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A NEW COST ALLOCATION METHOD FOR MULTIPURPOSE WATER PROJECTS: THE COST GAP METHOD

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Abstract

Four different cost allocation methods are compared, which allocate the joint costs of water resource projects among its participants on the basis of separable and nonseparable costs: the egalitarian nonseparable cost (ENSC) method, the separable costs remaining benefits (SCRB) method, the minimum costs remaining savings (MCRS) method and a new one, the so-called nonseparable cost gap (NSCG) method, which is derived from the τ-value, a game theoretical concept. All these methods, except the ENSC-method, can be described with the aid of lower and upper bounds for the core of the involved cost game. For convex cost games, these three methods use the same bounds for the core and hence coincide, but their cost allocation not necessarily belongs to the core. For a second class of cost games, the so-called 1-convex cost games, all methods, except the SCRB-method, coincide and their cost allocation turns out to be the centre of gravity of the core of the involved cost game.

1. INTRODUCTION

In this paper the joint cost allocation problem is studied by making use of concepts from the theory of cooperative games. Therefore, the joint cost allocation problem is modelled as a cooperative cost game, which is done in section 2. In the other sections, different methods for apportioning the joint costs of projects in a "fair" manner among the participants are considered and compared with the aid of game theoretical notions.

In the recent literature on the cost allocation problem, the most widely used concepts from cooperative game theory are

- the Shapley value [*Littlechild and Owen*, 1973; *Loehman et al.*, 1979; *Young*, 1982]
- the nucleolus [*Littlechild*, 1974; *Littlechild and Owen*, 1976; *Suzuki and Nakayama*, 1976; *Legros*, 1982]
- variants of the nucleolus [*Young et al.*, 1982]
- the core and variants of the core [*Shapley and Shubik*, 1963, 1966; *Young et al.*, 1982; *Heaney and Dickinson*, 1982].

The purpose of this paper is to emphasize a new cost allocation method, the so-called nonseparable cost gap (NSCG) method, which is derived from the τ-value, a game theoretical concept introduced by *Tijs* [1981]. The NSCG-method, which is defined in section 3, will be compared with several allocation methods which are most commonly considered in water resource projects, namely those allocation methods which are based on separable and nonseparable costs. Chief among these methods is the so-called separable costs, remaining benefits (SCRB) method, which is used for allocating the costs of multipurpose water reservoir projects in the United States [*Inter-Agency Committee on Water Resources*, 1958] and which is still the most widely used method in current multipurpose water development projects. In section 5 it is shown that the cost allocations by the NSCG- and SCRB-method coincide whenever the involved cost allocation problem gives rise to a (semi) convex cooperative cost game.

Heaney and Dickinson [1982] proposed the so-called minimum costs, remaining savings (MCRS) method, which can be viewed as a generalization of the SCRB-method. Both methods can be described with the aid of lower and upper bounds for the core of the involved cost game, but for the MCRS-method those bounds are as sharp as possible. In section 4, it is shown that also the NSCG-method is based on lower and upper bounds for the core, which are in general sharper than in the SCRB-method, but not as sharp as in the MCRS-method. The bounds for the core in the NSCG- and SCRB-method are given by simple formulas, while those in the MCRS-method are obtained by solving several linear programs. However, for convex cost games the three methods make use of the same bounds of the core and hence their outcomes for cost allocation problems which give rise to convex cost games are the same, but the single outcome does not necessarily belong to the core whenever there are more than four participants.

According to the three above-mentioned allocation methods, the non-separable cost is allocated among the participants in proportion to one or another criterion. The egalitarian nonseparable cost (ENSC) method is based on the rather naive criterion that the nonseparable cost should be proportioned equally. In section 5 the class of 1-convex cost games is introduced, for which the ENSC-method is preferred to any other cost allocation method since for those 1-convex cost games the cost allocation by the ENSC-method is in the centre of gravity of the core of the cost game. Furthermore, for those 1-convex cost games it turns out that the ENSC-method coincides with our NSCG- and the MCRS-method, but not with the SCRB-method.

2. THE GAME THEORETICAL APPROACH TO THE COST ALLOCATION PROBLEM

If persons, cities, firms, etc. have decided to undertake a joint project, then there arises the problem of apportioning the total project costs among the project participants in a fair manner. An analysis of this joint cost allocation problem can be carried out in a game theoretical context since the joint cost allocation problem can be modelled as a cooperative cost game by taking into account the strategic aspects of the problem.

