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BOUNDS FOR THE CORE AND THE τ-VALUE

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For a game in characteristic function form, the upper vector and the lower vector are introduced, which are upper and lower bounds for the core of such a game. With the aid of these vectors the core cover and the hypercube of a game are defined. Both sets, which contain the core, are compared with other core catchers. For the subclass of quasi-balanced games the τ-value is introduced. Some nice properties of the τ-value are derived, and a comparison with the Shapley value is made.

1. INTRODUCTION

Let N be the set $\{1,2,\ldots,n\}$ and 2^N the family of all subsets of N. Elements of N are called *players* and elements of 2^N *coalitions*.
A function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = \emptyset$ will be called an n-*person game* (*in characteristic function form*); v(S) is called the *worth* of the coalition $S \in 2^N$. The set of n-person games will be denoted by G^n. To each game $v \in G^n$ we assign two vectors b^v and a^v in $\mathbb{R}^n$, which we introduce at the end of this section. These vectors play an important role in the rest of the paper. With the aid of these vectors we define in section 2 the notions core cover and hypercube of a game and discuss some of their properties and their relation with the core of a game and with other core catchers. In section 3 the subclass of games with non-empty core cover and hypercube, respectively, are characterized. The τ-value of games with non-empty core cover is introduced in section 4 and some interesting properties are derived. Special attention is paid to monotone simple games. In the last section we discuss some examples.

Now we begin with some definitions. For $v \in G^n$ and $i \in N$, we call $v(N)-v(N-\{i\})$ the *marginal contribution of player* i in the game v. Let b^v be the vector in $\mathbb{R}^n$ with i-th coordinate the marginal contribution of player i. We will call b^v the *upper vector* of v.
For $S \in 2^N$ and $i \in S$ we call the expression

$$R^v(S,i) := v(S) - \sum_{k \in S-\{i\}} b_k^v$$

the *remainder for player* i *in the coalition* S. Note that $R^v(S,i)$ is the amount which remains for player i, when in the coalition S the amount $v(S)$ is divided in such a way that all players in S, unequal to i, obtain their marginal contribution. With the aid of these remainders we define the vector a^v. This is the vector in $\mathbb{R}^n$ with i-th coordinate the greatest possible remainder for player i. Hence $a_i^v := \max_{S \ni i} R^v(S,i)$ for each $i \in N$.

The vector a^v will be called the *lower vector* of v.

The names upper and lower vector are motivated by the following propositions, which imply that a^v is a lower bound of the *core*

$$C(v) := \{x \in \mathbb{R}^n ; \sum_{k\in N} x_k = v(N), \sum_{k\in S} x_k \geq v(S) \text{ for all } S \in 2^N\}$$

of the game v and b^v an upper bound (both with respect to the usual partial order in $\mathbb{R}^n$).

PROPOSITION 1.1. Let $v \in G^n$. Then $C(v)$ is a subset of the orthant $U(v) := \{x \in \mathbb{R}^n ; x \leq b^v\}$ with upper point b^v.

PROOF. Let $x \in C(v)$ and $i \in N$. Then

$$x_i = \sum_{k\in N} x_k - \sum_{k\in N-\{i\}} x_k \leq v(N) - v(N-\{i\}) = b_i^v. \qquad \square$$

PROPOSITION 1.2. Let $v \in G^n$, $S \in 2^N$ and $i \in S$ and $x \in C(v)$. Then $x_i \geq R^v(S,i)$. Consequently, $C(v)$ is a subset of the orthant $L(v) := \{x \in \mathbb{R}^n ; x \geq a^v\}$ with lower point a^v.

