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ARBITRATION GAMES; A SURVEY

by

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ABSTRACT

General two-person games in normal form are considered, where the players have non-cooperative and cooperative actions available, and where the problem arises, which Pareto optimal point in the payoff region to choose. We suppose that the players solve this problem with the aid of an arbitrator, using a bargaining solution, which is known to the players. A survey is given of known existence results and proof techniques concerning the value of arbitration games. Furthermore, new existence theorems are derived using the dummy game technique. This technique leads also to answers to computational problems and of questions concerning the structure of optimal strategy spaces, extending earlier results.

ZUSAMMENFASSUNG

Wir betrachten allgemeine Zweipersonenspiele in Normalform, wobei den Spielern nicht-kooperative und kooperative Aktionen zu Verfügung stehen und das Problem entsteht, welcher Pareto-optimale Punkt im Auszahlungsraum auszuwählen ist. Wir setzen voraus, daß die Spieler dieses Problem mit Hilfe eines Schiedsrichters zu lösen versuchen. Der Schiedsrichter verfügt über eine Schiedsvorschrift, die den Spielern bekannt ist. Es wird eine Übersicht geboten über bekannte Existenzsätze und Beweismethoden in dem Bereich der Vermittlungsspiele. Außerdem werden mit Hilfe der Technik der dummy-Nullsummenspiele neue Existenzsätzen bewiesen. Es werden auch Probleme numerischer Art untersucht und die Struktur des Raumes der optimalen Strategien beschrieben.

1. INTRODUCTION

In this paper, we consider several situations in which two decision makers (players) are involved, who have common actions available if they can agree on cooperation and where each of the decision makers has also available a set of non-cooperative actions, which are used if cooperation fails. Furthermore, we suppose that to each cooperative action and to each pair of non-cooperative actions there are associated rewards for both players, expressible in real numbers. Such situations can be modelled as a general two-person game in normal form, introduced recently in [19].

This class of games includes many classes of conflict situations studied earlier in the literature as we shall show in section 2 of this paper. Hence, these games can play a unifying role in the further development of the theory.

Starting in section 3, we look at the situation, where the players decide to cooperate with the aid of an arbitrator, using a bargaining solution known to both of them. Then the situation reduces to an arbitration game, which is, essentially, a non-cooperative two-person game in normal form and which has many similarities with a zero-sum game.

In section 4, the solution concepts 'value' and 'optimal strategy' are introduced for such arbitration games and the relationship with other solution concepts is discussed.

The existence of a value for such games is the subject of section 5. A survey is given of known existence results and proof techniques.

Subsequently, in section 6, a new technique - the dummy game approach - recently introduced in [24] for two-person games, where the cooperative actions available to the players are the correlated strategies, is applied to general two-person games. This technique gives the possibility to derive existence theorems, containing all earlier results and with conditions on strategy spaces and payoff functions which are almost as weak as the conditions in minimax theorems.

Finally, a subclass of general two-person games is considered, where the non-cooperative actions of the players are probability measures on finite sets. For these games, the dummy game approach gives insight in the structure of the solution sets. Also a multifunction is introduced, such that the search for the arbitration value is equivalent to the determination of the possible zero of that multifunction.

2. COOPERATIVE GAMES IN NORMAL FORM

One of the many nice ideas in the fundamental 1928-paper [14] of John von Neumann, is the insight that a great amount of conflict situations, where two opponents have opposite interests, can be reduced to a two-person zero-sum game in normal form (n.f). This idea is worked out further in his book [15] with O. Morgenstern. An extension was given by John Nash [12], who looked at non-zero-sum games in n.f.. Games in n.f., where players may correlate their strategies, where also looked at in the fifties (cf. H. Raiffa [18] and J. Nash [13]) but without formalisation of that situation. Surprisingly only recently in the book of B. Rauhut, N. Schmitz and E.W. Zachow [19], a formal model was presented of a general two-person game in normal form, in which cooperative as well as non-cooperative actions play a role. It will be obvious that many conflict situations where cooperation as well as non-cooperation are possible, can be modelled as a general two-person game in n.f.

Let us now be precise.

DEFINITION 2.1. A *general two-person game in normal form* Γ is a quadruplet $\langle X_1, X_2, C, K\rangle$, where X_1, X_2 and C are non-empty sets, with $X_1 \times X_2 \subset C$ and K is a bounded map from C into $\mathbb{R}^2$. For $i \in \{1,2\}$, the set X_i is called the *non-cooperative action space of player* i. The set C is called the *cooperative action space* of the players and $K : C \to \mathbb{R}^2$ is called the *payoff function*. For each $c \in C$, the i-th coordinate

$K_i(c)$ of $K(c)$ is called the payoff for player i, corresponding to action c.

In the following, we will suppose, unless otherwise stated, that $K(C)$ is a convex set, because mostly the players can mix their cooperative actions with the aid of lotteries. To avoid technical problems we suppose that K is bounded, but many results can be extended to the unbounded case.