A cooperative N-person cost game (in characteristic function form) consists of a finite set N of players along with a characteristic cost function c. Here the cost function c assigns to any subset S of players the real number c(S), which represents the least costs of a project, simply and solely undertaken by the members of S in order to fulfil their own purposes. In particular, $c(\emptyset) = 0$ where $\emptyset$ is the empty set. The cost function c so defined must be subadditive, i.e.

$$c(S) + c(T) \geq c(S \cup T) \quad \text{for all } S,T \subset N \text{ with } S \cap T = \emptyset \qquad (1)$$

since the ways of serving the purposes of S together with T, which does not overlap S, include the possibility of serving S alone and T alone.

Nonempty subsets S of the players set N are called coalitions. It is usual to index the players by the numbers 1,2,... and n if there are n players. Hence, in the remainder of the paper it is assumed that $N = \{1,2,\ldots,n\}$ where $n \in \mathbb{N}$, $n \geq 2$. Furthermore, the notation $c(1), c(12), c(234), \ldots$ is used instead of $c(\{1\}), c(\{1,2\}), c(\{2,3,4\}), \ldots$.

If the potential players in N decide to undertake a joint project, then the cost allocation problem consists of allocating the joint costs c(N) among the players in a fair manner. The cost allocated to player $i \in N$ will be denoted by y_i. Because we require that the principle of efficiency has to be met, a cost allocation y is defined to be a vector

= $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} y_j = c(N) \quad \text{and} \quad y_j \geq 0 \text{ for all } j \in N. \tag{2}$$

ıe problem is to choose a unique cost allocation in a fair manner. : is also reasonable to require that the principle of individual ıtionality has to be met, which states that the cost allocated to any .ayer is less than or equal to the cost of acting independently, i.e.

$$y_j \leq c(\{j\}) \quad \text{for all } j \in N \tag{3}$$

ɔte that there exist always individually rational cost allocations for cost game $(N;c)$ since $c(N) \leq \sum_{i=1}^{n} c(\{i\})$ by the subadditivity condition l) for c.

third principle which we shall consider is the principle of group ationality, which states that the total cost allocated to the members f a coalition S is not more than the alternate cost of S in the cost ame, i.e.

$$\sum_{j \in S} y_j \leq c(S) \quad \text{for all } \emptyset \neq S \subset N. \tag{4}$$

hose cost allocations y that satisfy (4) are called stable in the cost ame $(N;c)$. The core CORE(c) of a cost game $(N;c)$ is defined to be the et of all stable cost allocations in the cost game and hence, the core epresents those cost allocations that cannot be improved upon by any oalition. However, for some cost games the core may be empty. Note that ny stable cost allocation is individually rational since a coalition ay consist of a single player.

The principles of individual and group rationality were already equired by *Ransmeier* [1942; page 220] in his first "preliminary riterion of a satisfactory allocation" in his presentation of the cost .llocation problem concerning the Tennessee Valley Authority (TVA) ıroject during the 1930's. So, the work of *Ransmeier* [1942] foreshadowed :he idea of the core of a cooperative game, which was introduced and named explicitly in game theory by *Gillies* [1953]. *Ransmeier* dit not notice that the core of a cost game may be empty. This might be due to the fact that the TVA cost games were always convex, i.e.

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T) \quad \text{for all } S,T \subset N. \tag{5}$$

Convex cost games possess many nice properties, e.g. the core has a very regular structure [*Shapley*, 1971] and in fact is always large [*Sharkey*, 1982]. Convex cost games and some related versions (the so-called semi-convex and 1-convex cost games) are treated extensively in section 5. In their paper, *Straffin* and *Heaney* [1981] drew attention to the fact that independently developed ideas by the TVA during the 1930's are related to certain game theoretical concepts, such as the core and the nucleolus [*Schmeidler*, 1969]. In the next sections we shall show that some of those ideas by the TVA are also related to another game theoretical concept, the τ-value [*Tijs*, 1981].

3. COST ALLOCATION METHODS BASED ON SEPARABLE AND NONSEPARABLE COSTS: THE ENSC-, SCRB- and NSCG-METHOD

Another principle, which is often required in the literature on the evaluation of water resource projects, states that the cost allocated to any player is not less than the marginal cost of including him in the project [*Federal Inter-Agency River Basin Committee*, 1950; *Inter-Agency Committee on Water Resources*, 1958; *Water Resources Council*, 1962]. As such, we define the separable cost of player i in a cost game (N;c) by

$$SC_i(c) := c(N) - c(N-\{i\}) \tag{6}$$

Given that any player has been allocated his separable cost, there remains the problem of how to allocate the remaining costs in the game. Those remaining costs are called the nonseparable cost and are given by

$$NSC(c) := c(N) - \sum_{j=1}^{n} SC_j(c) \tag{7}$$

In general, the allocation of the nonseparable cost among the players can be based on the ratio of suitable chosen real numbers $\beta_i(c)$, $i = 1,2,\ldots,n$, which may depend on the involved cost game.

