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PROBABILISTIC BARGAINING SOLUTIONS

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Abstract. Probabilistic bargaining solutions are considered, i.e. bargaining solutions which in a bargaining game do not single out one feasible point or a subset of feasible points, but assign probabilities to sets of feasible points. Although, in essence, the normative, axiomatic approach to the bargaining problem as initiated by Nash /8/ is followed, an attempt is made to bring in a descriptive element by considering probabilistic solutions, since these may predict disagreement or non-Pareto optimal agreements with positive probability. An - in the authors' opinion weakest possible - analogon of Nash's axiom of independence of irrelevant alternatives (IIA) is proposed and some classes of solutions satisfying this axiom together with a basic other axiom, are described. It is also shown that former results are covered by this new approach, and indicated what, in the authors' view, is the essence in looking for stronger versions of the IIA axiom.

Zusammenfassung. Wir betrachten probabilistische Verhandlungslösungen, d.h. Lösungen eines Verhandlungsspiels, die nicht einen zulässigen Punkt oder eine Menge von zulässigen Punkten auswählen, sondern Teilmengen von zulässigen Punkten Wahrscheinlichkeiten zuordnen. Obwohl wir im wesentlichen dem normativen axiomatischen Ansatz von Nash /8/ folgen, wird der Versuch unternommen, mit der Einführung probabilistischer Lösungen ein deskriptives Element einzubringen, da mit diesen ein Scheitern von Verhandlungen oder nicht Pareto-optimale Verhandlungsergebnisse mit positiver Wahrscheinlichkeit vorausgesagt werden können. Ein - nach Meinung der Autoren schwächstmögliches - Analogon zu Nashs Axiom der Unabhängigkeit von irrelevanten Alternativen wird vorgeschlagen und einige Klassen von Lösungen werden beschrieben, die dieses Axiom zusammen mit einem grundlegenden weiteren Axiom erfüllen. Außerdem wird gezeigt, daß frühere Ergebnisse durch diesen neuen Ansatz abgedeckt werden, und angedeutet, was, aus der Sicht der Autoren, wesentlich ist für die Suche nach schärferen Versionen des Unabhängigkeits-Axioms von Nash.

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1. INTRODUCTION. DEFINITIONS AND KNOWN RESULTS

A (2-person) bargaining game is a subset S of $\mathbb{R}^2$ such that
(B1) S is a compact and convex subset of $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2;\ x \geq 0\}$,
(B2) S is comprehensive, i.e. if $x \in S$ and $y \in \mathbb{R}^2_+, y \leq x$, then $y \in S$,
(B3) $x > 0$ for some $x \in S$.

The game-theoretic interpretation of a bargaining game S is as follows. Two players (bargainers) are bargaining over the set of feasible outcomes S. If they cooperate and agree on an element $s \in S$, then each player i (i = 1,2) receives utility payoff s_i. If they fail to reach an agreement, then each player receives utility payoff 0. The origin 0 (= (0,0)) is therefore called the disagreement outcome.
The compactness of S is required just for mathematical convenience. Convexity stems from randomizing between riskless alternatives, and the requirement $S \subset \mathbb{R}^2_+$ reflects the fact that agreements which give one or both of the players less than the disagreement utility 0, are regarded as irrational. Comprehensiveness reflects the fact that both players may freely dispose of utility. (B3) can be seen as requiring that there is an incentive to bargain.
Putting the disagreement outcome equal to the origin is just a convenient normalization. By B, we denote the family of all bargaining games.