Such a game Γ is played as follows. If the players, in some way, can come to an agreement about a cooperative action $c \in C$, then both players obtain rewards $K_1(c)$ and $K_2(c)$, respectively. If they do not agree on cooperation, then each player is obliged to choose a non-cooperative action; if $x_i \in X_i$ is chosen by player i, then player i obtains a payoff $K_i(x_1,x_2)$, $i \in \{1,2\}$.

There are of course many ways to come to the choice of a cooperative action. One of them is studied since a long time, namely arbitration. In the sections 3-6, this form of cooperation will be considered.

Now we want to describe some conflict situations, studied earlier, which can be incorporated in the theory of general two-person games in n.f.

(i) Let $\Gamma = \langle X_1,X_2,C,K\rangle$ be a general two-person game in n.f. with $C = X_1 \times X_2$. Then Γ is essentially a *non-cooperative game* introduced by J. Nash [12] and such a game will be denoted, more shortly, by $\langle X_1,X_2,K_1,K_2\rangle$. In the special case of a two-person *zero-sum game* in n.f., where $K_2 = -K_1$, the game is denoted by the triplet $\langle X_1,X_2,K\rangle$, where K equals the realvalued function K_1. Here $K(C)$ may be non-convex.

(ii) A *bargaining pair* (cf. [11]), is a pair (a,S), where S is a compact convex subset of $\mathbb{R}^2$ and $a \in S$. The set of all bargaining pairs is denoted by $\mathcal{B}$. An element $(a,S) \in \mathcal{B}$ corresponds to a situation, where two players are involved and where the i-th coordinate a_i of a is the level of utility which player i receives if they do not cooperate, while S contains all the attainable utility pairs when they cooperate. Such a bargaining pair can be seen as a general two-person game in n.f. $\langle X_1,X_2,C,K\rangle$ where $X_1 = \{a_1\}$, $X_2 = \{a_2\}$, $C = S$ and $K : C \to \mathbb{R}^2$ is the identity.

(iii) For a set X, let us denote the set of discrete probability measures on X, by $X^{\sim}$. Let $\Gamma = \langle X_1,X_2,C,K\rangle$ be a general two-person game in n.f. with $C = (X_1 \times X_2)^{\sim}$ and where, for $i \in \{1,2\}$,

$$K_i(\mu) = \iint K_i(x_1,x_2)\,d\mu(x_1,x_2), \text{ for all } \mu \in (X_1 \times X_2)^{\sim}.$$

Elements of $(X_1 \times X_2)^{\sim}$ are called *correlated strategies*.

Such games with correlation were studied (informally) for the first time in H. Raiffa [18] and J. Nash [13].

(iv) Let $\langle X_1,X_2,K_1,K_2\rangle$ be a non-cooperative game where the players are allowed to devide the rewards afterwards amongst them and to throw away some rewards if they want. Then this results in a general two-person game in n.f. $\langle X_1,X_2,C,K_1,K_2\rangle$, where elements $c \in C$ are of the form "play $x_1 \in X_1$, $x_2 \in X_2$ and pay player $i \in \{1,2\}$ an amount t_i where $t_1+t_2 \le K_1(x_1,x_2)+K_2(x_1,x_2)$" and where we may suppose that $t_i \ge \inf\{K_i(x_1,x_2); (x_1,x_2) \in X_1 \times X_2\}$, for $i \in \{1,2\}$. Such games are also considered informally in Raiffa [18] p.370, when he looks at games where side payments are permissible.

3. ARBITRATION GAMES

Let $\Gamma = \langle X_1, X_2, C, K\rangle$ be a general two-person game in n.f. We will call the closure of the convex set $\{K(c);\ c \in C\}$ the *cooperative payoff space* of Γ and we denote this space by $\mathcal{R}(\Gamma)$. Thus $\mathcal{R}(\Gamma) = \mathrm{cl}\{K(c);\ c \in C\}$. By using cooperative actions, each point of $\mathcal{R}(\Gamma)$ can be approached as near as the players want. If there is cooperation between the players, then it is reasonable to suppose that the players are only interested in payoff pairs lying in the *Pareto set* $\mathcal{P}(\Gamma)$ of Γ, where

$$\mathcal{P}(\Gamma) := \{p \in \mathcal{R}(\Gamma)\ ;\ \text{for each } s \in \mathcal{R}(\Gamma) \text{ with } s \geq p, \text{ we have } s = p\}.$$

But the preferences of the two players with respect to the payoff pairs in $\mathcal{P}(\Gamma)$ are strictly opposite. For $p, q \in \mathcal{P}(\Gamma)$, p is more preferred than q by player 1 – notation $p \geq_1 q$ – if $p_1 \geq q_1$. The most preferred elements of player 1 and player 2, respectively, are

$$\underline{p}(\Gamma) := (\max_{p\in\mathcal{P}(\Gamma)} p_1, \min_{p\in\mathcal{P}(\Gamma)} p_2) \text{ and } \tilde{p}(\Gamma) := (\min_{p\in\mathcal{P}(\Gamma)} p_1, \max_{p\in\mathcal{P}(\Gamma)} p_2).$$

There are various manners to solve this conflict of preferences. One way will be considered in this paper. We will suppose that an arbitrator is called in. After that the only influence still open to the players is then the delivering of a threat strategy. Then the arbitrator indicates a Pareto point of the cooperative payoff region, depending on the delivered threat strategies. The role of the arbitrator can be described by a map $\phi : \mathcal{B} \to \mathbb{R}^2$, called *bargaining map*, which is known to both players. We will suppose that

$$\phi(a,S) \geq a \text{ and } \phi(a,S) \in \mathcal{P}(S), \text{ for all } (a,S) \in \mathcal{B}.$$

(Here $\mathcal{P}(S)$ is the Pareto set of S.)