PROOF. By the definition of $C(v)$ and by proposition 1.1, we have

$$x_i = \sum_{k\in S} x_k - \sum_{k\in S-\{i\}} x_k \geq v(S) - \sum_{k\in S-\{i\}} x_k \geq v(S) - \sum_{k\in S-\{i\}} b_k^v = R^v(S,i). \qquad \square$$

2. THE CORE COVER AND THE HYPERCUBE OF A GAME

For a game $v \in G^n$ we call $L(v) \cap U(v)$ the *hypercube* of v and denote this set by $H(v)$. So $H(v) = \{x \in \mathbb{R}^n ; a^v \leq x \leq b^v\}$. The set $\{x \in H(v); \sum_{k\in N} x_k = v(N)\}$ is denoted by $CC(v)$ and called the *core cover* of v. These sets were studied for the first time in Tijs and Lipperts [14]. In view of the propositions 1.1 and 1.2 we have

THEOREM 2.1. For each $v \in G^n$:

(i) $C(v) \subset CC(v) \subset H(v)$.

(ii) $H(v) \neq \emptyset \iff a^v \leq b^v$.

(iii) $CC(v) \neq \emptyset \iff a^v \leq b^v, \sum_{k\in N} a_k \leq v(N) \leq \sum_{k\in N} b_k$.

The question arises, whether a^v and b^v are sharp bounds for the core elements i.e. is a^v equal to $\inf C(v)$ and $b^v = \sup C(v)$, where

$$\inf C(v) := (\inf_{x\in C(v)} x_1, \inf_{x\in C(v)} x_2, \ldots, \inf_{x\in C(v)} x_n),$$

$$\sup C(v) := (\sup_{x\in C(v)} x_1, \sup_{x\in C(v)} x_2, \ldots, \sup_{x\in C(v)} x_n)?$$

It is not difficult to construct a game, where the bounds are not sharp. But for many classes of games, a^v and b^v are sharp lower and upper bounds, as the following theorem shows. For a proof of this theorem we refer to [14].

THEOREM 2.2. Inf $C(v) = a^v$, sup $C(v) = b^v$, if v satisfies one of the following conditions:

(i) v is an additive game.
(ii) v is a superadditive 2-person game.
(iii) v is a superadditive 3-person game with $C(v) \neq \emptyset$.
(iv) v is a monotone simple game with $v(N) = 1$ for which the set of veto players $J^v := \{i \in N;\ v(N-\{i\}) = 0\}$ is non-empty.
(v) v is a convex game.

Another question is: differs the core from the core cover?
It is easy to construct a game v with $CC(v) \neq C(v)$ (see example 5.3 in section 5), but it is surprising that the sets coincide for many classes of games as the following theorem shows. (For a proof see [14].)

THEOREM 2.3. $C(v) = CC(v)$ if v satisfies one of the following conditions:

(i) v is an additive game.
(ii) v is a 2-person game or a 3-person game.
(iii) v is a monotone simple game.
(iv) v is a constant-sum game.

It is interesting to compare CC(v) with other 'core catchers'.

One of them was given by Milnor [9] (cf. Luce and Raiffa [7], chapter 11). Milnor looked at the orthant

$$M(v) := \{x \in \mathbb{R}^n;\ \sum_{i \in N} x_i = v(N),\ x_i \le m_i^v \text{ for each } i \in N\}$$

where $m_i^v := \max_{S \ni i} (v(S)-v(S-\{i\}))$, the largest incremental contribution, which player i can make to any coalition. Obviously, $m_i^v \ge b_i^v$ for each $i \in N$, which implies that $C(v) \subset CC(v) \subset U(v) \subset M(v)$.

K. Kikuta showed in [6] and in earlier papers that the numbers $n_i^v := \min_{S \ni i} (v(S)-v(S-\{i\}))$ are also interesting in game theory, but simple examples show that they are not necessarily lower bounds (or upper bounds) for core elements.

For convex games (cf. [13]) $U(v) = M(v)$, but in general U(v) is a better core catcher than M(v).
M(v) possesses the additional property that it contains also the Shapley value ϕ^v, each stable set of v and also the kernel (cf. [8], p.319).

Another core catcher W(v) was introduced by Weber [15]. For each permutation θ of N, let x^θ be the *marginal worth vector* with i-th coordinate $v(p_i^\theta \cup \{i\})-v(p_i^\theta)$, where $p_i^\theta = \{j \in N;\ \theta(j) < \theta(i)\}$. Then W(v) is the convex hull of the set of all n! marginal worth vectors. Obviously, $\phi^v \in W(v)$, and also $C(v) \subset W(v)$ as was proved in [15].