In his 1950 paper, Nash proposes to tackle the bargaining problem (i.e., which agreement will the players reach?) in a normative way (which agreement should the players reach?). He introduces the concept of a bargaining solution and singles out a unique one by axioms. One of these axioms, symmetry, is dropped in Harsanyi, Selten /3/. De Koster, Peters, Tijs, Wakker /6/ allow a solution to assign zero utility to one of the players, and obtain two more (dictatorial) solutions. In describing these results, we follow the last mentioned authors.
Let $S \in B$. The Pareto optimal subset of S is the set

$$P(S) := \{x \in S;\ \text{if } y \in S \text{ and } y \geq x, \text{ then } y = x\}.$$

By $\bar{p}(S)$ ($\underline{p}(S)$) we denote the point in P(S) with maximal second (first) coordinate. Further, $\bar{w}(S) := (0,\bar{p}_2(S))$, $\underline{w}(S) := (\underline{p}_1(S),0)$, $\bar{W}(S) := \text{conv}\{\bar{p}(S),\bar{w}(S)\}$, $\underline{W}(S) := \text{conv}\{\underline{p}(S),\underline{w}(S)\}$. Finally, the weak Pareto optimal subset of S is the set $W(S) := \bar{W}(S) \cup P(S) \cup \underline{W}(S)$.
A scale transformation is a vector a in $\mathbb{R}^2$ such that $a > 0$. Let $\mathbb{R}^2_{++} := \{a \in \mathbb{R}^2; a > 0\}$ denote the family of all scale transformations. For $a = (a_1,a_2) \in \mathbb{R}^2_{++}$, we denote, for $x \in \mathbb{R}^2$, $a*x := (a_1x_1,a_2x_2)$, and for $T \subset \mathbb{R}^2$, $a*T := \{a*x;\ x \in T\}$. Note that $a*S \in B$ for all $a \in \mathbb{R}^2_{++}$, $S \in B$.

A <u>bargaining solution</u> (b.s.) is a map $\phi : B \to \mathbb{R}^2$ such that $\phi(S) \in S$ for all $S \in B$. The following axioms for a b.s. ϕ will be important.

<u>Pareto Optimality</u> (PO): $\phi(S) \in P(S)$ for all $S \in B$.

<u>Weak Pareto Optimality</u> (WPO): $\phi(S) \in W(S)$ for all $S \in B$.

<u>Scale Transformation Invariance</u> (STI): $\phi(a*S) = a*\phi(S)$ for all $a \in \mathbb{R}^2_{++}$, $S \in B$.

<u>Independence of Irrelevant Alternatives</u> (IIA): For all $S,T \in B$ with $S \subset T$ and $\phi(T) \in S$, we have $\phi(T) = \phi(S)$.

For $t \in (0,1)$, the bargaining solution $F^t : B \to \mathbb{R}^2$ is defined as follows: for every $S \in B$, the function $x \mapsto x_1^t x_2^{1-t}$ is maximal on S in $F^t(S)$. The <u>dictator solutions</u> D^1 and D^2 are defined by $D^1(S) := \underline{p}(S)$ and $D^2(S) := \bar{p}(S)$ for every $S \in B$. It is not hard to verify that all these solutions satisfy PO, STI and IIA, moreover, the following theorem is proved in /6/:

<u>THEOREM 1.1</u>. A bargaining solution $\phi : B \to \mathbb{R}^2$ satisfies PO, STI and IIA iff $\phi \in \{D^1, D^2, F^t;\ t \in (0,1)\}$.

Substituting in Theorem 1.1. for PO the <u>Symmetry</u> (SY) axiom (i.e., $\phi_1(S) = \phi_2(S)$ for all $S \in B$ which are symmetric w.r.t. the line $x_1 = x_2$) gives Roth's result (/12, theorem 2/) which says that there are exactly two solutions satisfying SY, STI and IIA, namely Nash's solution $F^{\frac{1}{2}}$, and the <u>disagreement solution</u> which assigns 0 to every $S \in B$. The solutions F^t were introduced in Harsanyi, Selten /3/. If, in Theorem 1.1., we allow WPO instead of PO, then we obtain two more (unfriendly dictator) solutions, namely the solution $\bar{w}$ which assigns $\bar{w}(S)$ to every $S \in B$, and the solution $\underline{w}$ which assigns $\underline{w}(S)$ to every $S \in B$ (cf. theorem 2.2. in Peters, Tijs, De Koster /11/).