The arbitration procedure takes place as follows.

Step 1. Independently of each other, the players deliver (*threat*) *strategies* $\hat{x}_1 \in X_1$ and $\hat{x}_2 \in X_2$, respectively, to the arbitrator.

Step 2. The arbitrator calculates the payoff $\phi(K(\hat{x}), \mathcal{R}(\Gamma))$, where $K(\hat{x}) := (K_1(\hat{x}_1, \hat{x}_2), K_2(\hat{x}_1, \hat{x}_2))$ and chooses a cooperative action $\hat{c} \in C$ with $K(\hat{c}) = \phi(K(\hat{x}), \mathcal{R}(\Gamma))$, if that is possible; otherwise $\hat{c} \in C$ is chosen as close to $\phi(K(\hat{x}), \mathcal{R}(\Gamma))$ as both players want.

Step 3. The players play the game Γ and are obliged to use the action $\hat{c}$, indicated by the arbitrator.

In the following we will often write $\phi K(\hat{x}_1, \hat{x}_2)$ instead of $\phi(K(\hat{x}_1, \hat{x}_2), \mathcal{R}(\Gamma))$.

If we normalize the above procedure, then we obtain the non-cooperative game in normal form

$$\Gamma_\phi = \langle X_1, X_2, \phi_1 K, \phi_2 K\rangle,$$

where $\phi_i K(x,y)$ is the i-th coordinate of $\phi(K_1(x_1,y_1),K_2(x_2,y_2)) \in \mathbb{R}^2$. This game Γ_ϕ is called the *arbitration game*, corresponding to the game Γ and the bargaining solution ϕ.

In the literature, much attention is paid to questions as: which bargaining solution is acceptable, fair or reflects the relative strenghts of the players etc.? (Cf. [2], [10], [20], [21], [23]). A bargaining solution ϕ^N with nice properties was introduced in [11], which we shall call the *Nash solution* and which assigns to an $(a,S) \in \mathcal{B}$ the unique point $\phi^N(a,S) = (\phi_1^N(a,S),\phi_2^N(a,S))$ of $P(S)$ with the property that

$$(\phi_1^N(a,S)-a_1)(\phi_2^N(a,S)-a_2) = \max_{p\in P(S)} (p_1-a_1)(p_2-a_2).$$

Another attractive bargaining solution ϕ^M was introduced by E. Kalai and R.W. Rosenthal [8]: $\phi^M(a,S)$ is the unique element in $[a,\mathcal{U}(S)] \cap P(S)$, where $\mathcal{U}(s) := (\max_{s\in S} s_1, \max_{s\in S} s_2)$ is the *utopia point* of S and $[a,\mathcal{U}(S)]$ the line segment with endpoints a and $\mathcal{U}(S)$. For other bargaining solutions see [3], [4], [7], [9], [28] and for continuity questions in this field, see [5] and [25].

It is not our purpose to discuss in this paper the advantages and disadvantages of the various bargaining solutions but we refer to [2], [10], [21], [23]. In the following, we restrict our attention mostly to the class of regular and upper semicontinuous (u.s.c.) bargaining solutions. Here a bargaining solution ϕ is called *regular* if, for all $(a,S),(b,T) \in \mathcal{B}$ with $S = T$ and $\phi(a,S) = \phi(b,T)$, we have $\phi(\lambda a+(1-\lambda)b,S) = \phi(a,S)$, for all $\lambda \in [0,1]$. A bargaining solution is called *upper semicontinuous* (u.s.c.) if, for each sequence $(a,S),(a(1),S_1),(a(2),S_2),\ldots$ in $\mathcal{B}$ with

$$\lim_{n\to\infty} (a(n),S_n) = (a,S) \text{ (i.e. } \lim_{n\to\infty} a(n) = a \text{ and } \lim_{n\to\infty} d_H(S_n,S) = 0),$$

we have $\phi_i(a,S) \geq \limsup_{n\to\infty} \phi_i(a(n),S_n)$, for $i \in \{1,2\}$. (Here d_H is the Hausdorff metric). The Nash solution ϕ^N and the Kalai-Rosenthal solution ϕ^M are regular and u.s.c. (cf. [5]).

In the next section, we introduce the concept: value for arbitration games.