Shapley [13] proved that $C(v) = W(v)$, if v is convex and the converse of this result was established by Ichiishi [5]. So $W(v)$ is a better core catcher than $CC(v)$ for convex $v \in G_n$ if $n \geq 4$. For superadditive 2-person games we have: $C(v) = CC(v) = W(v)$, and for superadditive 3-person games: $C(v) = CC(v) \subset W(v)$. Note that $C(v) \neq W(v)$ for all non-convex 3-person games v, so here $CC(v)$ is a better core catcher than $W(v)$. For each monotone simple game v we have $C(v) = CC(v)$, but for the monotone simple 3-person game v with $v\{i\} = 0$ for all $i \in \{1,2,3\}$, $v\{1,2\} = v\{1,3\} = v\{1,2,3\} = 1$ and $v\{2,3\} = 0$, we have

$$C(v) = \{(1,0,0)\} = CC(v) \neq W(v) = \mathrm{conv}\{(1,0,0),(0,1,0),(0,0,1)\}.$$

3. GAMES WITH NON-EMPTY CORE, CORE COVER OR HYPERCUBE

A family C of subsets of $2^N-\{\emptyset\}$ and a function $\omega : C \to [0,\infty)$ form a *balanced pair* (C,ω) if $1_N = \sum_{S\in C} \omega(S)1_S$; here $1_S : N \to \{0,1\}$ is the function with $1_S(i) = 1$ iff $i \in S$.

A game $v \in G_n$ is called a *balanced game* if for each balanced pair (C,ω) we have

$$\sum_{S\in C} \omega(S)\ v(S) \leq v(N). \tag{3.1}$$

Let us denote the family of balanced games by B^n. Independently of each other Bondareva [2] and Shapley [12] proved:

$$v \in B^n \iff C(v) \neq \emptyset \tag{3.2}$$

We will show that the n-person games with non-empty hypercube and also the n-person games with non-empty core cover are precisely the games, satisfying a specific subset of the balancedness inequalities in (3.1).

Let us call a game *semi-balanced* if

$$v(S) + \sum_{i\in S} v(N-\{i\}) \leq |S|v(N) \text{ for each } S \in 2^N-\{\emptyset\} \tag{3.3}$$

where $|S|$ is the number of elements of S.

Note that each balanced game is also semi-balanced, because (C_S,ω) with $C_S := \{S,\{N-\{i\}\}_{i\in S}\}$ and $\omega(T) := |S|^{-1}$ for all $T \in C_S$ is a balanced pair for each $S \in 2^N-\{\emptyset\}$. Let us denote the family of semi-balanced n-person games by SB^n. Then we have

THEOREM 3.1. $v \in SB^n \iff H(v) \neq \emptyset$.

PROOF. $H(v) \neq \emptyset \iff R(S,i) \leq b_i^v$ for all $i \in N$ and $S \ni i$

$\iff v(S) \leq \sum_{i\in S} b_i^v$ for all $S \in 2^N-\{0\} \iff (3.3)$. □

Let us call a semi-balanced game $v \in G_n$, *quasi-balanced* if it satisfies also the following (balancedness) inequalities

$$\sum_{i\in N} \Big(v(S_i) + \sum_{j\in S_i-\{i\}} v(N-\{j\})\Big) \leq \Big(1-n+\sum_{i\in N}|S_i|\Big)v(N) \tag{3.4}$$

for all n-tuples $(S_1,S_2,\dots,S_n)$ with $S_i \in 2^N$ and $i \in S_i$ $(i \in N)$

Let QB^n be the family of all quasi-balanced n-person games. It is straightforward to show, in view of theorem 2.1 (iii), that the following theorem holds.

THEOREM 3.2. $v \in QB^n \iff CC(v) \neq \emptyset$.

