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A Complement on Pole Placement

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***Abstract*—A complement is given on Wonham's description of the spectral assignability properties of $A+BF$ when F is varied under the restriction that a given subspace $\mathcal{V}$ be $(A+BF)$-invariant. We apply the theory to give a simple solution to the output stabilization problem.**

I. Introduction

In this correspondence, we shall follow the notational conventions of [1]. So, let $\mathcal{V}$ be an (A,B)-invariant subspace, and let us write $F(\mathcal{V})=\{F: \mathcal{X}\to\mathcal{U} \mid (A+BF)\mathcal{V}\subset\mathcal{V}\}$. A natural question is, what freedom do we have in placing the poles of $A+BF$ when F is restricted to $F(\mathcal{V})$. The spectrum of the restriction of $A+BF$ to $\mathcal{V}$ is described in [1, Corollary 5.2], but this only gives a partial answer. To obtain a full description, we shall determine the remaining part of the spectrum below. The usefulness of the thus provided description will be illustrated in a simple solution to the output stabilization problem.

II. A Spectral Lattice Diagram

For all clarity, let us state the basic fact we shall use in a separate lemma.

Lemma: Suppose $\mathcal{V}$ is an A-invariant subspace, so that, in a basis adapted to the chain $0\subset\mathcal{V}\subset\mathcal{X}$, we have

$$A=\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{pmatrix}, \quad B=\begin{pmatrix} B_1 \\ B_2\end{pmatrix}. \tag{1}$$

Then (A_{22},B_2) is controllable if and only if $\langle A|\mathcal{B}+\mathcal{V}\rangle=\mathcal{X}$.

Proof: This is easily seen from the rank criterion, applied to A and $\begin{pmatrix} B_1 & V_1 \\ B_2 & 0\end{pmatrix}$, where V_1 is any matrix such that $\operatorname{Ran}\begin{pmatrix} V_1 \\ 0\end{pmatrix}=\mathcal{V}$; note that $\operatorname{Ran}\begin{pmatrix} B_1 & V_1 \\ B_2 & 0\end{pmatrix}=\mathcal{B}+\mathcal{V}$. □

Now, let us introduce the following notation. If $\mathcal{V}$ and $\mathcal{W}$ are invariant subspaces for some linear mapping A, and $\mathcal{V}\subset\mathcal{W}$, then we shall write $\sigma(A|\mathcal{W}/\mathcal{V})$ (the *spectrum of A between $\mathcal{W}$ and $\mathcal{V}$*) for the spectrum of the mapping that is induced on the quotient space $\mathcal{W}/\mathcal{V}$ by the restriction of A to $\mathcal{W}$.

In matrix terms, this means the following. With respect to a basis adapted to the chain $0\subset\mathcal{V}\subset\mathcal{W}\subset\mathcal{X}$ the matrix of A looks like

$$A=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33}\end{bmatrix}. \tag{2}$$

Then $\sigma(A|\mathcal{W}/\mathcal{V})$ is just $\sigma(A_{22})$. The rule

$$\sigma(A)=\sigma(A_{11})\,\dot\cup\,\sigma(A_{22})\,\dot\cup\,\sigma(A_{33}) \tag{3}$$

can now also be written as

$$\sigma(A)=\sigma(A|\mathcal{V}/0)\,\dot\cup\,\sigma(A|\mathcal{W}/\mathcal{V})\,\dot\cup\,\sigma(A|\mathcal{X}/\mathcal{W}). \tag{4}$$

Of course, this rule can be extended to longer chains of invariant subspaces; in the theorem following, we shall have occasion to apply it to a five-term chain.

Theorem: Let $\mathcal{V}$ be an (A,B)-invariant subspace. Then $\mathcal{R}\equiv\langle A+BF|\mathcal{B}\cap\mathcal{V}\rangle$ and $\mathcal{S}\equiv\langle A+BF|\mathcal{B}+\mathcal{V}\rangle$ do not depend on the particular choice of $F\in F(\mathcal{V})$, so that $0\subset\mathcal{R}\subset\mathcal{V}\subset\mathcal{S}\subset\mathcal{X}$ is a chain of invariant subspaces for all such F. The freedom we have in placing the poles of $A+BF$ when F is restricted to $F(\mathcal{V})$ can be described as follows:

$$\left.\begin{array}{l} \sigma(A+BF|\mathcal{X}/\mathcal{S}) \text{ is fixed} \\ \sigma(A+BF|\mathcal{S}/\mathcal{V}) \text{ is free} \\ \sigma(A+BF|\mathcal{V}/\mathcal{R}) \text{ is fixed} \\ \sigma(A+BF|\mathcal{R}/0) \text{ is free.}\end{array}\right\} \tag{5}$$

Moreover, $\mathcal{S}=\mathcal{V}+\langle A|\mathcal{B}\rangle$ and $\sigma(A+BF|\mathcal{X}/\mathcal{S})=\sigma(A|\mathcal{X}/\mathcal{S})$.