4. VALUES AND OPTIMAL STRATEGIES FOR ARBITRATION GAMES

First let us consider a zero-sum game in normal form $\Gamma = \langle X_1,X_2,K\rangle$. The expression $\underline{v}(\Gamma) := \sup_{x_1\in X_1} \inf_{x_2\in X_2} K(x_1,x_2)$ is called the *lower value* of Γ and $\bar{v}(\Gamma) := \inf_{x_2\in X_2} \sup_{x_1\in X_1} K(x_1,x_2)$ the *upper value*. In general $\underline{v}(\Gamma) \leq \bar{v}(\Gamma)$. If $\underline{v}(\Gamma) = \bar{v}(\Gamma)$, then we say that the game Γ is *strictly determined* and the common number is called the *value* of the game and is denoted by $v(\Gamma)$. If the value of Γ exists, then $O_1(\Gamma) = \{\hat{x}_1 \in X_1;\ \inf_{x_2\in X_2} K(\hat{x}_1,x_2) = v(\Gamma)\}$ is the *optimal strategy space for player* 1 in the game Γ and $O_2(\Gamma) = \{\hat{x}_2 \in X_2;\ \sup_{x_1\in X_1} K(x_1,\hat{x}_2) = v(\Gamma)\}$ the *optimal strategy space for player* 2.

In minimax theorems one deals with sufficient conditions guaranteeing the existence of the value of a zero sum game. For a survey of minimax theorems, see [27].

Inspired by the above we now give a (new) definition of arbitration value and optimal threat strategy reflecting the alliedness of arbitration games with zero-sum games.

Let Γ_ϕ be an arbitration game, where $\Gamma = \langle X_1, X_2, C, K\rangle$. First we note that $\pi_1 : P(\Gamma) \to [\bar{p}_1(\Gamma), \underline{p}_1(\Gamma)]$ with $\pi_1(p_1,p_2) = p_1$ is an order preserving homeomorphism of the totally ordered space $(P(\Gamma), \geq_1)$ onto the ordered closed interval $([\bar{p}_1(\Gamma), \underline{p}_1(\Gamma)], \geq)$. This implies that the expressions $\inf(S)$, $\sup(S)$ make sense for each $S \subset P(\Gamma)$, where inf and sup are taken w.r.t. the order $\geq_1$. Thus the expressions

$$v^1(\Gamma_\phi) := \sup_{x_1 \in X_1} \inf_{x_2 \in X_2} \phi K(x_1,x_2)$$

$$v^2(\Gamma_\phi) := \inf_{x_2 \in X_2} \sup_{x_1 \in X_1} \phi K(x_1,x_2)$$

are meaningful. $v^i(\Gamma_\phi)$ is called the *security point of player* i. In general $v^1(\Gamma_\phi) \leq_1 v^2(\Gamma_\phi)$.

DEFINITION 4.1. We will say that the arbitration game Γ_ϕ is *strictly determined* if $v^1(\Gamma_\phi) = v^2(\Gamma_\phi)$. In that case, this common point on $P(\Gamma)$ is called the *(arbitration) value* of Γ_ϕ and is denoted by $v(\Gamma_\phi)$. If $v(\Gamma_\phi)$ exists, then

$$O_1(\Gamma_\phi) := \{\hat{x}_1 \in X_1;\ \inf_{x_2 \in X_2} \phi K(\hat{x}_1,x_2) = v(\Gamma_\phi)\},$$

$$O_2(\Gamma_\phi) := \{\hat{x}_2 \in X_2;\ \sup_{x_1 \in X_1} \phi K(x_1,\hat{x}_2) = v(\Gamma_\phi)\}$$

are the *optimal (threat) strategy spaces* of player 1 and 2, respectively.

We conclude this section with some

REMARKS. (i) To a zero-sum game $\Gamma = \langle X_1, X_2, K\rangle$ there corresponds, in an obvious way, the general two-person game $\Gamma^* = \langle X_1, X_2, (X_1 \times X_2)^\sim, (K,-K)\rangle$. Now $R(\Gamma^*) = P(\Gamma^*)$ and for each bargaining solution ϕ: $\phi(a, R(\Gamma^*)) = a$, for all $a \in R(\Gamma^*)$. Furthermore,

$$v^1(\Gamma^*_\phi) = (\underline{v}(\Gamma), -\underline{v}(\Gamma)),\quad v^2(\Gamma^*_\phi) = (\bar{v}(\Gamma), -\bar{v}(\Gamma))$$

which implies that the game Γ^*_ϕ possesses an arbitration value iff the zero-sum game Γ possesses a value; and then $O_i(\Gamma^*_\phi) = O_i(\Gamma)$, for $i \in \{1,2\}$.

(ii) A point $(\hat{x}_1, \hat{x}_2) \in X_1 \times X_2$ is a *Nash equilibrium point* in the non-cooperative game Γ_ϕ (where $\Gamma = (X_1, X_2, C, K)$) iff Γ_ϕ possesses an arbitration value and $\hat{x}_1 \in O_1(\Gamma_\phi)$, $\hat{x}_2 \in O_2(\Gamma_\phi)$. For all Nash equilibrium points the payoff to player i is the i-th coordinate $v_i(\Gamma_\phi)$ of $v(\Gamma_\phi)$. (Cf. [24], section 4).