It follows from (3.1), (3.3) and (3.4) that B^n, QB^n and SB^n are polyhedral cones in G_n. The dimension of these cones equals n.

If we provide G^n with the metric d, defined by $d(v,w) := \max\{|v(S)-w(S)|;\ S \in 2^N\}$ then

$$\|b^v - b^w\|_\infty \le 2d(v,w) \tag{3.5}$$

$$R^v(S,i) \le R^w(S,i)+(2|S|-1)d(v,w) \text{ if } S \ni i$$

$$\|a^v - a^w\|_\infty \le (2n-1)d(v,w) \tag{3.6}$$

From the inequalities (3.5) and (3.6) it follows that the multifunctions $H : SB^n \to \mathbb{R}^n$, $CC : QB^n \to \mathbb{R}^n$ are continuous. These inequalities will also be useful in the proof of theorem 4.1.

4. THE τ-VALUE FOR QUASI-BALANCED GAMES

For each $v \in QB^n$, on the line segment $[a^v,b^v]$ with endpoints a^v and b^v, there lies precisely one vector, for which the sum of the coordinates equals $v(N)$. Let us denote this vector by τ^v and let us call this vector the *τ-value of the game* v. The map $\tau : QB^n \to \mathbb{R}^n$, assigning to each $v \in QB^n$, the τ-value τ^v, possesses nice properties, some of which we collect in the following theorem.

THEOREM 4.1.

(i) For each $v \in QB^n$ and each $i \in \mathbb{N}$: $\tau_i^v \ge v\{i\}$ (*Individual rationality*).

(ii) For each $v \in QB^n$: $\sum_{k\in N} \tau_k^v = v(N)$ (*Efficiency*).

(iii) For each $v \in QB^n$ and each permutation $\theta : N \to N$ we have $\tau^{\theta_* v} = \theta_*(\tau^v)$ (*Symmetry*).
[Here $\theta_* v$ is the n-person game with $\theta_* v(S) = v(\theta(S))$ for each $S \in 2^N$, and $\theta_*(\tau^v)$ is the vector with i-th coordinate the $\theta(i)$-th coordinate of τ^v.]

(iv) For each $v \in QB^n$ and each $i \in N$ for which

$$v(S \cup \{i\}) = v(S) \quad \text{for all } S \in 2^N \tag{4.1}$$

we have $\tau_i^v = 0$ (*Dummy player property*).

(v) For each $v \in QB^n$, $k \in (0,\infty)$, $c \in \mathbb{R}^n$: $\tau^{kv+c} = k\tau^v + c$ (*S-equivalence property*).

(vi) $\tau : QB^n \to \mathbb{R}^n$ is a continuous function (*Continuity property*).

PROOF. (i) $\tau_i^v \ge a_i^v \ge R(\{i\},i) = v\{i\}$.

(ii) The efficiency property follows immediately from the definition of τ^v.

(iii) It is straightforward to show that $b^{\theta_* v} = \theta_*(b^v)$ and $a^{\theta_* v} = \theta_*(a^v)$. Then it follows from $\tau^v \in [a^v,b^v]$ and $\sum_{k\in N} \tau_k^v = v(N)$, that $\theta_*(\tau^v) \in [\theta_*(a^v),\theta_*(b^v)] = [a^{\theta_* v},b^{\theta_* v}]$ and $\sum_{k\in N} (\theta_*(\tau^v))_k = \sum_{i\in N} \tau_i^v = v(N)$. So $\theta_*(\tau^v) = \tau^{\theta_* v}$.

(iv) Suppose that for some $i \in N$, (4.1) holds. Then $b_i^v = 0$ and $v\{i\} = v\{\emptyset\} = 0$. Since $0 = b_i^v \ge a_i^v \ge v\{i\} = 0$, we have $a_i^v = b_i^v = 0$. But then the i-th coordinate of each vector on $[a^v,b^v]$ equals 0, especially $\tau_i^v = 0$.

