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TECHNICAL NOTE
On a Conjecture of Basile and Marro

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Abstract. An alternative characterization is given of the class of self-bounded controlled invariant subspaces that was introduced by Basile and Marro in Ref. 1. We also prove a result that was stated as a conjecture in the cited paper.

Key Words. Linear systems, controlled invariants, self-bounded controlled invariants, controllability subspaces, internal stabilizability.

1. Introduction

A recent addition to the so-called geometric approach to linear systems has been made (Ref. 1) by the same authors who introduced the basic concepts of this theory fourteen years ago (Ref. 2). Given a linear, finite-dimensional, time-invariant system

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, \]

and with the notation \( \mathcal{F} \) for the subspace \( \text{im } B \) of forcing actions, a subspace \( \mathcal{S} \) is said to be an \( (A, \mathcal{F}) \)-controlled invariant (Ref. 2) if, for every initial point \( x(0) \in \mathcal{S} \), there exists a control \( u(\cdot) \) such that the corresponding state trajectory \( x(\cdot) \) is completely contained in \( \mathcal{S} \). When furthermore a general subspace \( \mathcal{N} \) of \( \mathbb{R}^n \) is given, a subspace \( \mathcal{S} \) is said to be a self-bounded controlled invariant w.r.t. \( \mathcal{N} \) (Ref. 1) if \( \mathcal{S} \) is a controlled invariant contained in \( \mathcal{N} \) and if all trajectories in \( \mathcal{N} \) with starting point in \( \mathcal{S} \) are completely contained in \( \mathcal{S} \).

We shall use the following notations, consistent with Ref. 1. The class of \( (A, \mathcal{F}) \)-controlled invariants contained in \( \mathcal{N} \) is denoted by \( \text{CI}(A, \mathcal{F}, \mathcal{N}) \), and the class of \( (A, \mathcal{N}) \)-conditioned invariants (see Ref. 2) containing \( \mathcal{F} \) is

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written $\text{ci}(A, N, F)$. We also write $\text{SBCI}(A, F, N; D)$ for the class of all self-bounded $(A, F)$-controlled invariants that are contained in $N$ and that themselves contain $D$. Instead of $\text{SBCI}(A, F, N; \emptyset)$, we simply write $\text{SBCI}(A, F, N)$. Finally, if a class $\mathcal{C}$ of subspaces has a supremum (or an infimum), then we denote this supremum (or infimum) by $MC$ ($mC$). Note that, in this paper, supremum and infimum are always taken with respect to the usual lattice operations on the set of all subspaces of a given vector space.

2. Alternative Characterization

The following characterization of the class of self-bounded controlled invariants contained in a given subspace $N$ was given in Ref. 1. Write

$$J = MC(A, F, N).$$

Then, a subspace $\mathcal{S}$ belongs to $\text{SBCI}(A, F, N)$ if and only if

$$\mathcal{S} \cap F \subset \mathcal{S} \subset N,$$

$\mathcal{S}$ is controlled invariant. (2)

Using this, one can derive another useful characterization.

**Proposition 2.1.** A subspace $\mathcal{S}$ belongs to $\text{SBCI}(A, F, N)$ if and only if

$$\mathcal{J} \cap F \subset \mathcal{S} \subset N,$$

$$(A + BH)\mathcal{S} \subset \mathcal{S}, \quad \text{for all } H \text{ such that } (A + BH)\mathcal{J} \subset \mathcal{J}. \quad (5)$$

**Proof.** Since a subspace is a controlled invariant if and only if it is $(A + BH)$-invariant for some $H$, the "if" part is obvious. So, it remains to show that (5) follows from (2) and (3). Let $H$ be such that

$$(A + BH)\mathcal{J} \subset \mathcal{J};$$

and let $H'$ be such that

$$(A + BH')\mathcal{S} \subset \mathcal{S}.$$ Take $x \in \mathcal{S}$. Then,

$$(A + BH)x - (A + BH')x = B(H - H')x \in F \cap \mathcal{J} \subset \mathcal{S}. \quad (6)$$

Because

$$(A + BH')x \in \mathcal{S},$$
this shows that

\[(A + BH)x \in \mathcal{F}\]

as well, which is what we wanted to prove.