(iii) The following four assertions are equivalent for a general two-person game $\Gamma = \langle X_1, X_2, C, K\rangle$ and a bargaining solution ϕ (cf. theorems 4.1 and 4.3 in [24]).

(a) $v^1(\Gamma_\phi) = v^2(\Gamma_\phi)$ i.e. Γ_ϕ possesses a value,

(b) $(\sup_{x_1} \inf_{x_2} \phi_1 K(x_1,x_2), \sup_{x_2} \inf_{x_1} \phi_2 K(x_1,x_2)) \in P(\Gamma)$,

(c) the non-cooperative game Γ_ϕ possesses ε-equilibria for each $\varepsilon > 0$.

(d) the zero-sum games $\langle X_1, X_2, \phi_1 K\rangle$ and $\langle X_1, X_2, -\phi_2 K\rangle$ have a value.

In [24], the definition of arbitration value was based on formulation (b).

5. EXISTENCE OF ARBITRATION VALUES

We start with a survey of known existence theorems for arbitration games and of used proof techniques. Much attention was given to arbitration games corresponding to a finite bimatrix game (A,B), where $A = [a_{ij}]_{i=1,j=1}^{m\ \ n}$ and $B = [b_{ij}]_{i=1,j=1}^{m\ \ n}$ are m×n-matrices of real numbers and where correlation is allowed. This situation corresponds to the general two-person game $\Gamma(A,B) = \langle S^m, S^n, S^{m\times n}, E\rangle$, where

$$S^m = \{p \in \mathbb{R}^m;\ p \geq 0,\ \sum_{i=1}^m p_i = 1\},\quad S^n = \{q \in \mathbb{R}^n;\ q \geq 0,\ \sum_{j=1}^n q_j = 1\},$$

$$S^{m\times n} = \{Z = [z_{ij}]_{i=1,j=1}^{m\ \ n};\ z_{ij} \geq 0 \text{ for all } i,j \text{ and } \sum_{i=1}^m \sum_{j=1}^n z_{ij} = 1\},$$

$$E(p,q) = (pAq, pBq), \text{ for all } (p,q) \in S^m \times S^n,$$

$$E(Z) = (\sum_{i=1}^m \sum_{j=1}^n a_{ij} z_{ij},\ \sum_{i=1}^m \sum_{j=1}^n b_{ij} z_{ij}), \text{for all } Z \in S^{m\times n}.$$

J. Nash [13] proved in 1953 that the arbitration game $(\Gamma(A,B))_{\phi^N}$ possesses an arbitration value and optimal strategies. In his proof he used the fixed point theorem of Kakutani for multifunctions. G. Owen [17] reproved this result of J. Nash with the aid of the minimax theorem of J. von Neumann, using a suitable family of matrix games. H. Raiffa [18] proved in 1953 also with Kakutani's theorem, the more general result that $(\Gamma(A,B))_\phi$ possesses a value and optimal strategies for each regular and u.s.c. bargaining solution ϕ and this result was proved again by E. Burger [1] with the aid of the fixed point theorem of L.E.J. Brouwer.

In [26], S.H. Tijs and M.J.M. Jansen proved that an arbitration game $(\Gamma(A,B))_\phi$, where A and B are m×∞-matrices of real numbers and ϕ is an arbitrary regular and u.s.c. bargaining solution, has a value and that player 1 possesses an optimal strategy. This result was proved by approximating the semi-infinite bimatrix game with finite subgames, to which Raiffa's theorem is applied.

In [24], S.H. Tijs and M.J.M. Jansen derived a string of existence theorems with the dummy game approach for two-person arbitration games where correlation is allowed. In these theorems only weak conditions were posed on strategy spaces and payoff functions.

In [19], B. Rauhut, N. Schmitz and E.W. Zachow paid attention to games $\Gamma^0 = \langle S^m, S^n, C, E\rangle$, where C is a compact, convex set, containing $S^m \times S^n$ and where for the restriction E' of E to $S^m \times S^n$: $E'(p,q) = (pAq, pBq)$, for all $(p,q) \in S^m \times S^n$. They proved, with the aid of Kakutani's theorem, that $\Gamma^0_{\phi^M}$ and $\Gamma^0_{\phi^N}$ possess an arbitration value and optimal strategies for both players.

In the next section, we want to extend the successful dummy game approach of [24] to arbitration games based on general two-person games in normal form. This method gives the possibility to derive very general existence theorems, containing all existence results mentioned before. Furthermore, for subclasses of arbitration games, an insight into the structure of the solution sets is gained and also computational procedures can be based on it.