(v) The proof of the S-equivalence property is straightforward.
(vi) The continuity of τ follows easily with the aid of the inequalities (3.5) and (3.6). □

We note that the restriction of the Shapley value Φ (cf. [1], p.295 and [11]) to the cone QB^n also satisfies the properties (ii)-(vi) in theorem 4.1 but not property (i) as the following example shows.

EXAMPLE 4.2. Let v be the 3-person game with $v\{1,2\} = -7$, $v\{1,2,3\} = 1$ and $v(S) = 0$ if $S \neq \{1,2\}$ and $S \neq \{1,2,3\}$. Then $\tau^v = (\frac{1}{10},\frac{1}{10},\frac{4}{5}) \geq (v\{1\},v\{2\},v\{3\})$ and $\Phi^v = (-\frac{5}{6},-\frac{5}{6},2\frac{2}{3})$. So $\phi_1^v < v\{i\}$.

The Shapley value satisfies the additivity property: $\phi^{v+w} = \phi^v+\phi^w$. The τ-value satisfies all properties characterizing the Shapley value (cf. [1], p.296) except the additivity property as we see in the following example.

EXAMPLE 4.3. Let v be the 3-person game with $v\{1,2\} = \frac{1}{2}$, $v\{1,2,3\} = 1$ and $v(S) = 0$ if $S \neq \{1,2\},\{1,2,3\}$. Let w be the 3-person game with $w\{1,3\} = \frac{1}{4}$, $w\{1,2,3\} = 1$ and $w(S) = 0$ if $S \neq \{1,3\},\{1,2,3\}$. Then $\tau^v = (\frac{2}{5},\frac{2}{5},\frac{1}{5})$, $\tau^w = (\frac{4}{11},\frac{3}{11},\frac{4}{11})$ and $\tau^{v+w} = (\frac{8}{21},\frac{7}{21},\frac{6}{21}) \neq \tau^v+\tau^w$.

For a study, where maps were considered, satisfying another subcollection of the Shapley axioms, we refer to [4].

We note that τ is not the only map from QB^n into $\mathbb{R}^n$, satisfying the properties (i)-(vi) of theorem 4.1. Also the nucleolus satisfies these axioms (cf. [8], p.336). Another interesting map, satisfying these properties is the σ-value $\sigma : QB^n \to \mathbb{R}^n$, where σ^v is the unique vector on the line segment with endpoints $(v\{1\},v\{2\},\ldots,v\{n\})$ and b^v, for which the sum of the coordinates equals $v(N)$. (σ can be defined on the larger class of games v satisfying $\sum_{k\in N} v\{k\} \leq v(N) \leq \sum_{k\in N} b_k^v$). Note that σ^v is very easy to calculate in comparison with the calculation of ϕ^v and τ^v.

It is an open problem to find properties, which together with the properties (i)-(vi) characterize the τ-value (A related open problem for the nucleolus was mentioned in [8], p.336).

The following theorem is an immediate consequence of the definitions of τ^v and $CC(v)$.

THEOREM 4.4. For each $v \in QB^n$: $\tau^v \in CC(v)$.

The τ-value is not necessarily an element of the core as example 5.3 in the next section shows; but for 2- and 3-person quasi-balanced games the τ-value lies in the core as we see in the following theorem.

THEOREM 4.5. (i) For each $v \in QB^2$ we have

$$\tau^v = \phi^v = \sigma^v = (\tfrac{1}{2}(v\{1,2\}+v\{1\}-v\{2\}),\tfrac{1}{2}(v\{1,2\}-v\{1\}+v\{2\})) \in C(v).$$

(ii) For each $v \in QB^3$: $\tau^v \in C(v)$.

PROOF. The proof of (i) is straightforward and will be omitted.