From this result, it is immediately clear that the class \(SBCI(A, \mathcal{F}, \mathcal{N})\) is closed under subspace addition as well as under intersection. This is also shown, in a quite laborious manner, in Ref. 1, Theorem 2.2.

3. Some Useful Identities

From the remarks above, it follows that the class \(SBCI(A, \mathcal{F}, \mathcal{N})\) has an infimum. Basile and Marro show (Ref. 1, Corollary 2.1) that the following relation holds:

\[mSBCI(A, \mathcal{F}, \mathcal{N}) = MCI(A, \mathcal{F}, \mathcal{N}) \cap mci(A, \mathcal{N}, \mathcal{F}). \tag{7}\]

Through the interpretation of \(mSBCI(A, \mathcal{F}, \mathcal{N})\) as the reachable set from 0 by trajectories in \(\mathcal{N}\) (Ref. 1, Theorem 3.1), (7) becomes equivalent to a result proven earlier by Morse (Ref. 3). For further interpretations of \(mSBCI(A, \mathcal{F}, \mathcal{N})\) and related subspaces, compare also Ref. 4. In fact, Basile and Marro prove a more general result (Theorem 2.3, Proposition 2.1); if \(\mathcal{D} \subset MCI(A, \mathcal{F}, \mathcal{N})\), then

\[mSBCI(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) = MCI(A, \mathcal{F}, \mathcal{N}) \cap mci(A, \mathcal{N}, \mathcal{F} + \mathcal{D}). \tag{8}\]

It is, however, easy to derive (8) from the special case (7), if one uses the following simple identities.

**Proposition 3.1.** Suppose that

\[\mathcal{D} \subset MCI(A, \mathcal{F}, \mathcal{N}). \tag{9}\]

Then, the following relations hold:

\[MCI(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) = MCI(A, \mathcal{F}, \mathcal{N}), \tag{10}\]

\[SBCI(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) = SBCI(A, \mathcal{F} + \mathcal{D}, \mathcal{N}). \tag{11}\]

**Proof.** Since

\[CI(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) \supset CI(A, \mathcal{F}, \mathcal{N}),\]

we have

\[MCI(A, \mathcal{F} + \mathcal{D}, \mathcal{N}) \supset MCI(A, \mathcal{F}, \mathcal{N}) \supset \mathcal{D}.\]
Because any \((A, \mathcal{F} + D)\)-controlled invariant that contains \(D\) is also \((A, \mathcal{F})\)-controlled invariant, it follows that

\[
\text{MCI}(A, \mathcal{F} + D, N) \in \text{CI}(A, \mathcal{F}, N).
\]

This entails (10). Next, take

\[
\mathcal{I} \in \text{SBCI}(A, \mathcal{F}, N; D).
\]

Then,

\[
\mathcal{I} \in \text{CI}(A, \mathcal{F}, N) \subset \text{CI}(A, \mathcal{F} + D, N).
\]

Moreover, by (9) and (10),

\[
(\mathcal{F} + D) \cap \text{MCI}(A, \mathcal{F} + D, N) = (\mathcal{F} + D) \cap \text{MCI}(A, \mathcal{F}, N)
\]

\[
= (\mathcal{F} \cap \text{MCI}(A, \mathcal{F}, N)) + D \subset \mathcal{I}.
\]

So, we have

\[
\mathcal{I} \in \text{SBCI}(A, \mathcal{F} + D, N).
\]

Conversely, let

\[
\mathcal{I} \in \text{SBCI}(A, \mathcal{F} + D, N).
\]