The easiest way is to choose $\beta_i(c) = 1$ for all $i \in N$, so independently of the cost game. Then the nonseparable cost is proportioned equally and hence, this method will be called the egalitarian nonseparable cost (ENSC) method. Thus the cost allocated to player i by this method is given by

$$ENSC_i(c) = SC_i(c) + n^{-1} NSC(c). \qquad (8)$$

A major problem with this allocation method is that it may even fail to meet the individual rationality principle.

In the first stage (1933-1934) of the research into the TVA cost allocation problem, regular staff members of TVA favoured the ENSC-method but the TVA Board refrained from adopting this method since this method fails to consider the justifiability of the charges it assesses. However, in section 5 we shall describe a class of cost games, for which the ENSC-method is preferred to any other cost allocation method.

In the current literature on water resource projects, the separable costs remaining benefits (SCRB) method is emphasized which is obtained by choosing $\beta_i(c) = \min[b_i(c), c(\{i\})] - SC_i(c)$ for all $i \in N$ where $b_i(c)$ represents the benefit to player i in the game (N;c) by acting independently. By the SCRB-method, the nonseparable cost is allocated in proportion to each player's willingness to pay minus the separable cost already allocated. Here, player i is not willing to pay more than his benefit $b_i(c)$ or his alternate cost $c(\{i\})$ in order to participate in the joint project. If the benefit of any player exceeds his alternate cost, then $\beta_i(c) = c(\{i\}) - SC_i(c)$ for all $i \in N$. The figure $c(\{i\}) - SC_i(c)$, which is nonnegative by subadditivity of c, represents the alternate cost avoided by including player i in the joint project and hence, the corresponding cost allocation method is known as the alternate cost avoided (ACA) method. The ACA-method was first proposed by a TVA consultant in 1938. Since the benefits usually exceed the alternate costs, the cost allocated to player i by the SCRB-method is usually given by

$$SCRB_i(c) = SC_i(c) + [c(\{i\})-SC_i(c)][\sum_{j=1}^{n} (c(\{j\})-SC_j(c))]^{-1} NSC(c). \quad (9)$$

In case $NSC(c) \geq 0$, the subadditivity of c and (9) imply that

$$SC_i(c) \leq SCRB_i(c) \leq c(\{i\}) \quad \text{for all } i \in N.$$

So, (3) is satisfied and hence, the SCRB-method is then individually rational, but in general not stable. Other criteria for fairness of the SCRB-method are examined by *Loughlin* [1977].

Given that any player has been allocated his separable cost, the allocation of the remaining nonseparable cost by the SCRB-method is mainly based on the remaining alternate costs of the one-person coalitions. However, there is no reason why the remaining alternate costs of other coalitions should not be taken into account in the allocation of the nonseparable cost. Hence, for any cost game (N;c) we introduce its cost gap function g^c which assigns to any coalition S the remaining alternate cost of S given by

$$g^c(S) := c(S) - \sum_{j \in S} SC_j(c) \quad (10)$$

The figure $g^c(S)$ is called the cost gap of coalition S in the game (N;c). Note that the cost gap of the grand coalition is equal to the nonseparable cost, i.e. by (7) and (10) we have

$$g^c(N) = NSC(c).$$

Further, we let $g^c(\emptyset) = 0$. In general we shall look only at cost games for which the cost gap function is nonnegative, so we assume that

$$g^c(S) \geq 0 \quad \text{for all } S \subset N. \quad (11)$$

Now our purpose is to describe the so-called nonseparable cost gap (NSCG) method, which is derived from the game theoretical concept of the τ-value [*Tijs*, 1981]. Consider a cost game (N;c) with a nonnegative cost gap function g^c and a player $i \in N$. Let T be a coalition to which player i belongs. Then player i will reject any cost allocation, which charges to him an amount that is more than the figure $SC_i(c) + g^c(T)$. The motive of player i for this rejection is as follows. Player i can threaten to try to form the coalition T and to allocate the alternate cost c(T) among its members in such a way that all members of T, except i, are charged only their separable costs, while player i himself is charged the remaining cost which equals

$$c(T) - \sum_{j \in T-\{i\}} SC_j(c) \quad \text{or equivalently} \quad SC_i(c) + g^c(T).$$