Take $v \in QB^3$. There is an $\alpha \in [0,1]$ such that $\tau_1^v = a_1^v + \alpha(b_1^v - a_1^v)$. Since $b_1^v \geq a_1^v$, we obtain $\tau_1^v \leq a_1^v + (b_1^v - a_1^v) = b_1^v$. Then $\tau_2^v + \tau_3^v = v(N) - \tau_1^v \geq v(N) - b_1^v = v(N-\{1\}) = v\{2,3\}$. Similarly, one can show that $\tau_1^v + \tau_3^v \geq v\{1,3\}$ and $\tau_1^v + \tau_2^v \geq v\{1,2\}$. These inequalities imply that $\tau^v \in C(v)$. □

Because of their importance in applications, in the remainder of this section we give special attention to the subclass MS^n of G^n of monotone simple games. A game $v \in G^n$ is called a *monotone simple game* if

(M.1) $v(S) \leq v(T)$ for all $S,T \in 2^N$ with $S \subset T$,

(M.2) $v(S) \in \{0,1\}$ for all $S \in 2^N$,

(M.3) $v(N) = 1$.

For such a game $v \in MS^n$ the set

$$J^v := \{i \in N;\ v(N-\{i\}) = 0\} = \{i \in N;\ b_i^v = 1\}$$

is called the set of veto players. In the next theorem we see that a game $v \in MS^n$ is a quasi-balanced game if and only if there are veto players.

THEOREM 4.6. Let $v \in MS^n$. Then

(i) $v \in QB^n$ iff $J^v \neq \emptyset$.

(ii) If $J^v \neq \emptyset$, then: $\tau_i^v = |J^v|^{-1}$ if $i \in J^v$, $\tau_i^v = 0$ if $i \notin J^v$.

(iii) If $|J^v| \geq 2$, then $\tau^v = \sigma^v$.

PROOF. (i) Obviously, $b_i^v \in \{0,1\}$ for each $i \in N$ in view of (M.2). If $v \in QB^n$, then $\sum_{k \in N} b_k^v \geq v(N) = 1$ by theorem 2.1. Hence, $J^v = \{i \in N;\ b_i^v = 1\} \neq \emptyset$. Conversely, suppose $i \in J^v \neq \emptyset$. Then $v(N-\{i\}) = 0$ and also $v(S) = 0$ for all $S \subset N-\{i\}$ by (M.1). Let $e^i \in \mathbf{R}^n$ be the vector with $e_k^i = 0$ for $k \neq i$ and $e_i^i = 1$. Then $\sum_{k \in T} e_k^i = 0 = v(T)$ for all $T \in 2^N$ with $T \subset N-\{i\}$ and $\sum_{k \in T} e_k^i = 1 \geq v(T)$ for all $T \in 2^N$ with $i \in T$. Hence, $e^i \in C(v)$. Then $e^i \in CC(v)$ by theorem 2.1 and thus $v \in QB^n$ in view of theorem 3.2.

(ii) and (iii) If $J^v = \{i\}$ for some $i \in N$, then $b_i^v = a_i^v = 1$ and $b_k^v = a_k^v = 0$ for all $k \in N-\{i\}$. Then $\tau^v = e^i$, and (ii) is satisfied for this case. If $|J^v| \geq 2$, the $a_i^v = 0$ for all $i \in N$ and $v\{i\} = 0$ for all $i \in N$. Hence $\tau^v = \sigma^v$, implying (iii). Furthermore, $\tau_i^v = |J^v|^{-1}$ if $i \in J^v$ and $\tau_i^v = 0$, otherwise. So also in this case (ii) holds. □

In the following we denote the set $\{v \in MS^n;\ J^v \neq \emptyset\}$ by MS_*^n. The question, for which $v \in MS_*^n$ the τ-value coincides with the Shapley value, is answered in the following theorem.

THEOREM 4.7. Let $v \in MS_*^n$. Then $\tau^v = \phi^v$ iff $v(J^v) = 1$.

PROOF. (i) Suppose $v(J^v) = 1$. Then $v(S) = 1$ iff $S \supset J^v$.

By the dummy player property of ϕ we obtain $\phi_i^v = 0$ for $i \notin J^v$ and by the symmetry property and the efficiency property: $\phi_i^v = |J^v|^{-1}$ for $i \in J^v$. In view of theorem 4.6 (ii) we have then: $\phi^v = \tau^v$.