6. THE DUMMY GAME APPROACH

In the following, $\Gamma = \langle X_1, X_2, C, K\rangle$ is a general two-person game in n.f. and $\phi : \mathcal{B} \to \mathbb{R}^2$ is u.s.c. and regular. Let $\phi^\Gamma : \mathcal{R}(\Gamma) \to \mathcal{P}(\Gamma)$ be the (arbitration) map, defined by

$$\phi^\Gamma(r) := \phi(r, \mathcal{R}(\Gamma)), \text{ for all } r \in \mathcal{R}(\Gamma).$$

Then ϕ^Γ is continuous and, for all $r, s \in \mathcal{R}(\Gamma)$, $\alpha \in [0,1]$, we have $\phi^\Gamma(r) \geq r$ and $\phi^\Gamma(\alpha r + (1-\alpha)s) = \phi^\Gamma(r)$ if $\phi^\Gamma(r) = \phi^\Gamma(s)$ (cf. [25] lemma 3.2). That the dummy game approach is successful is partly a consequence of the cone property of ϕ^Γ: for each $p \in \overset{\circ}{\mathcal{P}}(\Gamma)' := \mathcal{P}(\Gamma) \setminus \{\underline{p}(\Gamma), \bar{p}(\Gamma)\}$ the set $(\phi^\Gamma)^{-1}(p)$ is the intersection of $\mathcal{R}(\Gamma)$ and a convex cone with vertex p (cf. [24], lemma 3.2). Let us call the elements of the set $D := \{(d_1, d_2) \in \mathbb{R}^2 ;\ d \leq 0,\ d_1 + d_2 = -1\}$ *directions*. We will say that a direction d is a *suitable direction* for $p \in \overset{\circ}{\mathcal{P}}(\Gamma)$ if for all $r \in \{p + \alpha d;\ \alpha \in [0,\infty)\} \cap \mathcal{R}(\Gamma)$ we have $\phi^\Gamma(r) = p$. We will denote the set of suitable directions by D(p). D(p) is, for each $p \in \overset{\circ}{\mathcal{P}}(\Gamma)$, a non-empty compact, convex set (cf. [24], lemma 3.3).

DEFINITION 6.1 For each $p \in \overset{\circ}{\mathcal{P}}(\Gamma)$ and $d \in D(p)$, the zero-sum game $\langle X_1, X_2, K_{p,d}\rangle$, where

$$K_{p,d}(x_1,x_2) := d_2(p_1 - K_1(x_1,x_2)) - d_1(p_2 - K_2(x_1,x_2)), \text{ for all } (x_1,x_2) \in X_1 \times X_2,$$

will be called the *dummy game* of Γ_ϕ corresponding to the pair (p,d).

Important relations between the dummy games and the arbitration game Γ_ϕ are given in the following theorem. A proof can be given by modifying proofs in [24], and using the fact that

$$\phi K(x_1,x_2) \leq_1 p \text{ if } K_{p,d}(x_1,x_2) \leq 0 \text{ and } \phi K(x_1,x_2) \geq_1 p \text{ if } K_{p,d}(x_1,x_2) \geq 0$$

(cf. [24], lemma 5.1).

THEOREM 6.2. *Let* Γ *and* ϕ *be as above. Let* $p \in \overset{\circ}{\mathcal{P}}(\Gamma)$, $d \in D(p)$.

(i) *If* $\underline{v}(\Gamma_{p,d}) \geq 0$, *then* $v^1(\Gamma_\phi) \geq_1 p$.

(ii) *If* $\bar{v}(\Gamma_{p,d}) \leq 0$, *then* $v^2(\Gamma_\phi) \leq_1 p$.

(iii) *If* $\underline{v}(\Gamma_{p,d}) \geq 0$, *for all* $p \in \overset{\circ}{\mathcal{P}}(\Gamma)$ *and* $d \in D(p)$, *then* $v(\Gamma_\phi) = \underline{p}(\Gamma)$.

(iv) *If* $\bar{v}(\Gamma_{p,d}) \leq 0$, *for all* $p \in \overset{\circ}{\mathcal{P}}(\Gamma)$, $d \in D(p)$, *then* $v(\Gamma_\phi) = \bar{p}(\Gamma)$.

(v) *If* $v(\Gamma_{p,d}) = 0$, *then* $v(\Gamma_\phi) = p$, $O_1(\Gamma_{p,d}) \subset O_1(\Gamma_\phi)$ *and* $O_2(\Gamma_{p,d}) \subset O_2(\Gamma_\phi)$.

(vi) *If* $\underline{v}(\Gamma_{q,d}) \geq 0$ *for all* $q \in \mathcal{P}(\Gamma)$ *with* $p \geq_1 q$ *and all* $d \in D(q)$ *and if* $\bar{v}(\Gamma_{q,d}) \leq 0$, *for all* $r \in \mathcal{P}(\Gamma)$ *with* $r \geq_1 p$ *and all* $d \in D(r)$, *then* $v(\Gamma_\phi) = p$.

In view of the following theorem it is possible to derive existence theorems for arbitration games by exploiting minimax theorems to dummy games. We will give two examples in the theorems 6.4 and 6.5.

