A survey of semidefinite programming approaches to the generalized problem of moments and their error analysis

E. de Klerk and M. Laurent

Erratum: Correction of Theorems 9 and 11 and their proofs

Recall the setting of the GMP:

$$\text{(GMP)} \qquad val := \inf_{\mu \in \mathcal{M}_+(K)} \Big\{ \int_K f_0(x) d\mu(x) : \int_K f_i(x) d\mu(x) = b_i \ (i \in [m]) \Big\},$$

where $K = \{x \in \mathbb{R}^n : g_j(x) \geq 0 \ (j \in [k])\}$. We let $\mathcal{Q}(g)$ denote the quadratic module generated by $g = \{g_1, \ldots, g_k\}$ and, for an integer r, $\mathcal{Q}^r(g)$ is its truncation at degree $2r$. Then consider the parameter

$$val^{(r)}_{outer} := \inf_{\mu \in (\mathcal{Q}^r(g))^*} \Big\{ \int_K f_0(x) d\mu(x) : \int_K f_i(x) d\mu(x) = b_i \ (i \in [m]) \Big\}.$$

Clearly, $val^{(r)}_{outer} \leq val^{(r+1)}_{outer} \leq val$ and thus $\lim_{r \to \infty} val^{(r)}_{outer} = \sup_r val^{(r)}_{outer} \leq val$. Theorem 9 below gives conditions ensuring the asymptotic convergence to val.

Recall the Slater-type condition

$$\text{(S)} \qquad \exists z_0, z_1, \ldots, z_m \in \mathbb{R} \text{ such that } \sum_{i=0}^m z_i f_i(x) > 0 \ \forall x \in K$$

and the dual problem

$$val^* := \sup_{y \in \mathbb{R}^m} \Big\{ b^T y : f_0(x) - \sum_{i=1}^m y_i f_i(x) \geq 0 \ \forall x \in K \Big\}.$$

By weak duality, $val^* \leq val$ holds. Moreover, we have $val^* = val \in \mathbb{R}$ if the program defining val is feasible and Slater condition (S) holds (by Corollary 1).

Theorem 9. *Assume K is compact, program* (GMP) *is feasible, the quadratic module $\mathcal{Q}(g)$ is Archimedean, and Slater condition (S) holds. Then, we have*

$$\lim_{r \to \infty} val^{(r)}_{outer} = val.$$

Proof of Theorem 9. Since $\lim_{r \to \infty} val^{(r)}_{outer} = \sup_r val^{(r)}_{outer} \leq val$ and $val^* = val$, it suffices now to show that $val^* \leq \sup_r val^{(r)}_{outer}$. For this, let $\delta > 0$ and let $y \in \mathbb{R}^m$ be a δ-optimal solution for val^*. That is, $f_0 - \sum_{i=1}^m y_i f_i \geq 0$ on K and $b^T y \geq val^* - \delta$. Pick $\epsilon > 0$. Then, we have

$$(1) \qquad f_0 - \sum_{i=1}^m y_i f_i + \epsilon \sum_{i=0}^m z_i f_i > 0 \ \text{ on } K.$$

By Theorem 8, there exists $r := r_{\delta,\epsilon} \in \mathbb{N}$ such that $f_0 - \sum_{i=1}^m y_i f_i + \epsilon \sum_{i=0}^m z_i f_i \in \mathcal{Q}^r(g)$. Let μ be feasible for the program defining the bound $val^{(r)}_{outer}$. Then, evaluating μ at the above polynomial, we obtain

$$\int_K (f_0 - \sum_{i=1}^m y_i f_i + \epsilon \sum_{i=0}^m z_i f_i)(x) d\mu(x) \geq 0.$$

Using the fact that $\int_K f_i d\mu = b_i$ for $i \in [m]$ and $y^T b \geq val^* - \delta$, we get

$$(1 + \epsilon z_0) \int_K f_0 d\mu \geq y^T b - \epsilon \sum_{i=1}^m z_i b_i \geq val^* - \delta - \epsilon \sum_{i=1}^m z_i b_i.$$

Since this holds for any feasible μ, we obtain

$$(1 + \epsilon z_0) val^{(r_{\delta,\epsilon})}_{outer} \geq val^* - \delta - \epsilon \sum_{i=1}^m z_i b_i$$

and thus

$$(1 + \epsilon z_0) \sup_r val^{(r)}_{outer} \geq val^* - \delta - \epsilon \sum_{i=1}^m z_i b_i.$$

Letting ϵ and δ tend to 0, we obtain

$$\sup_r val^{(r)}_{outer} \geq val^*,$$

as desired, and the proof is complete. □

Remark. The missing part in the proof of Theorem 9 in the published paper lies in the fact that one needs to upper bound $\mu(K)$ (for any feasible μ) by an absolute constant (in order to be able to let ϵ tend to 0, see the displayed equation at the bottom of page 46).

Hence, the current proof is correct, for instance, if problem (GMP) contains a constraint of the form $\int_K d\mu \leq b$ (for some $b \in \mathbb{R}$) (which gives $\mu(K) \leq b$).

The statement in Theorem 10 (and its proof) should be adapted in the same way as for Theorem 9 (without the Archimedean assumption), now using Theorem 10 (by Schmüdgen) instead of Theorem 8 (by Putinar).

Theorem 11. *Assume K is compact, program* (GMP) *is feasible, and Slater condition (S) holds. Then, we have*

$$\lim_{r\to\infty} \overline{val}^{(r)}_{outer} = val.$$