(ii) Suppose now that $v(J^v) = 0$. Let $u_1, u_2, \ldots, u_k$ be the elements of J^v and

$u_{k+1},\dots,u_n$ the elements of $N \setminus J^v$. Let $\tilde{\theta}$ be the permutation $(u_1,u_2,\dots,u_n)$ of N. Then

$$\phi_i^v \geq \frac{1}{n!}x_i^\theta \text{ for all } i \in N \text{ and there is a } u^* \in N \setminus J^v$$

with $x_{u^*}^\theta = 1$. But then $\phi_{u^*}^v > 0$, while $\tau_{u^*}^v = 0$ by theorem 4.6. So $\phi^v \neq \tau^v$. □

It is easy to prove that a game $v \in MS_*^n$ is a convex game if and only if $v(J^v) = 1$. So we may conclude with the aid of the theorem of Ichiishi [5]:

THEOREM 4.8. Let $v \in MS_*^n$. Then $W(v) = C(v)$ iff $v(J^v) = 1$.

In [3], Dubey gave a characterization of the restriction of the Shapley value to MS^n. One of the three characterizing properties was the transfer property (cf. Weber [15], p.16):

$$\phi^{v \vee w} + \phi^{v \wedge w} = \phi^v + \phi^w.$$

[Here $v \vee w(S) := \max\{v(S),w(S)\}$, $v \wedge w(S) := \min\{v(S),w(S)\}$ for all $S \in 2^N$].
In the next theorem we answer the question for which pairs $v,w \in MS_*^n$ the τ-value satisfies the transfer property

THEOREM 4.9. Let $v,w \in MS_*^n$. Then

(i) $v \wedge w \in MS_*^n$.

(ii) $v \vee w \in MS_*^n$ iff $J^v \cap J^w = \emptyset$.

(iii) If $v \vee w \in MS_*^n$ then

$$\tau^{v \vee w} + \tau^{v \wedge w} = \tau^v + \tau^w \text{ iff } J^v \subset J^w \text{ or } J^w \subset J^v.$$

PROOF. Obviously, $v \wedge w \in MS^n$ and $v \vee w \in MS^n$. Then $v \wedge w \in MS_*^n$, because $J^{v \wedge w} = J^v \cup J^w \neq \emptyset$. From $J^{v \vee w} = J^v \cap J^w$ we may conclude (ii). Now we prove (iii). Suppose first that $J^v \subset J^w$. Then $J^{v \wedge w} = J^w$ and $J^{v \vee w} = J^v$. If $i \in J^v$, then $\tau_i^{v \vee w} + \tau_i^{v \wedge w} = |J^v|^{-1} + |J^w|^{-1} = \tau_i^v + \tau_i^w$. If $i \in J^w \setminus J^v$, then $\tau_i^{v \vee w} + \tau_i^{v \wedge w} = 0 + |J^w|^{-1} = \tau_i^v + \tau_i^w$. If $i \notin J^w$, then $\tau_i^{v \vee w} + \tau_i^{v \wedge w} = 0 + 0 = \tau_i^v + \tau_i^w$.
So $J^v \subset J^w$ implies the transfer property for v and w.
Similarly, the transfer property is implied by $J^w \subset J^v$. Suppose now that the transfer property holds. We have to prove that $J^v \subset J^w$ or $J^w \subset J^v$. Suppose $J^v \not\subset J^w$. Take $i \in J^v \setminus J^w$. Then $\tau_i^{v \vee w} + \tau_i^{v \wedge w} = \tau_i^v + \tau_i^w$ implies $0 + |J^v \cup J^w|^{-1} = |J^v|^{-1} + 0$. Hence $J^w \subset J^v$. □

5. SOME EXAMPLES

In this section we calculate for three games the τ-value and the Shapley value. In the first two examples the τ-value seems a more 'natural' payoff distribution than the Shapley value; the converse is true in the third example.

