperative R&D. Aside from enabling the participants to overcome a cost-of-development barrier impenetrable to any of them alone, one advantage of cooperative R&D is the elimination of duplication of effort. As pointed out in Benfratello and Sembenelli (2002), Research Joint Ventures (RJVs) are commonly seen as a potential solution to the small amount of resources invested in R&D activities in Europe and to the low productivity of these resources. Framework Programs for Science and Technology (FPST) and the EUREKA program are examples of these policies in the European Union.

Recent years have witnessed the development of a large literature analyzing cooperation in R&D activities. For a survey of different kinds of inter-firm partnerships see Hagedoorn et al. (2000) and Hagedoorn (2002). In the theoretical literature papers studying cooperation in R&D include for instance Katz (1986), D'Aspremont and Jacquemin (1988), Kamien et al. (1992), Suzumura (1992), Petit and Tolwinski (1999), and Cellini and Lambertini (2002). Several forms of RJVs have been studied in the literature where, while sharing the outcome of their R&D efforts, firms can decide unilaterally on their R&D investments, or they can coordinate them in order to maximize the sum of overall profits (RJV competition and RJV cartelization). In the latter case they internalize the effect that the R&D effort of one firm has on the profits of the other firms in the RJV

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[1] For a review of the theoretical and empirical literature on the R&D activity see for instance Kamien and Schwartz (1982), Cohen and Levin (1989), Reinganum (1989), Cohen (1995), and Sena (2004).

(the so-called combined-profits externality), and consequently the amount of profitable R&D investments that firms are able to carry out is higher. However, if firms are analyzing the investment decision in a R&D project of a given size, and this project is profitable for an RJV independent of the way that firms adopt investments decisions, a question arises: is there some additional criterion that could help them to decide on which is the best organizational structure?

In many papers it is assumed that R&D expenditures lead to immediate effects (see, e.g., among many others, Cellini and Lambertini, 2002), either in product differentiation or in cost reduction. However, in reality it takes time to develop a breakthrough, which is the topic of this paper. As pointed out in Martin (1994), p. 362, "Like all investment projects, R&D involves time in an essential way. A firm seeking to develop a new technique or product must sink its funds into the project for some time before it profits from lower costs or revenues from the sale of a new product". The issue of time to complete in an R&D project was also analyzed in, e.g., Miltersen and Schwartz (2004) in a two-decision maker framework and Kort (1998) in a one-decision maker framework, but where these papers concentrate on uncertainty, we undertake a game theoretic approach within a deterministic framework. By including time to complete in the R&D process we show that cooperation in R&D will not only allow firms in the RJV to undertake larger R&D projects, but also to finish them in a shorter time, accelerating the acquisition of the gains associated with the innovation.

Another issued studied in previous works relates to the optimal number of firms for the best RJV performance. As the number of firms in an RJV increases, the required per-firm R&D effort for a given project is lower, but at the same time competition in the output market increases, and thus the final gain reduces. De Bondt et al. (1992) and Poyago-Theotoky (1995) study the optimal number of firms in an industry in order to maximize the effective R&D. Both papers analyze an industry with R&D aimed at cost reduction, and find that the optimal size of the RJV depends on the degree of spillovers. For the case of perfect spillovers and R&D cooperation, the optimal size includes all the

firms in the industry (Poyago-Theotoky, 1995), while with RJV competition the optimal size, being independent of the R&D cost function, is two firms (De Bondt et al., 1992). We employ a more general R&D cost function and show that its shape is crucial for the optimal RJV size. In particular we find that in case the cost function consists of both a linear and a quadratic term (Poyago-Theotoky (1995) and De Bondt et al. (1992) only have a quadratic term in the cost function), the optimal number of firms in the RJV increases with the increase of the parameter associated with the quadratic part relative to the linear parameter.

In this paper we jointly address cooperation in R&D in the form of RJV and time to complete. We consider an industry with N symmetric firms where market competition takes place à la Cournot. Firms join an RJV aimed at process innovation, so that they could reduce their unitary and constant production cost once the innovation is achieved, but for which a positive time interval of successive R&D efforts is required. While competing in the output market, firms in the RJV can decide independently or coordinately on their R&D investments. In the latter case we will also consider the case of collusion in the output market.

The paper is organized as follows. The model is introduced in section 2. In section 3 we solve the model for the non-cooperative and cooperative cases. Section 4 analyzes and compares results for the different cases, and their optimality from a social point of view is studied in section 5. Finally, in section 6 conclusions are presented.

2   The Model

Consider an industry with N symmetric firms from which n of them ($n \leq N$) constitute an RJV aimed at developing a process innovation. The R&D improvement made in the joint venture due to the participation of firm i at time t is denoted by $x_i(t)$. The costs of performing this improvement are given by $ax_i(t) + bx_i^2(t)$, where a and b are positive constants. The fact that these costs are convex in the research progress can be motivated by realizing that

increasing the research improvement is more difficult when the improvement is already large. Then more qualified people and/or equipment are needed, which is more expensive. R&D effort is typically irreversible, so that $x_i(t) \geq 0$.

The firms being active in the joint venture are aware that the total R&D improvement needed to complete the project at time t is known and equals $Y(t)$.

Recalling that the number of firms that participate in the joint venture is n, these definitions lead to the following state equation:

$$\dot{Y}(t) = -\sum_{j=1}^{n} x_j(t), \quad Y(0) = Y_0 > 0. \tag{1}$$

where Y_0 is the anticipated total R&D improvement required to complete the project.

The project is finished once a sufficient amount of R&D improvement is undertaken, which is the case when

$$Y(T) = 0. \tag{2}$$

The value of the innovation for the individual firm, obtained at the moment that (2) holds, depends on two factors: the number of firms that can make use of this innovation and the behavior of these firms in the output market. We consider initially an industry with N firms, which play a Cournot game in a homogeneous product market facing a linear inverse demand function

$$D^{-1}(Q) = A - BQ,$$

where $Q = \sum_{i=1}^{N} q_i$ and $q_i, i = 1, ..., N$, is the per-firm output. Once the innovation is completed, the constant unit cost of firms in the RJV is reduced from c to $\hat{c}$, where $(c, \hat{c})$ are the corresponding costs before and after the innovation is achieved, respectively, and $A > c$. We assume that competition in the output market is of the same kind before and after collaboration in the R&D project. We also consider that the innovation is of the drastic type,[2] so that after the process innovation only n firms will be active in the output market. Hence, when

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[2]See Tirole (1988).

firms act non-cooperatively in the output market, the per-firm gain associated with the innovation is

$$R_{nc}(n) = \frac{1}{r}\left(\frac{(A-\hat{c})^2}{B(n+1)^2} - \frac{(A-c)^2}{B(N+1)^2}\right), \qquad (3)$$

where r is the discount rate ($r > 0$ and constant). When firms cooperate in the output market, the value of the innovation for the individual firm is

$$R_c(n) = \frac{1}{r}\left(\frac{1}{n}\frac{(A-\hat{c})^2}{4B} - \frac{1}{N}\frac{(A-c)^2}{4B}\right). \qquad (4)$$

Hereafter, and without loss of generality we consider that $n = N$, that is, all the firms present initially in that market join the RJV.[3]

Within this Cournot setting it holds that $R'(n) < 0$. Alternatively, we could consider the case where $R'(n) > 0$. The latter holds in case of a positive network externality, which implies that the value of the innovated product increases with the number of users.