Then, (12) holds again, showing this time that \(\mathcal{I} \supset D\), so that

\[
\mathcal{I} \in \text{CI}(A, \mathcal{F}, N),
\]

and also that

\[
\mathcal{I} \supset \mathcal{F} \cap \text{MCI}(A, \mathcal{F}, N),
\]

so that, in fact,

\[
\mathcal{I} \in \text{SBCI}(A, \mathcal{F}, N; D).
\]

If

\[
D \subset \text{MCI}(A, \mathcal{F}, N),
\]

one notes the following, using (7), (10), (11):

\[
\text{mSBCI}(A, \mathcal{F}, N; D) = \text{mSBCI}(A, \mathcal{F} + D, N)
\]

\[
= \text{MCI}(A, \mathcal{F} + D, N) \cap \text{mci}(A, N, \mathcal{F} + D)
\]

\[
= \text{MCI}(A, \mathcal{F}, N) \cap \text{mci}(A, N, \mathcal{F} + D).
\]

So, in this way, it is possible to derive (8) from (7). It should be noted, though, that the proof of the crucial identity (7) in Ref. 1 can be considered
as being more straightforward than the original proof of Ref. 3. For a completely different proof, see Ref. 5, Corollary 4.10.

4. Proof of the Conjecture

Finally, let us prove a result that is given as a conjecture in Ref. 1.

Proposition 4.1. If there exists an internally stabilizable \((A, \mathcal{F})\)-controlled invariant contained in \(\mathcal{N}\) and containing \(\mathcal{D}\), then \(\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})\) is internally stabilizable.

Proof. Let us denote the class of internally stabilizable \((A, \mathcal{F})\)-controlled invariants contained in \(\mathcal{N}\) by \(\text{ISCI}(A, \mathcal{F}, \mathcal{N})\). This class has a supremum (Ref. 6, p. 114; a more direct proof is given in Ref. 7, p. 26; see also Ref. 8, Lemma 3.2). So, the assumption in the statement of the proposition is, in effect,

\[
\mathcal{D} \subseteq \text{MISCI}(A, \mathcal{F}, \mathcal{N}).
\]

The subspace \(\text{MISCI}(A, \mathcal{F}, \mathcal{N})\) is always self-bounded. This is obvious from the construction in Ref. 6, p. 114; or one can use Theorem 3.1 in Ref. 1, together with the well-known link between controllability and pole placement, to show that

\[
\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) \subseteq \text{MISCI}(A, \mathcal{F}, \mathcal{N}).
\]

So, it follows from (14) that, in fact,

\[
\text{MISCI}(A, \mathcal{F}, \mathcal{N}) \subseteq \text{SBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}).
\]

This immediately entails

\[
\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) \subseteq \text{MISCI}(A, \mathcal{F}, \mathcal{N}).
\]

Take \(H\) such that \(\text{MISCI}(A, \mathcal{F}, \mathcal{N})\) is \((A + BH)\)-invariant and such that the restriction of \(A + BH\) to this subspace is stable. It then follows from the relation (16), via the same argument that was used in the proof of Proposition 2.1, that \(\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})\) is also \((A + BH)\)-invariant; and, obviously, the restriction of \(A + BH\) to \(\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})\) is stable. □

It is not true, in general, that \(\text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})\) is the smallest internally stabilizable controlled invariant subspace in \(\mathcal{N}\) that contains \(\mathcal{D}\). In fact, such a subspace may not even exist, since the class \(\text{ISCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D})\) of internally stabilizable controlled invariants in \(\mathcal{N}\) containing \(\mathcal{D}\) is not
generally closed under intersection. For instance, when
\[ \mathcal{N} = \mathbb{R}^n, \]
the whole state space, and the pair \((A, B)\) is controllable, then
\[ \text{mSBCI}(A, \mathcal{F}, \mathcal{N}; \mathcal{D}) = \mathbb{R}^n, \]
so this subspace is not of much help in solving the important problem of finding low-dimensional internally stabilizable controlled invariants containing a given subspace \(\mathcal{D}\). It is shown in Ref. 9 that this so-called \textit{stable cover problem} is crucial in low-order compensator design.

References