EXAMPLE 5.1. We consider the 3-person market (cf. von Neumann and Morgenstern [10], p.564) with one seller (player 1) and two buyers (player 2 and 3). We suppose that

the seller possesses a commodity with value u for him and with values v_1 and v_2 for the players 2 and 3, respectively. This market corresponds with the 3-person game v with $v\{1\} = u$, $v\{2\} = v\{3\} = 0$, $v\{1,2\} = v_1$, $v\{1,3\} = v_2$, $v\{2,3\} = 0$ and $v\{1,2,3\} = v_2$.
Let us suppose first that $u < v_1 < v_2$. For this game the upper vector b^v equals $(v_2,0,v_2-v_1)$ and the lower vector a^v equals $(v_1,0,0)$. The core and the core cover coincide with the line segment with endpoints $(v_2,0,0)$ and $(v_1,0,v_2-v_1)$. The Weber set W(v) equals the convex hull of the points $(v_2,0,0)$, $(v_1,0,v_2-v_1)$, (u,v_1-u,v_2-v_1) and $(u,0,v_2-u)$. The τ-value $\tau^v = (\frac{1}{2}(v_1+v_2),0,\frac{1}{2}(v_2-v_1))$ seems a very 'natural' payoff: in the payoff distribution τ^v the weak buyer, player 2, obtains nothing and the price paid by player 3 to player 1 is the average of the values v_1 and v_2. The Shapley value $\phi^v = \frac{1}{6}(2u+v_1+3v_2,v_1-u,3v_2-2v_1-u)$ assigns also a positive amount to the weak buyer, and $\phi^v \notin C(v)$. If $u < v_1 = v_2$, then $\tau^v = b^v = a^v = (v_2,0,0)$, $\phi^v = \frac{1}{6}(2u+4v_2,v_2-u,v_2-u)$.

EXAMPLE 5.2. Now we look at the "glove" game <N,v> for which the player set N is divided in two disjoint non-empty subsets L and R, where the players in L possess a left hand glove, and those in R a right hand glove. The worth of a coalition $S \in 2^N-\{\emptyset\}$ is given by $v(S) := \min\{|S \cap L|,|S \cap R|\}$. Suppose first that $1 \leq |L| < |R|$. Then $b_i^v = a_i^v = 1$ for each $i \in L$ and $b_i^v = a_i^v = 0$ for each $i \in R$. So $CC(v) = C(v) = \{b^v\} = \{\tau^v\}$. Note that $\tau_i^v = 0$ for all $i \in L$ and that $\tau^v \in C(v)$. For each $i \in N$, $\phi_i^v > 0$ and $\phi^v \notin C(v)$.
If $|L| = |R|$, then $a^v = (0,0,\ldots,0)$, $b^v = (1,1,\ldots,1)$ and $\tau^v = \phi^v = (\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$.

EXAMPLE 5.3. Let <N,v> be the 99-person game with $v(N) = 1$, $v(S) = \frac{1}{2}$ if $\{1,2\} \subset S \neq N$, $v\{2,3,\ldots,n\} = v\{1,3,\ldots,n\} = \frac{1}{4}$ and $v(S) = 0$ otherwise. Then $b^v = (\frac{3}{4},\frac{3}{4},\frac{1}{2},\ldots,\frac{1}{2})$, $a^v = (0,0,0,\ldots,0)$.
Hence $\tau^v = \frac{1}{200}(3,3,2,\ldots,2)$. The players 1 and 2 are 'powerful' in this game, which is not reflected in the payoff distribution τ^v. The players 1 and 2 obtain more, if the Shapley value is the payoff distribution: $\phi^v = (\frac{99}{392},\frac{99}{392},\frac{1}{196},\ldots,\frac{1}{196})$. Note that $\tau^v \notin C(v)$, because $\tau_1^v+\tau_2^v < \frac{1}{2} = v\{1,2\}$, and $\phi^v \in C(v)$. Note further that for this game $C(v) \neq CC(v)$ because $\tau^v \in CC(v)$.
Furthermore, τ^v is unequal to the nucleolus of this game, because the nucleolus lies in the core.

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