Finally, note that other assumptions on the output market can be included by a direct modification of the functions (3) and (4). As this will lead only to a different amount of the final gain, it is straightforward to extend the analysis to these other settings. In particular, the case of firms operating in a market with product differentiation will be briefly reviewed in Section 4.

3   Optimal RJV R&D policy

For this problem it holds that there exists a maximal amount of R&D improvement needed to complete the project, given that the innovation project is still profitable. We determine this critical level and study how it is affected by the

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[3]Note however that the case $n < N$ is quantitatively similar to an increase in the final function value, since the per-firm gain of competition is a decreasing function of n for the three analyzed cases. On the contrary, the case of $n > N$, that is, the incorporation of new firms in the RJV, and subsequently in the output market, would lead to a decrease in the final function. All these situations point to the existence of incentives to enter and exit the market.

parameters of the model depending on the behavior of the firms (both in the R&D phase and in the output market). With respect to the behavior in the R&D phase, firms can adopt an attitude of collaboration, by coordinating their R&D efforts in order to maximize the sum of overall profits, or no collaboration, by deciding unilaterally on their R&D efforts. The latter case corresponds to the RJV competition model discussed in Kamien et al. (1992), where firms compete but share their R&D efforts avoiding duplication of R&D activities, while the former corresponds to RJV cartelization.

In case firms cooperate in the R&D phase, they maximize

$$W(Y(0)) = \max_{\{x_1,\dots,x_n\}} \left[-\int_0^T e^{-rt} \sum_{j=1}^n \left[ax_j(t) + bx_j^2(t) \right] dt + nR(n)e^{-rT} \right], \quad (5)$$

where

$$T = \inf\left[t \mid Y(t) = 0 \right].$$

In case firms do not cooperate, the value of firm i, being a participant in the RJV, is given by

$$V_i(Y(0)) = \max_{x_i} \left[-\int_0^T e^{-rt} \left(ax_i(t) + bx_i^2(t) \right) dt + R(n)e^{-rT} \right]. \quad (6)$$

3.1 Non-cooperation in R&D

In case firms in the RJV do not cooperate during the R&D phase, the aim is to find the Markov perfect Nash equilibrium.[4] The value of the project for firm i is given by (6), so that the HJB-equation is

$$rV_i(Y) = \max_{x_i} \left\{ -ax_i - bx_i^2 + V_i' \left(-x_i - \sum_{j \neq i} x_j \right) \right\}, \quad (7)$$

with the boundary condition

$$V_i(0) = R_{nc}(n). \quad (8)$$

The first order condition is

$$x_i = -\frac{1}{2b} \left(a + V_i' \right), \quad (9)$$

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[4]See, e.g., Dockner et al., 2000.

and substitution of (9) into (7), while assuming symmetry, gives the following differential equation (note that the value of the firm is the same for all firms so that we can drop the subscript i) for the value function:

$$rV = \left(\frac{2n-1}{4b}\right)(V')^2 + \frac{na}{2b}V' + \frac{a^2}{4b}. \tag{10}$$

The general solution of (10) in implicit form[5] is

$$\frac{2n-1}{2br}V' + \frac{na}{2br}\ln|V'| + \alpha_1 = Y, \tag{11}$$

where α_1 is the constant of integration, and

$$V' = -\left(\frac{na}{2n-1}\right) - \frac{1}{2}\sqrt{\left(\frac{2a(n-1)}{2n-1}\right)^2 + \frac{16brV}{2n-1}}. \tag{12}$$

The constant α_1 is obtained from (11) valued at $Y = 0$ where, from (8), we know that the value of the project[6] equals $R_{nc}(n)$. Thus,

$$\alpha_1 = -\frac{2n-1}{2br}V'(0) - \frac{na}{2br}\ln|V'(0)|. \tag{13}$$

Let $\bar{Y}_{nc-nc}$ be the largest value for Y for which it is profitable to carry out the project. Since the control variable is continuous, it holds that

$$V'\left(\bar{Y}_{nc-nc}\right) = -a. \tag{14}$$

Substituting (13) and (14) into (11) gives

$$\bar{Y}_{nc-nc} = \frac{2n-1}{2br}(-a) + \frac{na}{2br}\ln|-a| - \frac{2n-1}{2br}V'(0) - \frac{na}{2br}\ln|V'(0)|. \tag{15}$$

Note that for $\bar{Y}_{nc-nc}$ the value of the project $V(\bar{Y}_{nc-nc}) = 0$. Only projects that require an initial amount of total R&D investments below this critical level will be interesting for the firm. Thus, the optimal R&D investment policy is

$$x_{nc-nc} = \left\{ \begin{array}{c} -\frac{1}{2b}(a + V'(Y)) \\ 0 \end{array} \right\} \text{ for } Y \left\{ \begin{array}{c} < \\ \geq \end{array} \right\} \bar{Y}_{nc-nc}.$$

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[5] See Appendix A.

[6] Here we employ that equation (11) defines V as a function of Y (and therefore, also V' through (12)), which is ensured by the implicit function theorem.

Projects with a required total amount of R&D improvement below $\bar{Y}_{nc-nc}$ have a positive value, since the cost of the R&D effort carried out during the planning horizon $[0, T]$ falls below the final gain. Note that the assessment of the project is carried out through the net present value rule. Then, $\bar{Y}_{nc-nc}$ is the value for which a R&D project has a zero net present value.

3.2 Cooperation in R&D

When firms cooperate in R&D, the problem for the RJV is given by (5). Performing the same steps as in the non-cooperative case, we have that the value of the project, W, for the RJV must satisfy the following differential equation:

$$rW = \frac{n}{4b}\left(W'\right)^2 + \frac{na}{2b}W' + \frac{na^2}{4b}. \tag{16}$$

In order to solve (16), analogous to the non-cooperative case we obtain that

$$\frac{n}{2br}W' + \frac{na}{2br}\ln|W'| + \alpha_i = Y, \quad i = 2, 3 \tag{17}$$

where $\alpha_2, \alpha_3 \in \mathbb{R}$ are the constants of integration when firms in the RJV compete or cooperate in the output market, respectively. The root of (16) that guarantees a positive control is

$$W' = -a - 2\sqrt{\frac{b}{n}rW}.$$

Therefore, the general solution of (16) is

$$\frac{n}{2br}\left(-a - 2\sqrt{\frac{b}{n}rW}\right) + \frac{na}{2br}\ln\left|-a - 2\sqrt{\frac{b}{n}rW}\right| + \alpha_i = Y, \quad i = 2, 3 \tag{18}$$

with the boundary condition $W(0) = nR_c(n)$ in case firms in the RJV also cooperate in the output market, or $W(0) = nR_{nc}(n)$ when there is no cooperation once the R&D phase is finished. These two cases are denoted by the subscripts (c-c) and (c-nc), respectively.