Another approach to the bargaining problem is the multisolution approach. A <u>multisolution</u> (m.s.) $\Phi : B \to \mathbb{R}^2$ is a correspondence (i.e., a set-valued map) which assigns to each $S \in B$ a subset $\Phi(S) \subset S$. The mentioned axioms can be reformulated for multisolutions; only the reformulation of the IIA-axiom needs special attention.

<u>IIA for multisolutions</u>: For all $S,T \in B$ with $S \subset T$ and $\Phi(T) \cap S \neq \emptyset$, we have $\Phi(S) = \Phi(T) \cap S$; (cf. Kaneko /5/).

(For another definition of IIA, see Aumann /1/). Note that a bargaining solution ϕ can be identified with an m.s. $\tilde{\phi}$ by $\tilde{\phi}(S) := \{\phi(S)\}$ for all $S \in B$. We write ϕ instead of $\tilde{\phi}$. Let $\bar{W}$ ($\underline{W}$) denote the multisolution which assigns $\bar{W}(S)$ ($\underline{W}(S)$) to every $S \in B$. We then have

<u>THEOREM 1.2</u>. An m.s. $\phi : B \to \mathbb{R}^2$ satisfies WPO, STI and IIA iff $\phi \in \{D^1, D^2, \underline{w}, \bar{w}, \underline{W}, \bar{W}, F^t;\ t \in (0,1)\}$.

<u>PROOF</u>. See theorems 3.1 and 3.2 in /11/. □

So far we have given a survey of the main known results w.r.t.

(multi)solutions satisfying STI and IIA. In this paper, we confine our attention to the IIA* axiom. (Probabilistic versions of axioms will be denoted by an asterix.) However, probabilistic solutions satisfying other characterizing axioms may be investigated; we refer to a remark in the final section. We will also consider only solutions satisfying STI*. In experiments (e.g., Roth, Malouf /13/) and in everyday practice, often axioms which are regarded as basic in a normative theory of bargaining, are violated. In this paper, we try to appraise the possibility of negotiations ending up in disagreement, or in non-Pareto optimal agreements. More broadly speaking, we will attempt here to set up a theory, which is still axiomatic and normative, but may also have some descriptive value. Dropping the WPO axiom in theorem 1.2 is not sufficient; we obtain some more solutions, but these are either too indecisive (e.g. the m.s. which assigns the whole set S to every $S \in B$), or highly improbable (e.g. the m.s. which assigns $\text{conv}\{0,\underline{w}(S)\}$ to every $S \in B$), or too restrictive (e.g. the disagreement solution, which always assigns disagreement). Therefore, we propose to look at probabilistic solutions. Such a solution may, e.g., predict disagreement with probability strictly between 0 and 1.

In section 2, we define the new model and characterize a family of probabilistic solutions satisfying IIA*, by laying restrictions on the possible probability measures. In section 3, we characterize another family of solutions by using a stronger version of the IIA* axiom. Section 4 concludes with some final comments, among which some other arguments for considering probabilistic solutions.

To conclude this section, we remark that in social choice theory too probabilistic solutions play an interesting rôle (cf. Gibbard /2/, for example). And in bargaining, Myerson's choice mechanism (Myerson /7/) bears resemblance to our probabilistic solution concept.

2. PROBABILISTIC SOLUTIONS. THE IIA-AXIOM

For $S \in B$, let $\sigma(S)$ denote the σ-algebra of Borel sets in S. A probability measure on S is a map $\mu_S : \sigma(S) \to [0,1]$ such that $\mu_S(S) = 1$ and $\mu_S(\bigcup_i E_i) = \sum_i \mu_S(E_i)$ if $E_1, E_2, \ldots$ is a sequence of pairwise disjoint elements in $\sigma(S)$. The support of μ_S, denoted $\text{supp}(\mu_S)$, is defined by

$$\text{supp}(\mu_S) := \{x \in S;\ \mu_S(E) \neq 0 \text{ for all open } E \text{ in } \sigma(S) \text{ with } x \in E\}.$$

A (probabilistic bargaining) solution is a map μ assigning to each

$S \in B$ an element $\mu(S)$ in $M(S)$, where $M(S)$ denotes the family of all probability measures on S. In the sequel, we write μ_S instead of $\mu(S)$. For $S \in B$ and $E \in \sigma(S)$, $\mu_S(E)$ is interpreted as the probability predicted by μ that the final outcome of the bargaining game S will be in E.