Proof: Take any $F\in F(\mathcal{V})$, and write $\mathcal{R}=\langle A+BF|\mathcal{B}\cap\mathcal{V}\rangle$ and $\mathcal{S}=\langle A+BF|\mathcal{B}+\mathcal{V}\rangle$. Clearly, we have the inclusions $0\subset\mathcal{R}\subset\mathcal{V}\subset\mathcal{S}\subset\mathcal{X}$, and with respect to a basis adapted to this chain of $(A+BF)$-invariant subspaces, we obtain

$$A+BF\equiv Z=\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ 0 & Z_{22} & Z_{23} & Z_{24} \\ 0 & 0 & Z_{33} & Z_{34} \\ 0 & 0 & 0 & Z_{44}\end{bmatrix}. \tag{6}$$

The matrix of B with respect to the same basis looks like

$$B=\begin{bmatrix} B_1 \\ 0 \\ B_3 \\ 0\end{bmatrix}. \tag{7}$$

We now see immediately that $\langle A+B\tilde{F}|\mathcal{B}\cap\mathcal{V}\rangle\subset\mathcal{R}$ for all $\tilde{F}\in F(\mathcal{V})$ and that $\langle A+B\tilde{F}|\mathcal{B}+\mathcal{V}\rangle\subset\mathcal{S}$ for all $\tilde{F}$. Because our initial F was chosen arbitrarily from $F(\mathcal{V})$, this implies that $\mathcal{R}$ and $\mathcal{S}$ do not depend on $F\in F(\mathcal{V})$. With respect to $\mathcal{S}$ we can even say more: $\langle A+BF|\mathcal{B}+\mathcal{V}\rangle$ does not depend on F and so $\mathcal{S}=\langle A|\mathcal{B}+\mathcal{V}\rangle$.

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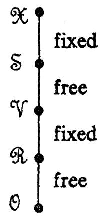
For $\sigma(A+BF|\mathcal{R}/\mathcal{O})$, we refer to [1]. Applying the lemma to $A+BF|\mathcal{S}$ and $\mathcal{V}$ shows that (Z_{33}, B_3) is controllable. Moreover, it is clear that Z_{22} and Z_{44} are the same for any $F \in \boldsymbol{F}(\mathcal{V})$. This proves (5).

We have already shown that $\mathcal{S} = \langle A|\mathcal{B}+\mathcal{V}\rangle$, and it is an easy exercise to verify that $\langle A|\mathcal{B}+\mathcal{V}\rangle = \langle A|\mathcal{B}\rangle + \mathcal{V}$.

Taking quotients with respect to $\mathcal{S}$, we have $\overline{A+BF} = \overline{A}$ because $\mathcal{B} \subset \mathcal{S}$; and consequently, $\sigma(A+BF|\mathcal{X}/\mathcal{S}) = \sigma(A|\mathcal{X}/\mathcal{S})$. □

We can get the standard spectral assignability result for unrestricted F from the Theorem, by taking $\mathcal{V}$ equal to any subspace that is $(A+BF)$-invariant for all F (such as $\mathcal{X}$, $\mathcal{O}$, or $\langle A|\mathcal{B}\rangle$).

Our findings can be conveniently summarized in a "spectral lattice diagram" for $\sigma(A+BF)$, $F \in \boldsymbol{F}(\mathcal{V})$:



(8)

III. Output Stabilization

The output stabilization problem (OSP) is the following [1, Section 4.4]: given, in addition to the pair (A,B), an output map D: $\mathcal{X} \to \mathcal{Z}$, together with a symmetric partition $\mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b$, find F: $\mathcal{X} \to \mathcal{U}$ such that

$$\mathcal{X}_b(A+BF) \subset \ker D. \tag{9}$$

The result on solvability of OSP appears in [1], but its proof might seem involved. However, it follows naturally from the above theory.

Corollary [*1, Theorem 4.4*]*:* OSP is solvable if and only if

$$\mathcal{X}_b(A) \subset \langle A|\mathcal{B}\rangle + \mathcal{V}^* \tag{10}$$

where $\mathcal{V}^*$ is the largest (A,B)-invariant subspace of $\ker D$.

Proof: By [1, Lemma 4.5], OSP is solvable if and only if there exists an (A,B)-invariant subspace $\mathcal{V} \subset \ker D$ and $F \in \boldsymbol{F}(\mathcal{V})$ such that

$$\sigma(A+BF|\mathcal{X}/\mathcal{V}) \subset \mathbb{C}_g. \tag{11}$$

By the theorem, such an F exists (for a given $\mathcal{V}$) if and only if

$$\sigma(A|\mathcal{X}/(\langle A|\mathcal{B}\rangle + \mathcal{V})) \subset \mathbb{C}_g \tag{12}$$

or, equivalently (using [1, Lemma 4.5]),

$$\mathcal{X}_b(A) \subset \langle A|\mathcal{B}\rangle + \mathcal{V}. \tag{13}$$

Clearly, (13) holds for some (A,B)-invariant subspace $\mathcal{V} \subset \ker D$ if and only if it holds for $\mathcal{V}^*$. □

IV. Final Remark

It would be interesting to investigate whether the subspaces $\mathcal{S}$, $\mathcal{V}$, and $\mathcal{R}$ can be traced solely from their properties as elements of the lattice

$$\{\mathcal{V} | (A+BF)\mathcal{V} \subset \mathcal{V} \text{ for all } F \in \boldsymbol{F}(\mathcal{V})\}. \tag{14}$$

If the answer is yes, a more general study might be undertaken into the connections between the structural properties of the lattice

$$\{\mathcal{V} | (A+BF)\mathcal{V} \subset \mathcal{V} \text{ for all } F \in \mathcal{F}_0\} \tag{15}$$

on one hand (where $\mathcal{F}_0$ is any subset of the class of all linear maps from $\mathcal{X}$ to $\mathcal{U}$), and the spectral assignability properties of $A+BF$ for $F \in \mathcal{F}_0$ on the other. For instance, taking $\mathcal{F}_0 = \{F | \ker F \supset \ker C\}$ would amount to studying the problem of stabilization by direct output feedback.

References

[1] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach.* New Springer-Verlag, 1979.

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