In order to determine the constant of integration α, we must take into account the behavior of firms in the output market. When there is no cooperation in output (c-nc), we know that when $Y = 0$ the RJV value of the project, W_{c-nc},

is $nR_{nc}(n)$. Therefore, from (18) we obtain that

$$\alpha_2 = \frac{n}{2br}\left(a + 2\sqrt{brR_{nc}\left(n\right)}\right) - \frac{na}{2br}\ln\left|-a - 2\sqrt{brR_{nc}\left(n\right)}\right|. \qquad (19)$$

Let $\bar{Y}_{c-nc}$ be the maximal R&D improvement for the RJV for which it is still profitable to carry out the R&D project. From the continuity of the control variable we know that at this level

$$W'_{c-nc}\left(\bar{Y}_{c-nc}\right) = -a,$$

while the optimal control is zero. Then, evaluating (18) at $\bar{Y}_{c-nc}$ gives

$$\begin{aligned}
\bar{Y}_{c-nc} = {} & \frac{n}{2br}\left(-a\right) + \frac{na}{2br}\ln\left|-a\right| \\
& + \frac{n}{2br}\left(a + 2\sqrt{brR_{nc}\left(n\right)}\right) - \frac{na}{2br}\ln\left|-a - 2\sqrt{brR_{nc}\left(n\right)}\right|. \quad (20)
\end{aligned}$$

As before, only projects that require an initial amount of total R&D improvement below this critical level will be interesting for the firm as a participant in the RJV. Thus, the optimal per-firm policy is

$$x_{c-nc} = \left\{ \begin{array}{c} -\frac{1}{2b}(a + W'_{c-nc}(Y)) \\ 0 \end{array} \right\} \text{ for } Y \left\{ \begin{array}{c} < \\ \geq \end{array} \right\} \bar{Y}_{c-nc}.$$

In case firms cooperate in both phases of the game (c-c) we know that when $Y = 0$ the RJV value of the project, W_{c-c}, is $nR_c(n)$. This leads to

$$\alpha_3 = \frac{n}{2br}\left(a + 2\sqrt{brR_c\left(n\right)}\right) - \frac{na}{2br}\ln\left|-a - 2\sqrt{brR_c\left(n\right)}\right|. \qquad (21)$$

Denoting by $\bar{Y}_{c-c}$ the total maximal R&D improvement that the RJV could carry out profitably in the fully cooperative case, then as before it holds that $W'_{c-c}(\bar{Y}_{c-c}) = -a$, where the optimal control is zero. Finally, we can determine $\bar{Y}_{c-c}$ by substituting the last condition in (18) to obtain that

$$\begin{aligned}
\bar{Y}_{c-c} = {} & \frac{n}{2br}\left(-a\right) + \frac{na}{2br}\ln\left|-a\right| \\
& + \frac{n}{2br}\left(a + 2\sqrt{brR_c\left(n\right)}\right) - \frac{na}{2br}\ln\left|-a - 2\sqrt{brR_c\left(n\right)}\right|. \quad (22)
\end{aligned}$$

In this case, the optimal per-firm R&D improvement rate for the fully cooperative case is

$$x_{c-c} = \left\{ \begin{array}{c} -\frac{1}{2b}(a + W'_{c-c}(Y)) \\ 0 \end{array} \right\} \text{ for } Y \left\{ \begin{array}{c} < \\ \geq \end{array} \right\} \bar{Y}_{c-c}.$$

4 Analysis of results

We first compare the level of profitable total R&D improvement in the three cases analyzed above for a given number of firms in the industry, as well as the optimal per-firm improvement rate for a given project size.[7] Results are shown in the following propositions, which are proved in Appendix B.

Proposition 1 *Let $\bar{Y}_s, (s = c - c, c - nc, nc - nc)$, be the maximal total R&D improvement that the RJV can face profitably. Then, for a given number of firms in the RJV, it holds that*

$$\bar{Y}_{c-c} > \bar{Y}_{c-nc} > \bar{Y}_{nc-nc}.$$

Proposition 2 *Given a level of pending work, $\tilde{Y}$, the per-firm R&D improvement rates $x_s(\tilde{Y}), (s = c - c, c - nc, nc - nc)$, satisfy*

$$x_{c-c}(\tilde{Y}) > x_{c-nc}(\tilde{Y}) > x_{nc-nc}(\tilde{Y}).$$

As expected, the higher the degree of cooperation, the larger the total R&D improvement needed to achieve the innovation that the RJV could bear profitably. When firms cooperate in the R&D phase they internalize the effect that the R&D effort of one firm has on the profits of the other firms in the RJV (the combined-profits externality according to Kamien et al., 1992), and consequently they are able to obtain more profit from cooperation. Note that the inequality $\bar{Y}_{c-nc}(n) > \bar{Y}_{nc-nc}(n)$ can be seen as the increase in efficiency that firms obtain when they coordinate their R&D decisions, since the final gain is equal. In the fully cooperative case the higher final gain is due to the fact that

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[7]All the results below hold for $n \geq 2$, while for $n = 1$ results coincide in the three cases.

firms are acting as a joint monopolist in the output market, and thus they are able to acquire a major part of the surplus created.

Next we analyze the behavior of the maximal profitable total R&D improvement with respect to changes in the final gain.[8] To do so, we differentiate $\bar{Y}_s$, $(s = c - c, c - nc, nc - nc)$, with respect to the final gain $R(n)$:

$$\frac{\partial \bar{Y}_{c-c}}{\partial R_c(n)} > 0, \frac{\partial \bar{Y}_{c-nc}}{\partial R_{nc}(n)} > 0, \text{ and } \frac{\partial \bar{Y}_{nc-nc}}{\partial R_{nc}(n)} > 0.$$

In the three cases the corresponding critical level $\bar{Y}_s$ is increasing in $R(n)$, so that the larger R, the larger the total amount of required improvement that the RJV can face. Nonetheless, it can be observed from

$$\frac{\partial^2 \bar{Y}_{c-c}}{\partial R_c(n)^2} < 0, \frac{\partial^2 \bar{Y}_{c-nc}}{\partial R_{nc}(n)^2} < 0, \text{ and } \frac{\partial^2 \bar{Y}_{nc-nc}}{\partial R_{nc}(n)^2} < 0,$$

that the increase in $\bar{Y}$ is diminishing in R. This holds because the costs related to the achieved improvement are convex. This implies that in case we duplicate the final gain, duplicating the effort needed in order to get the final gain would imply multiplying the associated costs more than twice.