Most axioms for deterministic solutions translate in an obvious way for probabilistic solutions:
$(W)PO^*$: $\mu_S(P(S))$ $(\mu_S(W(S))) = 1$ for all $S \in B$.
STI^*: $\mu_{a*S}(a*E) = \mu_S(E)$ for all $a \in \mathbb{R}^2_{++}, S \in B, E \in \sigma(S)$.
In our view, the weakest possible translation of the IIA axiom is:
IIA^*: For all $S,T \in B$ with $S \subset T$, $\mu_S(E) \geq \mu_T(E)$ for all $E \in \sigma(S)$.
In order not to violate the spirit of IIA, this formulation seems the least to ask. Note, that all the solutions (b.s. and m.s.) in section 1 can be identified with probabilistic solutions (take the values of these solutions as supports for probability measures inducing a homogeneous or degenerate distribution on their supports). Then all these solutions satisfy IIA^*.
In general, this axiom IIA^* is too weak, since it admits too many solutions. In the next section, we indicate how to look for stronger versions, and we will study one of these. In this section, we put a restriction on the allowed probability measures. First a convention: for $S \in B$ and $x \in S$, we write $\mu_S(x)$ instead of $\mu_S(\{x\})$. Then we have:
LEMMA 2.1. Let $S \in B$, and let μ be a solution satisfying STI^* and IIA^*. Then we have $\mu_S(x) = 0$ if $x \notin P(S) \cup \{0,\underline{w}(S),\bar{w}(S)\}$, for every $x \in S$.
PROOF. Let $x \in S$. Suppose there exists a set $S_x \subset S$ such that $y \geq x$ for all $y \in S_x$, S_x is countably infinite, and for every $y \in S_x$ there exists an $a \in \mathbb{R}^2_{++}$ such that $a*y = x$. By STI^* and IIA^*, if $y \in S_x$ and $a*y = x$ for some $a \in \mathbb{R}^2_{++}$, we have $\mu_S(y) = \mu_{a*S}(x) \geq \mu_S(x)$ since $a \leq (1,1)$. If $\mu_S(x) > 0$, then summing for all $y \in S_x$ would yield $\infty = \mu_S(S_x) \leq \mu_S(S) = 1$, a contradiction.

So $\mu_S(x) = 0$. The proof is finished by observing that such a set S_x exists for every $x \notin P(S) \cup \{0,\underline{w}(S),\bar{w}(S)\}$. □

Let $F(S) \subset M(S)$ be the family of all probability measures on $S \in B$ with finite support. A solution μ is called finite if $\mu_S \in F(S)$ for every $S \in B$. We restrict our attention, in the remainder of this section, to finite solutions satisfying STI^* and IIA^*. For brevity's sake only, we call such a solution an FIS solution. We will give a characterization of all FIS solutions.
Let the (probabilistic) solutions $d,\bar{w},\underline{w},D^1,D^2,F^t$ $(0 < t < 1)$ be defined as follows. For every $S \in B$ and $E \in \sigma(S)$, $d_S(E) := 1_E(0)$, $\bar{w}_S(E) := 1_E(\bar{w}(S))$, $\underline{w}_S(E) := 1_E(\underline{w}(S))$, $D^1_S(E) := 1_E(D^1(S))$, $D^2_S(E) := 1_E(D^2(S))$,