THEOREM 6.3. *Let* Γ *and* ϕ *be as above. Suppose that for each* $\alpha,\beta \geq 0$ *the zero-sum game* $\langle X_1,X_2,\alpha K_1+\beta(-K_2)|_{X_1\times X_2}\rangle$ *possesses a value. Then* Γ_ϕ *possesses an arbitration value.*

[Here $f|_{X_1\times X_2}$ is the restriction of f to $X_1\times X_2$.]

PROOF. We have to show that $v^1(\Gamma_\phi) = v^2(\Gamma_\phi)$. We know that $v^1(\Gamma_\phi) \leq_1 v^2(\Gamma_\phi)$. Suppose $v^1(\Gamma_\phi) <_1 v^2(\Gamma_\phi)$. Then there exists a $p \in \mathcal{P}(\Gamma)$ with $v^1(\Gamma_\phi) <_1 p <_1 v^2(\Gamma_\phi)$. Take $d \in D(p)$. Then $v(\Gamma_{p,d}) = \underline{v}(\Gamma_{p,d}) < 0$ by (i) of theorem 6.2 and $v(\Gamma_{p,d}) = \bar{v}(\Gamma_{p,d}) > 0$ by (ii) of theorem 6.2, a contradiction. □

Note that the condition in theorem 6.3 implies that all dummy games possess a value.

THEOREM 6.4. *Let* Γ *and* ϕ *be as above. Suppose further that*

(i) X_1 *and* X_2 *are compact and convex subsets of topological vector spaces,*

(ii) $K_1|_{X_1\times X_2}$ *and* $K_2|_{X_1\times X_2}$ *are continuous functions,*

(iii) $\alpha K_1|_{X_1\times X_2}+\beta(-K_2)|_{X_1\times X_2}$ *is quasi concave-convex, for each* $\alpha,\beta \geq 0$.

Then the arbitration game Γ_ϕ *possesses a value and optimal strategies for both players.*

PROOF. The theorem is a direct consequence of theorem 6.3, because the properties (i), (ii) and (iii) imply that for each $\alpha,\beta \geq 0$ the zero-sum game $\langle X_1,X_2,\alpha K_1+\beta(-K_2)\rangle$ satisfies the conditions of the minimax theorem of H. Nikaido [16]. Hence, $v(\Gamma_\phi)$ exists. With a standard argument (cf. the proof of theorem 5.3 in [24]) one can prove that there exist optimal strategies. □

Using the minimax theorem of M. Sion [22], theorem 6.3 implies

THEOREM 6.5. *Let* Γ *and* ϕ *be as above.* Γ_ϕ *possesses an arbitration value and an optimal strategy for player* 1, *if*

(i) X_1 *is a compact and convex subset of a topological vector space,*

(ii) X_2 *is a convex set,*

(iii) $K_1|_{X_1\times X_2}$, $-K_2|_{X_1\times X_2}$ *are semicontinuous functions (upper semicontinuous in the first coordinate and lower semicontinuous in the second coordinate),*

(iv) *for each* $\alpha,\beta \geq 0$ *the function* $\alpha K_1+\beta(-K_2)|_{X_1\times X_2}$ *is quasi concave-convex.*

One easily verifies that all existence results mentioned in section 5 are a special case of theorem 6.4 or 6.5.

In the rest of this section, we consider arbitration games $\Gamma(A,B,C)_\phi$ where A and B are m×n-matrices, ϕ is a regular and u.s.c. arbitration function and $\Gamma(A,B,C)$ is the general two-person game $\langle S^m,S^n,C,K\rangle$ with $K|_{S^m\times S^n}(p,q) = (pAq,pBq)$, for all $(p,q) \in S^m \times S^n$, and C a compact, convex set containing $S^m \times S^n$. Such an arbitration game possesses an arbitration value and optimal strategies for both players in view of theorem 6.4, because in this case all dummy games are essentially mixed extensions of a finite matrix game. In [6], a special subclass of such games is studied, namely games with $C = S^{m\times n}$. Many of the results in [6] can be extended to the whole class of general games in n.f. in an almost straightforward way. We will give some extensions. From the next theorem it follows that the optimal strategy spaces of the above games are polytopes. Before stating the theorem we need some notation.

Suppose that the arbitration value v of $\Gamma(A,B,C)_\phi$ is an element of $\mathring{P}(\Gamma(A,B,C))$. Then $d^+, d^- \in D(v)$ are the elements with $d_2^+ = \max_{d\in D(v)} d_2$ and $d_2^- = \min_{d\in D(v)} d_2$.