Proposition 2 states that the R&D effort is higher the higher the degree of the cooperation, leading to a faster time to completion and thus a quicker achievement of the process innovation. This is a relevant issue since the sooner the innovation is achieved and translated to production, the sooner the corresponding increase of private benefits (higher firm profits) and social benefits (lower prices and higher output) takes place. By only taking into account the size of a project, firms could be indifferent on the RJV organizational structure that firms choose to engage. However, the required time to complete also matters. In order to illustrate this point, Table 1 shows the required time to complete[9] a project requiring the maximal total R&D improvement (column 3)

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[8] As the ratio $(A - c)/B$ yields a measure of market size and profitability, and since the final gain is proportional to this ratio, the response of $\bar{Y}$ to a change in R can also be interpreted as the way the market size influences the critical level of R&D projects. In general, R&D investment incentives in cost reduction are positively related to the level of output, because the innovation costs can then be spread over a larger number of products.

[9] We have used Matlab software for computing the time to complete. The quadrature algorithms do not allow to start the integration process from the exact value of $\bar{Y}$, so that we

and the required time to complete a R&D project with a given size (column 4).

With respect to the R&D improvement rate, x_s, $(s = c - c, c - nc, nc - nc)$, by differentiating it with respect to Y, where the expressions of $\partial V'(Y)/\partial Y$ and $\partial W'(Y)/\partial Y$ are obtained from (11) and (17) and the implicit function theorem, respectively, we obtain that the control variable is a strictly decreasing concave function of Y:

$$\frac{\partial x_s}{\partial Y} < 0, \quad \text{and} \quad \frac{\partial^2 x_s}{\partial Y^2} < 0. \tag{23}$$

Therefore, the smaller the total pending work (reducing Y), the larger the optimal effort, that is, the intensity of the research effort increases (at a decreasing rate) as the pending work diminishes until the project is completed at T, where $Y(T) = 0$ and the firms in the RJV reduce their unit production cost.

Departing from (1), we can also analyze how the pending work and the per-firm R&D improvement rate evolve over time.[10] While the pending work, Y_s, $(s = c - c, c - nc, nc - nc)$, is a decreasing concave function of time, i.e.,

$$\frac{\partial Y_s}{\partial t} < 0, \quad \frac{\partial^2 Y_s}{\partial t^2} < 0, \tag{24}$$

the R&D improvement rate, x_s, $(s = c - c, c - nc, nc - nc)$, is an increasing and convex function of time,

$$\frac{\partial x_s}{\partial t} > 0, \quad \frac{\partial^2 x_s}{\partial t^2} > 0, \tag{25}$$

so that, as pointed out previously, at the moment that the investment effort is maximal the time to completion of the innovation ends. This behavior is in accordance with the empirical results pointed out in Scherer (1967): "...the time pattern of R&D outlays is typically bell-shaped, with the peak rate of spending occurring at the time when the end product is put into production."

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compute the time to complete with an error of 0.001 with respect to these values. While differences are minimal for the (c-nc) and (c-c) cases, for the case of (nc-nc) the time to complete could differ with small changes in the error value, because the optimal control is quite low at the initial stages of the R&D phase, so that initial R&D improvement units need high times to complete.

[10]See Appendix B.

The explicit consideration of a positive time to develop a breakthrough also stresses the importance of value of the discount rate. By differentiating $\bar{Y}_s$ with respect to r, we obtain that

$$\frac{\partial \bar{Y}_s}{\partial r} < 0, \quad (s = c - c, c - nc, nc - nc).$$

We conclude that in the three cases studied the maximal total amount of R&D investment being profitable to undertake decreases when r increases. This is the case since the stream of costs extends over the entire planning horizon whereas the final gain is obtained at the end of the R&D period.

4.1 Optimal size of the RJV

In this section we analyze the optimal size of the RJV. As the number of firms in the RJV increases, the required per-firm R&D effort for a given project is lower. However, at the same time the number of firms in the output market increases, so that the final gain reduces.

Contributions that study the optimal number of firms in an industry are, for instance, De Bondt et al. (1992) and Poyago-Theotoky (1995). Both papers analyze an industry where firms produce a differentiated or homogeneous product with R&D aimed at cost reduction. Decreasing returns of the innovative activity are modeled through a quadratic cost function for the R&D investments. While Poyago-Theotoky (1995) explicitly considers the formation of an RJV with a subset of the firms in the industry, De Bondt et al. (1992) analyzes a whole industry acting non-cooperatively in both R&D and output stages. In both papers the optimal size of the RJV[11] depends on the degree of spillovers. For the perfect spillover case Poyago-Theotoky (1995) finds that the equilibrium size of the RJV is equal to the total number of firms in the industry, which is also the socially optimal size of an RJV.[12] On the contrary, in De Bondt et al.

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[11]Concerning De Bondt et al. (1992), we associate the RJV case with their perfect spillover case.

[12]In her setting the equilibrium size is given by the number of firms that maximizes the profits of any individual member of the RJV while the socially optimum size corresponds to the number of firms that maximize joint industry profits for RJV and non-RJV firms.

(1992) the optimal size corresponds to the duopoly case.[13] Other models that exhibit perfect spillovers, although with R&D aimed at product differentiation rather than cost reduction, such as Cellini and Lambertini (2002), find that, when firms compete in R&D and output, R&D investments are positively related to the number of firms. This effect is due to the fact that a larger number of firms increases the incentive to invest in R&D in order to decrease competition through product differentiation. Note that in such a framework there is no incentive at all to perform R&D in case of a monopoly.

From the perspective of the firms (welfare is considered later), we define the optimal size of the RJV by the number of firms n^* for which the RJV exhibits the maximal profitable amount of R&D improvement, i.e.,

$$\bar{Y}_s(n^*) > \bar{Y}_s(n^* - 1) \quad \text{and} \quad \bar{Y}_s(n^*) > \bar{Y}_s(n^* + 1),$$

where $s = c - c, c - nc, nc - nc$, and let $\Delta \bar{Y}_s(n) = \bar{Y}_s(n + 1) - \bar{Y}_s(n)$, $(s = c - c, c - nc, nc - nc)$. If $\Delta \bar{Y}_s(n) > 0$, an RJV consisting of more firms would be desirable since, although per-firm final gains are lower with more firms acting in the output market, the increase in R&D efficiency associated with sharing R&D efforts, which have decreasing returns because of the quadratic structure of the cost function, will allow the RJV to face larger R&D projects. On the contrary, when $\Delta \bar{Y}_s(n) < 0$ the R&D performance of the RJV would increase with less firms participating in the RJV.

Because of the complexity of the expressions for $\Delta \bar{Y}_s(n)$, we rely on numerical simulations for the cases (nc-nc) and (c-nc) to state the following result on the sign of $\Delta \bar{Y}$:

Corollary 1 *In the cases (nc-nc) and (c-nc) there exists at most one positive real root n_s for the equation $\Delta \bar{Y}_s(n) = 0$, $s = nc - nc, c - nc$. It holds that*

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[13]In De Bondt et al. (1992) the monopoly case is not analyzed explicitly, but by deriving the optimal R&D effort for the monopolist and comparing it with the duopoly case, it can be shown that while R&D investments are maximal with only one firm in the output market, for not "too large" values of the parameter Γ in their cost function, total industry profits are higher if R&D is conducted for two firms, so that the efficiency effect leads to an optimal size of the RJV of $n = 2$.