$F^t_S(E) := 1_E(F^t(S))$ for all $t \in (0,1)$, where $1_E(x) := 1$ if $x \in E$, $1_E(x) := 0$ if $x \notin E$, for all $x \in S$. So e.g., the solution d assigns probability 1 to the disagreement outcome 0, for every $S \in B$. It is straightforward to verify that all these solutions are FIS solutions. Moreover, it follows from lemma 2.1 that, if μ is an FIS solution and $S \in B$, then $\mu_S(x) \neq 0$ for a finite, positive number of elements x in $\{0,\bar{w}(S),\underline{w}(S),D^1(S),D^2(S),F^t(S);\ t \in (0,1)\}$, and $\mu_S(x) = 0$ for all other $x \in S$. Let T be a finite subset of (0,1), and let q_1,q_2,q_3,q_4,q_5,q_t $(t \in T)$ be numbers in [0,1] with $\Sigma^5_{i=1}q_i + \Sigma_{t\in T}q_t = 1$. Then the solution μ, defined by

(*) $\mu_S(E) := q_1 d_S(E) + q_2\bar{w}_S(E) + q_3\underline{w}_S(E) + q_4 D^1_S(E) + q_5 D^2_S(E) + \Sigma_{t\in T}q_t F^t_S(E)$

for all $S \in B$, $E \in \sigma(S)$, is an FIS solution. The reverse is also true, that is, every FIS solution is of this form. This is the content of the following theorem, the proof of which will only be briefly outlined, for space's sake. A complete proof can be found in Peters /9/.

THEOREM 2.2. Let μ be an FIS solution. Then there exist a finite subset T of (0,1) and numbers $q_1,q_2,\ldots,q_5,q_t$ $(t \in T)$ in [0,1] summing up to 1, such that, for all $S \in B$ and $E \in \sigma(S)$, μ satisfies (*).

OUTLINE OF THE PROOF: Let $\Delta,\square \in B$ be defined by $\Delta := \mathrm{conv}\{(0,0),(1,0),(0,1)\}$ and $\square := \mathrm{conv}(\Delta \cup \{(1,1)\})$. From STI* and IIA*, we have $\mu_\Delta(0) \geq \mu_\square(0)$ as well as $\mu_\square(0) = \mu_{(\frac{1}{2},\frac{1}{2})*\square}(0) \geq \mu_\Delta(0)$, so $\mu_\Delta(0) = \mu_\square(0)$. Let $q_1 := \mu_\Delta(0)$. From IIA*, $\mu_\Delta((0,1)) \geq \mu_\square((0,1))$ and $\mu_\Delta((1,0)) \geq \mu_\square((1,0))$. Let $q_2 := \mu_\square((0,1))$, $q_3 := \mu_\square((1,0))$, $q_4 := \mu_\Delta((1,0))-q_3$, $q_5 := \mu_\Delta((0,1))-q_2$. Let $T \subset (0,1)$ be defined by $T := \{t \in (0,1);\ \mu_\Delta((t,1-t)) \neq 0\}$, and let, for every $t \in T$, $q_t := \mu_\Delta((t,1-t))$. Then $q_1,q_2,\ldots,q_5,q_t$ $(t \in T)$ sum up to 1, and moreover, (*) is satisfied for $\square$ and Δ. First, let $S \in B$ be such that the set $A(S) := \{0,\bar{w}(S),\underline{w}(S),D^1(S),D^2(S),F^t(S);\ t \in T\}$contains exactly $|T|+5$ elements. Then one proves, using STI* and IIA*, that $\mu_S(0) \geq q_1,\ldots,\mu_S(D^2(S)) \geq q_5$, $\mu_S(F^t(S)) \geq q_t$ for all $t \in T$, which implies equality in all these inequalities since the numbers $q_1,q_2,\ldots,q_5,q_t$ $(t \in T)$ sum up to 1 and $\mu_S(S) = 1$. So also for such S, μ satisfies (*). The proof of the general case involves some subtleties, like approaching the Pareto optimal subset in a differentiable way and using a geometric characterization of the bargaining solutions F^t as in Proposition 4.1. in Peters and Tijs /10/. □