THEOREM 6.6 (cf. theorem 2.2 and lemmas 2.5 and 2.6 in [6]).
If $v(\Gamma(A,B,C)_\phi) \in \mathring{P}(\Gamma(A,B,C))$, *then*

$$O_1(\Gamma(A,B,C)_\phi) = \{x^* \in S^m;\ K_{v,d^+}(x^*,y) \geq 0, \text{ for all } y \in S^n\}$$

$$O_2(\Gamma(A,B,C)_\phi) = \{y^* \in S^n;\ K_{v,d^-}(x,y^*) \leq 0, \text{ for all } x \in S^m\}.$$

If $v(\Gamma(A,B,C)_\phi) = \bar{p}(\Gamma(A,B,C))$, *then* $O_1(\Gamma(A,B,C)_\phi) = S^m$ *and* $O_2(\Gamma(A,B,C)_\phi)$ *is a polyhedral set.*
If $v(\Gamma(A,B,C)_\phi) = \underline{p}(\Gamma(A,B,C))$, *then* $O_2(\Gamma(A,B,C)_\phi) = S^n$ *and* $O_1(\Gamma(A,B,C)_\phi)$ *is polyhedral.*

For the computation of the value of an arbitration game $\Gamma(A,B,C)_\phi$ the multifunction $m : \mathring{P}(\Gamma(A,B,C)) \to \mathbb{R}$ defined by

$$m(p) := \{v(\Gamma(A,B,C)_{p,d});\ d \in D(p)\} \qquad (p \in \mathring{P}(\Gamma(A,B,C)),$$

can play a useful role. Thus m assigns to a $p \in \mathring{P}(\Gamma)$ the collection of all values of the dummy zero-sum games corresponding to p. For all $p \in \mathring{P}(\Gamma)$, $m(p)$ is a nonempty compact and convex subset of $\mathbb{R}$. Further properties of m are given in the following

THEOREM 6.7. *Let us denote the arbitration value of* $\Gamma(A,B,C)_\phi$ *by* v. *If* $v \in \mathring{P}(\Gamma(A,B,C))$, *then*

(i) v *is the only zero of* m: $0 \in m(p)$ *iff* $p = v$,

(ii) $m(p) > 0$ (*i.e.* $t > 0$ *for all* $t \in m(p)$) *iff* $v >_1 p$,

(iii) $m(p) < 0$ *iff* $p >_1 v$.

If $v \notin \mathring{P}(\Gamma(A,B,C))$, *then*

(iv) $m < 0$ (*i.e.* $m(p) < 0$ *for all* $p \in \mathring{P}(\Gamma(A,B,C))$ *iff* $v = \bar{p}(\Gamma(A,B,C))$,

(v) $m > 0$ *iff* $v = \underline{p}(\Gamma(A,B,C))$.

From this result it follows that the search for the arbitration value of a game of the described type is equivalent with the determination of the possible zero of the multifunction m. How to find such zeros is described in section 4 in [6].

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ARBITRATION GAMES; A SURVEY

S.H. Tijs; M.J.M. Jansen

We consider two-person games in normal form, where the players may cooperate and where then the problem arises, which Pareto optimal point in the cooperative payoff region to choose. We suppose that the players solve this problem with the aid of an arbitration function, which is continuous and profitable, and for which the original of each Pareto point is a convex set.

Let us be more precise. Let $\langle X,Y,C,K\rangle$ be a cooperative two-person game Γ, where X and Y are the strategy spaces for player 1 and player 2, respectively, C is the set of cooperative actions and $K : C \to \mathbb{R}^2$ is the (bounded) payoff map. Without loss of generality, one may suppose that $X \times Y \subset C$ and that K(C) is a convex set. $R(\Gamma) = \mathrm{cl}\ K(C)$ is called the payoff region and $P(\Gamma)$ is the Pareto set of $R(\Gamma)$. Let $\phi : R(\Gamma) \to P(\Gamma)$ be a regular arbitration function.

In our model the situation procedes as follows.

Step 1. Independently of each other, the players assign an $x \in X$ and a $y \in Y$ and deliver it to an arbitrator.

Step 2. The arbitrator calculates the payoff $\phi(K_1(x,y),K_2(x,y))$ and chooses a cooperative action $c \in C$ such that $K(c) = \phi(K(x,y))$ if that is possible; otherwise c is chosen in such a way that K(c) is as close to $\phi K(x,y)$ as both players want.

Step 3. Then the game Γ is played, where the players are obliged to choose c.

From a strategic point of view, this course of things is equivalent to the non-cooperative game in normal form $\Gamma_\phi = \langle X,Y,\phi_1 K,\phi_2 K\rangle$, where $\phi_i K(x,y)$ is the i-th coordinate of $\phi K(x,y)$. This game is called the *arbitration game* corresponding to Γ and ϕ.

Let $v_1 = \sup_x \inf_y \phi_1 K(x,y)$ and $v_2 = \sup_y \inf_x \phi_2 K(x,y)$. If $v = (v_1,v_2) \in P(\Gamma)$, then we call v the *arbitration value* of Γ_ϕ.

If $\phi_1 K(x^*,y) \geq v_1$ for all $y \in Y$ ($\phi_2 K(x,y^*) \geq v_2$ for all $x \in X$), then we call x^* (y^*) an *optimal strategy* for player 1 (2).

The following problems are discussed.

1. The existence of values and ε-optimal strategies.
2. The computation of values and ε-optimal strategies.
3. The structure of the optimal strategy spaces.

Central in the theory is the approximation of an arbitration game with families of dummy zero-sum games.

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