$n_{nc-nc} \in (0, 2)$ and $n_{c-nc} \in (0, \infty)$. In the case (c-c) we have that $\Delta Y_{c-c}(n) > 0$ for all n.

Corollary 1 states that when firms compete in the output market, there is a non-monotonic relation between the number of firms in the RJV and the R&D performance, irrespective of their attitude in the RJV. This non-monotonic relation leads to the existence of an optimal size of the RJV. We conclude that this result is placed between the Schumpeterian hypothesis, which relates R&D investment to profits associated with the innovation, and therefore with the extent of market power, being maximal for the monopoly, and Arrowian positions where R&D investments are negatively associated with market power and profits.

The optimal size of the RJV depends on the trade-off between benefits from sharing with a larger number of participants and the profit loss associated with increased competition in the output market. To illustrate, we perform a numerical analysis with respect to $\bar{Y}$ for the cases (nc-nc) and (c-nc). Results are shown in Table 2.

An increase in the cost function always leads to a reduction in the maximal profitable total R&D improvement since total profitability of the innovation project reduces. However, what determines a better R&D performance for a larger number of firms in the RJV is not the total cost but the weight of the quadratic term in the cost function. An increase in b/a raises the incentive to have more firms in the RJV. The reason is that an individual firm can cut down on R&D improvement if the number of firms increase. This especially leads to a considerable cost reduction if b is large. Hence, we obtain that the optimal size for the RJV will not only depend on the degree of spillovers (in our RJV setting always maximal), but also on the relative weight of the quadratic vs. the linear term in the cost function. This result is the more important because the inclusion of a linear term in the cost function seems reasonable, at least in some cases, to avoid a null marginal cost for initial R&D efforts.

In order to assess the influence of the linear term (a) in our cost function

and also to connect our results with those of De Bondt et al. (1992) and Poyago-Theotoky (1995), where only the quadratic term appears in their cost function, we drop out this term by taking the limit of $\bar{Y}_{nc-nc}(n)$ and $\bar{Y}_{c-nc}(n)$ when a goes to zero, and obtain that

$$\bar{Y}_{nc-nc}^{a=0}(n) = \lim_{a \to 0} \bar{Y}_{nc-nc}(n) = \frac{\sqrt{2n-1}}{(n+1)} \sqrt{\frac{(A-\hat{c})^2 - (A-c)^2}{bBr^2}}, \quad (26)$$

and

$$\bar{Y}_{c-nc}^{a=0}(n) = \lim_{a \to 0} \bar{Y}_{c-nc}(n) = \frac{n}{(n+1)} \sqrt{\frac{(A-\hat{c})^2 - (A-c)^2}{bBr^2}}. \quad (27)$$

It is straightforward to see that $\bar{Y}_{nc-nc}^{a=0}(n)$ has a maximum at $n = 2$, and that $\partial \bar{Y}_{c-nc}^{a=0}(n)/\partial n > 0$, $\forall n$. We conclude that the optimal size of the RJV for the (nc-nc) case is achieved at $n^* = 2$ (De Bondt et al., 1992). However, in the general case, i.e., with $a \neq 0$, this size depends on the ratio b/a. For the (c-nc) case and $a = 0$ it is optimal for the RJV to contain all the firms in the industry, which coincides with Poyago-Theotoky (1995), while for the general case ($a \neq 0$) the optimal size can be any $n^* \in [1, n]$.

For the total cooperative case (c-c) it holds that the larger the number of firms in the RJV, the better the RJV's R&D performance. This result confirms fears on incentives of firms to curtail competition after the R&D cooperation phase, since a fully cooperative agreement between firms would allow them to obtain a positive gain by carrying out R&D projects that would be non-profitable under a different scenario. One way to take advantage of the best R&D performance associated with the fully cooperative case while avoiding antitrust regulation is to form an inter-industry or international RJV, so that firms can benefit from R&D cost reduction without negative effects of competition in their respective output markets. In this sense, Steurs (1995) concludes that inter-industry R&D cooperation is more likely to result in higher R&D investment than intra-industry R&D cooperation. To see why in the fully cooperative case the cost saving effect dominates the profit loss in the output market as n increases, we rearrange (22) to obtain that

$$\frac{\bar{Y}_{c-c}}{n} = \frac{1}{2br} \left\{ a \ln|-a| + 2\sqrt{brR_c(n)} - a \ln\left|-a - 2\sqrt{brR_c(n)}\right| \right\}. \quad (28)$$

17

Now, from (4) we have that $R_c(n) = (1/n)R_c(1)$ and considering, for instance, the case of a duopoly, $(n = 2)$, leads to

$$\frac{\bar{Y}_{c-c}}{2} = \frac{1}{2br}\left\{ a\ln|-a| + 2\sqrt{br\frac{R_c(1)}{2}} - a\ln\left|-a - 2\sqrt{br\frac{R_c(1)}{2}}\right|\right\},$$

being the per-firm final gain, which is exactly one half of the gain for the monopoly. Owing to the quadratic R&D structure the firms prefer the case where both $\bar{Y}_{c-c}$ and R_c are halved. A similar reasoning holds for further increases of n. Moreover, the joint effort rate for the cooperative case is higher when the number of firms is larger. This implies a shorter time to completion for a given project.

We summarize the former results in the following proposition:

Proposition 3 *Let n^* be the optimal size for the RJV, then:*

1. *Case (nc-nc): $n^* = 2$ if $a = 0$, and $n^* \leq 2$ if $a > 0$.*

2. *Case (c-nc): $n^* = \infty$ if $a = 0$, and n^* decreases with a/b.*

3. *Case (c-c): $n^* = \infty$ $\quad\forall a$.*

4.2 Output market with differentiated products

We now briefly discuss the effect of having some degree of differentiation in the output market. As pointed out previously, this assumption will only affect the final gain associated with the innovation. Obviously, when some positive degree of differentiation exists in the output market, per-firm profits are higher than those with a homogeneous output. Therefore, the maximal total R&D improvement that an RJV can carry out profitably is higher, as seen before.

Here, we assume that the n firms in the industry sell n differentiated products, and that the inverse demand function for a variety i is $p_i = A - Bq_i - D\sum_{j\neq i} q_j$, $i = 1,\ldots,n$, where $D \in [0, B]$ is the symmetric degree of substitutability between any pair of varieties, so that when $D = B$ products are completely homogeneous, and when $D = 0$ products are totally independent. In

the latter case each firm acts as a monopolist. The final gain functions for the non-cooperative and cooperative case are, respectively:

$$R_{nc} = \frac{1}{r} \left(\frac{B(A - \hat{c})^2}{[2B + (n-1)D]^2} - \frac{B(A - c)^2}{[2B + (n-1)D]^2} \right), \tag{29}$$

and

$$R_c = \frac{1}{r} \left(\frac{(A - \hat{c})^2}{4[B + (n-1)D]} - \frac{(A - c)^2}{4[B + (n-1)D]} \right). \tag{30}$$

It is straightforward to see that as the degree of product differentiation increases (D decreases), gains from innovation, (29) and (30), also increase. Therefore, the more differentiated the products are, the larger the maximal total R&D improvement that an RJV can carry out profitably, i.e.

$$\frac{\partial \bar{Y}_s}{\partial D} < 0, \quad (s = nc - nc, c - nc, c - c).$$

In a similar way we obtain that for a given level of pending work, $\tilde{Y}$, the per-firm R&D improvement rate negatively depends on the degree of product differentiation D:

$$\frac{\partial x_s(\tilde{Y})}{\partial D} < 0, \quad (s = nc - nc, c - nc, c - c).$$

The implication is that an RJV with firms operating in a more differentiated product market will complete R&D projects in a shorter time.