All the bargaining solutions (single-valued multisolutions) appearing in section 1, return as special cases of the FIS solutions described above. Requiring the additional axiom of Symmetry (the definition of this axiom for probabilistic solutions is obvious) singles out a one-parameter family of solutions μ^p $(p \in [0,1])$ where, for every $S \in B$ and $E \in \sigma(S)$, $\mu_S^p(E) := pd_S(E)+(1-p)F_S^{\frac{1}{2}}(E)$. As special cases, μ^0 and μ^1 predict the symmetric Nash solution and disagreement with certainty, respectively.

In the next section, we will discuss a stronger version of the IIA^*-axiom.

3. A STRONGER VERSION OF THE IIA^* AXIOM

Let $S,T \in B$ with $S \subset T$, and let μ be a probabilistic solution. The IIA^* axiom requires that $\mu_S(E) \geq \mu_T(E)$ for all $E \in \sigma(S)$. If $\mu_T(S) = 1$, then, by IIA^*, μ_S is completely determined by μ_T. If not, i.e. if $\mu_T(T\setminus S) > 0$, then this remaining probability mass has to be distributed on S. One can think of infinitely many ways to do this. One way is the following:

Conditional IIA^* ($CIIA^*$): For all $S,T \in B$ with $S \subset T$, and for all $E \in \sigma(S)$, $\mu_S(E)\mu_T(S) = \mu_T(E)$.

Another way to state $CIIA^*$ is: for all $S,T \in B$ with $S \subset T$ and for all $E \in \sigma(S)$, if $\mu_T(S) \neq 0$, then $\mu_S(E) = \mu_T(E)\mu_T(S)^{-1}$. By Bayes' formula, this last expression implies $\mu_S(E) = \mu_T(E|S)$, hence the term "conditional".

The axiom $CIIA^*$ is much stronger than IIA^*. We are able to describe all solutions satisfying $CIIA^*$ and STI^*. For convenience, let us call such a solution a CS solution.

DEFINITION 3.1. For all $t \in (0,\infty)$, the solutions $\underline{d}^t, \bar{d}^t, \underline{w}^t, \bar{w}^t$ are defined as follows. For all $S \in B$ and $E \in \sigma(S)$,

$$\underline{d}_S^t(E) := \underline{w}_1(S)^{-t}\int_{[0,\underline{w}_1(S)]} 1_{\{x\in E;\, x_2=0\}}(x)\,dx_1^t,$$

$$\bar{d}_S^t(E) := \bar{w}_2(S)^{-t}\int_{[0,\bar{w}_2(S)]} 1_{\{x\in E;\, x_1=0\}}(x)\,dx_2^t,$$

$$\underline{w}_S^t(E) := \begin{cases} \underline{p}_2(S)^{-t}\int_{[0,\underline{p}_2(S)]} 1_{\{x\in E;\, x_1=\underline{p}_1(S)\}}(x)\,dx_2^t & \text{if } \underline{p}_2(S) > 0 \\ \underline{w}_S(E) & \text{if } \underline{p}_2(S) = 0, \end{cases}$$

$$\bar{w}_S^t(E) := \begin{cases} \bar{p}_1(S)^{-t}\int_{[0,\bar{p}_1(S)]} 1_{\{x\in E;\, x_2=\bar{p}_2(S)\}}(x)\,dx_1^t & \text{if } \bar{p}_1(S) > 0 \\ \bar{w}_S(E) & \text{if } \bar{p}_1(S) = 0. \end{cases}$$

For all $t,s \in (0,\infty)$, the solutions $h^{t,s}$ are defined as follows. For all $S \in B$ and $E \in \sigma(S)$

$$h_S^{t,s}(E) := \Big(\iint_S 1_S((x_1,x_2))\,dx_1^t dx_2^s\Big)^{-1}\iint_S 1_E((x_1,x_2))\,dx_1^t dx_2^s.$$