We finally consider the optimal size of the RJV with differentiated products. Taking again the limit of $\bar{Y}_{nc-nc}$ and $\bar{Y}_{c-nc}$ when a tends to zero we obtain that

$$\lim_{a \to 0} \bar{Y}_{nc-nc}(n) = \frac{\sqrt{2n-1}}{[2B + (n-1)D]} \sqrt{\frac{B\left((A - \hat{c})^2 - (A - c)^2\right)}{br^2}},$$

and

$$\lim_{a \to 0} \bar{Y}_{c-nc}(n) = \frac{n}{[2B + (n-1)D]} \sqrt{\frac{B\left((A - \hat{c})^2 - (A - c)^2\right)}{br^2}}.$$

Now, Proposition 3 can be generalized directly[14] in the following way:

Proposition 4 *Let n^* be the optimal size for the RJV, with participating firms selling products with a degree of differentiation of $D \in [0, B]$, then:*

_____

[14]For the (nc-nc) case, note that if $2B/D \notin \mathbb{N}$ when $a = 0$, then n^* must be determined according to the sign of $\Delta \bar{Y}_{nc-nc}(n)$ at the integer part of $2B/D$.

1. *Case (nc-nc):* $n^* = \frac{2B}{D}$ *if* $a = 0$, *and* $n^* \leq \frac{2B}{D}$ *if* $a > 0$.

2. *Case (c-nc):* $n^* = \infty$ *if* $a = 0$, *and* n^* *decreases with* a/b.

3. *Case (c-c):* $n^* = \infty \quad \forall a$.

Note that in the fully non-cooperative case the upper bound $(2B/D)$ for the optimal size of the RJV tends to infinity as products become more differentiated $(D \to 0)$. When $D = 0$ firms are monopolists, so that the larger the number of firms in the RJV, the higher the savings from sharing R&D efforts, while they will not suffer negative effects on the per-firm final gain because of the independence of their output markets.

5   Welfare analysis

The welfare performance of the R&D policies analyzed above can be gauged and compared in terms of several alternative criteria. We follow Suzumura (1992), and assume that competition in the output market lies beyond the regulatory power of the government, so that the value of the innovation of the social planner is

$$R_j^S = \frac{1}{r} \left(\int_0^{Q^j(\hat{c})} P(Q)dQ - \hat{c}Q^j(\hat{c}) - \int_0^{Q^j(c)} P(Q)dQ - cQ^j(c) \right), \quad (31)$$

where $Q^j(c), (j = nc, c)$, is the industry output under Cournot and Monopoly situations for a unit production cost of c, respectively. The objective function of the social planner is

$$S(Y(0)) = \max_{\{x_1, \ldots, x_n\}} \left[-\int_0^T e^{-rt} \sum_{j=1}^n \left[ax_j(t) + bx_j^2(t) \right] dt + R_j^S e^{-rT} \right]. \quad (32)$$

Note that the problem for the social planner is similar, except for the final gain given by (31), to that of the RJV when firms cooperate. In the previous section we have seen that the maximal profitable R&D project positively depends on the final gain. Since the social planner also considers the improvement in

consumer surplus associated with the innovation, which leads to higher production and lower prices, it holds that $R_c^S > nR_c(n)$ and $R_{nc}^S > nR_{nc}(n)$, so that $\bar{Y}_j^S > \bar{Y}_{c-j}, (j = nc, c)$, where $\bar{Y}_j^S$ is the social critical level of the R&D project. Hence, the social planner would start R&D projects that firms in the RJV do not consider profitable, pointing to a socially inefficient behavior in the three analyzed cases.

Moreover, the social planner would commit a larger amount of resources to profitable projects than any RJV structure, leading to lower total times to complete a given project. Policy implications are, therefore, that the government should promote cooperative agreements between firms, as well as take actions such as direct aids (e.g., investment grants or investment tax credits) to reach a higher level of R&D spending, in order to improve social welfare.

With respect to the optimal size of the RJV from the point of view of the social planner, it always comprises all the firms in the industry. For the case of competition in the output market, this results comes directly by observing that the gain in social welfare, i.e., the change in the consumer surplus together with the change in total industry profits, is positive when n increases:

$$\frac{\partial}{\partial n} \left(\frac{n^2((A - \hat{c})^2 - (A - c)^2)}{2B(n+1)^2} + \frac{n((A - \hat{c})^2 - (A - c)^2)}{B(n+1)^2} \right) > 0.$$

If firms compete in the output market, total industry profits decrease with more firms, but aggregate quantity is higher and product price lower, so that the consumer surplus increases and offsets the loss in the industry profits. Finally, in the case of collusion in the output market, total industry profits remain constant with more firms, as well as the consumer surplus, but in this case, as shown in the former section, it is always desirable for the RJV to incorporate more firms because of the gain in R&D efficiency.

Finally, note that this analysis extends in a straightforward way when there exists some degree of differentiation in the output market. As seen before, in that case firms in the RJV increase their R&D investments with the degree of product differentiation. However, the social planner will also find the RJV's R&D policy socially inefficient.

6 Concluding remarks

We have taken a differential game approach to analyze the RJV's R&D efforts when the completeness of a process innovation requires a known given amount of R&D improvement, Y_0. Depending on the organizational structure of the RJV we study the maximal amount of R&D improvement that an RJV can undertake profitably, and determine the optimal effort for every level of pending work during the research process.

Results obtained show that the coordination of the firm's R&D efforts is optimal from a double perspective. First, cooperation increases the ability to manage larger innovation projects. Second, under cooperation it is optimal to spend more resources in a given R&D project, which leads to a faster completion of the innovation process and, consequently, to a sooner appropriation of the private benefits (higher firm profits) and social benefits (lower prices and higher output) associated with the innovation. With respect to the timing of R&D investments, we have shown that the R&D improvement rate is an increasing and strictly convex function of time, so that R&D investments are maximal at the final stages of the process innovation.

We further established some results regarding the optimal size for the RJV. We show that this optimal size depends heavily on the cost function of R&D investments. While in the fully cooperative case it is always desirable to increase the number of firms, in the other scenarios the optimal number of firms depends on the relative weight of the quadratic and linear terms in the cost function.

Finally, a comparison with the social planner's optimal R&D efforts shows that society would consider it profitable to carry out projects that an RJV, irrespectively of its organizational structure, would reject. And for profitable projects, society would address more resources to every stage of the project, pointing to a permanent delay in the introduction of new technologies from the social point of view.

Appendix A: Derivation of $V(Y)$

In order to solve (10) we rewrite it as:

$$V = \Phi\left(V'\right) = \left(\frac{2n-1}{4br}\right)\left(V'\right)^2 + \frac{na}{2br}V' + \frac{a^2}{4br}. \tag{33}$$

By definition, $dV = V'dY$. Differentiating (33) we obtain that $dV = \Phi'\left(V'\right)dV'$. From these relations we get

$$dY = \frac{dV}{V'} = \frac{\Phi'\left(V'\right)}{V'}dV'. \tag{34}$$

Integrating (34) we finally obtain

$$Y = \int \frac{\Phi'\left(V'\right)}{V'}dV' + \alpha$$
$$= \left(\frac{2n-1}{2br}\right)V' + \frac{na}{2br}\ln|V'| + \alpha$$

where $\alpha \in \mathbb{R}$.

Equation (33) defines two solutions for V' :

$$\Psi_1 = -\left(\frac{na}{2n-1}\right) - \frac{1}{2}\sqrt{\left(\frac{2a\left(n-1\right)}{2n-1}\right)^2 + \frac{16brV}{2n-1}} \tag{35}$$

and

$$\Psi_2 = -\left(\frac{na}{2n-1}\right) + \frac{1}{2}\sqrt{\left(\frac{2a\left(n-1\right)}{2n-1}\right)^2 + \frac{16brV}{2n-1}}. \tag{36}$$

Next we study which of these two solutions is feasible for the studied problem. The minimal R&D effort that each firm can perform is zero and let $\bar{Y}_{nc-nc}$ be the maximal R&D improvement that each firm considers the RJV can face profitably under a non-cooperative attitude in the R&D phase. Therefore, from the continuity of the control variable, we know that at $\bar{Y}_{nc-nc}$ it is satisfied that

$$V'\left(\bar{Y}_{nc-nc}\right) = -a, \tag{37}$$

i.e., for $\bar{Y}_{nc-nc}$ the optimal effort is zero, and the value of the project is zero, i.e., $V\left(\bar{Y}_{nc-nc}\right) = 0$.

In order to assure positive values for the control variable, and from (9), it is required that

$$V' < -a. \tag{38}$$

First we analyze the solution given by (36). Solving (36) for $V = 0$ we have that

$$\Psi_2(0) = -\frac{a}{2n-1},$$

which contradicts (37). Since b and r are positive, we also have that V' is an increasing function of V, so that the optimal control would always be negative.

Now we study the solution obtained from (35). Solving it for $V = 0$, we obtain that $V' = -a$, that is, the value reached at $\bar{Y}_{nc-nc}$. Moreover, V' is now a decreasing function of V, and this guarantees (38). Then we take (35) as the differential equation to be satisfied by the value function V for each firm in the RJV.

Appendix B: Proofs of the propositions and main results

Proof of Proposition 1. We first prove that $\bar{Y}_{c-c} > \bar{Y}_{c-nc}$. First, we define the function

$$f(\xi) \equiv -\frac{n}{2br}\xi - \frac{na}{2br}\ln|\xi|. \tag{39}$$

By subtracting $\bar{Y}_{c-nc}$ from $\bar{Y}_{c-c}$, we conclude that $\bar{Y}_{c-c} > \bar{Y}_{c-nc}$ holds if $f(\xi_1) > f(\xi_2)$, where $\xi_1 = -a - 2\sqrt{brR_c(n)}$ and $\xi_2 = -a - 2\sqrt{brR_{nc}(n)}$. Then, since $f(\cdot)$ is a (strictly) decreasing function for $\xi < -a$, $R_c(n) > R_{nc}(n)$ implies that $\xi_1 < \xi_2$, so that $f(\xi_1) > f(\xi_2)$.

Note also that $f(\xi_1) = \alpha_3$ and $f(\xi_2) = \alpha_2$, where α_2 and α_3 are the constants of integration given in (19) and (21), respectively. Therefore, we also have that $\alpha_3 > \alpha_2$, which is a result that we will use below.

Next, we prove that $\bar{Y}_{c-nc} > \bar{Y}_{nc-nc}$. We obtain from (15) and 20), that the former inequality holds if

$$M > ln(N), \tag{40}$$

24

where

$$M = \frac{1}{na}\left(a(n-1) + \sqrt{4brn^2 R_{nc}(n)} - \sqrt{(a(n-1))^2 + 4br(2n-1)R_{nc}(n)}\right),$$

$$N = \frac{a + \sqrt{4br R_{nc}(n)}}{\frac{na}{2n-1} + \frac{1}{2}\sqrt{\left(\frac{2a(n-1)}{2n-1}\right)^2 + \frac{16br R_{nc}(n)}{2n-1}}}.$$

By taking exponentials of both sides of (40), and then the second-degree Taylor polynomial for the left-hand side of the resulting expression we obtain

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} > 1 + M + \frac{M^2}{2} > N, \tag{41}$$

which after some calculations can be shown to hold.

Proof of Proposition 2. As $x_s(\tilde{Y}), s = c-c, c-nc, nc-nc$, is given by the expression $-(1/2b)(a+\gamma(\tilde{Y})), \gamma(\tilde{Y}) = W'_{c-c}(\tilde{Y}), W'_{c-nc}(\tilde{Y}), V'(\tilde{Y})$, we will show that $W'_{c-c} < W'_{c-nc} < V'$, in order to construct the proof for Proposition 2.

We first look at the case $W'_{c-c} < W'_{c-nc}$, i.e., $x_{c-c}(\tilde{Y}) > x_{c-nc}(\tilde{Y})$. Let $f(\cdot)$ be as defined in (39), which attains a minimum at $\xi = -a$ (note that the functions W'_{c-c}, W'_{c-nc} and V' will only reach values less or equal than $-a$ for any (given) admissible level of pending work $\tilde{Y}$). Now, the expressions for the solution of the value function in the fully cooperative case (given in (18) with α as in (21)) and for the value function when cooperation arises only in the R&D stage (given in (18) with α as in (19)) can be rewritten, respectively, into

$$f(W'_{c-c}(\tilde{Y})) = -(\tilde{Y} - \alpha_3),$$

and

$$f(W'_{c-nc}(\tilde{Y})) = -(\tilde{Y} - \alpha_2).$$

Since $f(\cdot)$ decreases on the admissible range for $W'_s(Y), s = c-c, c-nc$, and $\alpha_3 > \alpha_2$, it follows that $W'_{c-c}(\tilde{Y}) < W'_{c-nc}(\tilde{Y})$.