So for $S \in B$, $\underline{d}_S^t$ and $\bar{d}_S^t$ are nonatomic probability measures with supports $\mathrm{conv}\{0,\underline{w}(S)\}$ and $\mathrm{conv}\{0,\bar{w}(S)\}$ respectively, and $\underline{w}_S^t$ and $\bar{w}_S^t$ nonatomic probability measures with supports $\underline{W}(S)$ and $\bar{W}(S)$ (if $\underline{W}(S) \neq \{\underline{w}(S)\}$, $\bar{W}(S) \neq \{\bar{w}(S)\}$), respectively, and $h_S^{t,s}$ is a nonatomic

probability measure with support S.
The following theorem describes all CS solutions. The proof is too lenghty to give here, and can be found in Peters /9/.

THEOREM 3.2. The solution μ is a CS solution iff
$\mu \in \{d,\underline{w},\bar{w},D^1,D^2,F^t;\ t \in (0,1)\} \cup \{\underline{d}^t,\bar{d}^t,\underline{w}^t,\bar{w}^t,h^{t,s};\ t,s \in (0,\infty)\}$.

Of course, all these solutions are also IIA* solutions. A drawback of the CIIA* axiom is, that it only admits "degenerate" FIS solutions, e.g., disagreement can only occur with probability always 0, or always 1. Note, moreover, that $\{d,\underline{w},\bar{w},D^1,D^2,F^t;\ t \in (0,1)\}$ is the family of all finite CS solutions, so also for finite solutions, CIIA* is strictly stronger than IIA*. Note also, that all the solutions appearing in section 1 can be identified with CS solutions (cf. our remark after the definition of IIA* in section 2).

Despite the drawback mentioned in the previous paragraph, we have described here all CS solutions, not only because of the mathematical elegance of the result, but also because of the striking similarities occurring in the definitions of the bargaining solutions F^t and of the probabilistic solutions defined in Definition 3.1. These similarities call for further investigation. We think of a probabilistic approach to the bargaining process in addition to the axiomatic one described above.

4. FINAL COMMENTS

First of all, other characterizing axioms may be considered, e.g. the Individual Monotonicity axiom (cf. Kalai, Smorodinsky /4/, Roth /12/). One probabilistic version of this axiom could be:

Individual Monotonicity (IM*): A solution μ satisfies IM* iff it satisfies IM_1^* and IM_2^*.

IM_1^*: For all $S,T \in B$ with $S \subset T$, if $\bar{w}_2(T) = \bar{w}_2(S)$, then (i) for every $E \in \sigma(T)$ there is an $E' \in \sigma(S)$ such that $x_1' \le \sup\{x_1;\ x \in E\}$ for all $x' \in E'$ and such that $\mu_S(E') \ge \mu_T(E)$, (ii) for every $E \in \sigma(S)$ there is an $E' \in \sigma(T)$ such that $x_1' \ge \inf\{x_1;\ x \in E\}$ for all $x' \in E'$ and such that $\mu_T(E') \ge \mu_S(E)$.

IM_2^*: Analogous to IM_1^*.

Solutions which satisfy IM* are e.g., the disagreement solution d, the solution K which always predicts the Kalai-Smorodinsky solution point (i.e. for $S \in B$, the intersection of P(S) and the straight line through 0 and $(\underline{w}_1(S),\bar{w}_2(S))$) with certainty, but also the solution which predicts disagreement and the Kalai-Smorodinsky solution point each with probability ½. However, there will be many more.

Further, other (stronger) versions of the IIA* axiom for probabilistic solutions may be considered, cf. the first paragraph of section 3. The results of section 3 deserve further consideration, cf. the last paragraph of section 3.

Finally, two other arguments for considering probabilistic solutions are, first, a notion of incomplete information: the Pareto boundary of a bargaining game S may be "vague", not exactly known to the players, and secondly: there is a possibility of the players making mistakes, which may account for non-Pareto optimal agreements having positive probability.

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