Finally, in order to prove that $W'_{c-nc} < V$, i.e., $x_{c-nc}(\tilde{Y}) > x_{nc-nc}(\tilde{Y})$, we

define the functions

$$g(\xi) \equiv \frac{2n-1}{2br}\xi + \frac{na}{2br}\ln|\xi| + \alpha_1, \tag{42}$$

$$h(\xi) \equiv \frac{n}{2br}\xi + \frac{na}{2br}\ln|\xi| + \alpha_2. \tag{43}$$

with α_1 as in (13). In order to compare g and h, we find the intersection point between both functions, $\bar{\xi} \in \mathbb{R}/g\left(\bar{\xi}\right) = h(\bar{\xi})$, so that

$$\bar{\xi} = \frac{2br}{n-1}(\alpha_2 - \alpha_1), \tag{44}$$

and now we check that $\bar{\xi} > -a$. Substituting the expressions for α_1 and α_2 into (44), and rearranging terms in the inequality $\bar{\xi} > -a$, we obtain expression (40), which was shown to be satisfied. Then, as g and h are continuous functions on their domains, we have that for $\xi \leq -a$ it holds that either $g > h$ or $g < h$. Note that both functions, g and h, valued at $\xi = -a$ are, respectively, $\bar{Y}_{nc-nc}$ and $\bar{Y}_{c-nc}$, so that, for $\xi \leq -a$ it holds that $g < h$. Then, for any given level of pending work $\tilde{Y} \leq \bar{Y}_{nc-nc}$, we have that

$$g^{-1}(\tilde{Y}) = V'(\tilde{Y}) > h^{-1}(\tilde{Y}) = W'_{c-nc}(\tilde{Y}). \tag{45}$$

Proof of Corollary 1. As pointed before, the results for the (nc-nc) and (c-nc) cases rely on numerical simulations. For the (c-c) case, and in order to prove that $\Delta\bar{Y}_{c-c}(n) > 0$ we calculate the derivative of $\bar{Y}_{c-c}(n)$ with respect to n, and show that it is always positive:

$$\frac{\partial\bar{Y}_{c-c}(n)}{\partial n} = \frac{1}{2br}\left\{a\ln\left(\frac{a}{a+2\Psi}\right) + \Psi\sqrt{2} + \frac{a}{2+\frac{a}{\Psi}}\right\},$$

where

$$\Psi = \sqrt{\frac{b}{n}\frac{(A-\hat{c})^2 - (A-c)^2}{4B}}.$$

This derivative is positive if

$$\Psi\sqrt{2} + \frac{a}{2+\frac{a}{\Psi}} > \Psi + \frac{a}{2+\frac{a}{\Psi}} = \frac{2\Psi(a+\Psi)}{a+2\Psi} > a\ln\left(\frac{a}{a+2\Psi}\right).$$

26

Finally, this inequality holds if

$$M > ln(N), \tag{46}$$

where now

$$M = \frac{2\Psi(a + \Psi)}{a(a + 2\Psi)} \quad \text{and} \quad N = \left(\frac{a + 2\Psi}{a}\right).$$

As before, we take exponentials of both sides of (46), and then the second-degree Taylor polynomial for the left-hand side of the resulting expression, to obtain that

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} > 1 + M + \frac{M^2}{2} > N,$$

which after some calculations can be shown to hold.

Proof of Expressions (24) and (25). As the analysis is initially similar for the three cases (nc-nc, c-nc, c-c), we omit subscripts where no confusion could arise. We first prove that $\partial Y(t)/\partial t < 0$ and $\partial^2 Y(t)/\partial t^2 < 0$. From (1) and the symmetric behavior assumption, we have that

$$\dot{Y} = -nx \quad \text{and} \quad \ddot{Y} = -n\frac{\partial x}{\partial Y}\dot{Y}. \tag{47}$$

As the control variable is assumed to be non-negative and with (23), the strict concavity of $Y(t)$ follows.

We now consider $\partial x(t)/\partial t > 0$, and $\partial^2 x(t)/\partial t^2 > 0$. Differentiating (1) with respect to t and rearranging terms we have that

$$\dot{x} = -n\ddot{Y} = n^2\frac{\partial x}{\partial Y}\dot{Y}, \tag{48}$$

$$\ddot{x} = n^2\dot{Y}\left[\frac{\partial^2 x}{\partial Y^2}\dot{Y} - n\left(\frac{\partial x}{\partial Y}\right)^2\right]. \tag{49}$$

The sign of (48) comes from (23) and the sign of $\dot{Y}$ shown above. On the other hand, the sign of (49) will depend on the sign of the expression inside square brackets, which, after substituting the expression for the control variable $x = -(1/2b)(a + \gamma(Y))$, where $\gamma(Y) = W'_{c-c}(Y), W'_{c-nc}(Y)$, and $V'(Y)$, becomes:

$$-\frac{1}{2b}\frac{\partial^2\gamma(Y)}{\partial Y^2}\left(\frac{n}{2b}(a + \gamma(Y))\right) - n\left(-\frac{1}{2b}\frac{\partial\gamma(Y)}{\partial Y}\right)^2. \tag{50}$$

27

Now, from (11) and (18) and the implicit function theorem, we can calculate $\partial\gamma(Y)/\partial Y$ and $\partial^2\gamma(Y)/\partial Y^2$, $\forall Y \in [0, \bar{Y})$.

We first consider the cases (c-nc) and (c-c) because of their symmetry, that is, with $\gamma(Y) = W'_s(Y)$, $s = c - nc, c - c$, and obtain that

$$\frac{\partial W'_s}{\partial Y} = \frac{2brW'_s}{n(a + W'_s)},$$

$$\frac{\partial^2 W'_s}{\partial Y^2} = \frac{2abr}{n(a + W'_s)^2} \frac{2brW'_s}{n(a + W'_s)}.$$

By substituting these expressions into (50), and after some calculations (taking into account that $W'_s(Y) < -a$) it can be shown that the sign of (50) is negative. Therefore, coming back to (49) we have that $\ddot{x} > 0$.

Finally, for the case (nc-nc), where $\gamma(Y) = V'(Y)$, we have that:

$$\frac{\partial V'}{\partial Y} = \frac{2brV'}{an + (2n - 1)V'},$$

$$\frac{\partial^2 V'}{\partial Y^2} = \frac{2abrn}{(an + (2n - 1)V')^2} \frac{2brV'}{an + (2n - 1)V'}.$$

Substituting the former expressions into (50), and after some calculations it can be shown that the sign of (50) is negative for the (nc-nc) case. Therefore, coming back to (49) we also have that $\ddot{x} > 0$.

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$n = 2$	$\bar{Y}$	$T(\bar{Y})$	$T(Y_0 = 200)$
nc-nc	409.31	145.44	12.14
c-nc	512.91	32.97	5.64
c-c	554.49	33.93	5.24

Table 1: Time to complete and maximal profitable total R&D improvement for parameter values: $a = 1$, $b = 0.1$, $r = 0.05$, $A = 100$, $B = 10$, $c = 70$ and $\hat{c} = 30$.

a	n^*_{nc-nc}	$\bar{Y}_{nc-nc}$	n^*_{c-nc}	$\bar{Y}_{c-nc}$
0.001	2	728.637	52	1212.553
0.01	2	718.302	19	1122.574
0.1	2	655.978	7	909.253
1	1	433.332	3	520.865
10	1	142.37	1	142.37

Table 2: Optimal size of the RJV for parameter values: $b = 0.1$, $r = 0.05$, $A = 100$, $B = 10$, $c = 70$ and $\hat{c} = 30$.