This motive of player i applies to any coalition T which contains player i and hence, player i is not willing to pay more than the amount

$$\min_{T; i \in T} [SC_i(c)+g^c(T)] \text{ or equivalently } SC_i(c) + \min_{T; i \in T} g^c(T).$$

In view of the above reasoning, we define the concession amount of player i in a cost game (N;c) by

$$\lambda_i(c) := \min_{T; i \in T} g^c(T). \tag{12}$$

The concession amount $\lambda_i(c)$ of player i is seen as his maximal contribution to the nonseparable cost NSC(c). Now we shall also assume that the total of these maximal contributions is at least the non-separable cost, so

$$\sum_{j=1}^{n} \lambda_j(c) \geq NSC(c). \tag{13}$$

The nonseparable cost gap method is obtained whenever the nonseparable cost is allocated among the players in proportion to their concession amounts. Hence, if the cost game (N;c) satisfies (11) and (13), then the cost allocated to player i by the NSCG-method is given by

$$NSCG_i(c) = SC_i(c) \qquad \text{if } NSC(c) = 0$$
$$= SC_i(c) + \lambda_i(c)[\sum_{j=1}^{n}\lambda_j(c)]^{-1}NSC(c) \quad \text{if } NSC(c) > 0. \qquad (14)$$

Let's treat two examples to compare the three above-mentioned cost allocation methods.

First of all, consider the example given by *Young et al.* [1982, page 464], where three neighbouring municipalities 1, 2 and 3 supply themselves with municipal water by building a joint water supply facility. The corresponding cost game $(N;c)$ is given by $N = \{1,2,3\}$ and

$c(1) = 6.5$	$c(12) = 10.3$	$c(123) = 10.6$
$c(2) = 4.2$	$c(13) = 8.0$	
$c(3) = 1.5$	$c(23) = 5.3$	

where the costs are in dollars $\times 10^6$. Since (5) is satisfied, the cost game is convex. Table 1 compares the cost allocations by the several methods. Note that $NSC(c) = 10.6 - 8.2 = 2.4$. Observe that the cost allocations by the SCRB- and NSCG-method coincide. As will be shown in section 5, this is due to the convexity of c. Further, the convexity of c implies [*Shapley*, 1971] that the core of the cost game is the convex hull of the cost allocations (6.5,3.8,0.3), (6.1,4.2,0.3), (6.5,2.6,1.5), (5.3,4.2,1.1) and (5.3,3.8,1.5). The core is drawn in figure 1, using triangular coordinate paper.

[About here table 1 and figure 1]

Secondly, consider the example given by *Young* [1982, page 2], where $N = \{1,2,3\}$ and

$c(1) = 15$	$c(12) = 35$	$c(123) = 78$
$c(2) = 20$	$c(13) = 61$	
$c(3) = 55$	$c(23) = 65$	

This cost game is not convex since $c(13) + c(23) < c(123) + c(3)$.

Table 2 shows the cost allocations by the several methods. Note that $NSC(c) = 78 - 73 = 5$. The core of the cost game turns out to be the convex hull of the cost allocations

(13,17,48), (13,20,45), (15,17,46) and (15,20,43).

Hence, the cost allocation by the NSCG-method is the centre of gravity of the core. The core is drawn in figure 2.

[About here table 2 and figure 2]

4. COST ALLOCATION METHODS BASED ON BOUNDS OF THE CORE

In this section it is shown that both the SCRB- and the NSCG-method can be described with the aid of lower and upper bounds for the core. This result gives rise to a comparison of both methods with the MCRS-method as proposed by *Heaney and Dickinson* [1982].

First of all, it is asserted that the separable costs can be seen as a lower bound for any stable cost allocation, i.e.

$$SC_i(c) \leq y_i \quad \text{for all } i \in N \quad \text{whenever } y \in CORE(c). \tag{15}$$

This result is a direct consequence of (6), (2) and (4), applied to $S = N-\{i\}$. [*Tijs and Lipperts,* 1982] We say that the figure $SC_i(c)$ is a sharp lower bound for the core if there exists a stable cost allocation y with $y_i = SC_i(c)$. In both examples of the previous section, the separable costs are sharp lower bounds for the core.

Further, in view of (4) applied to the one-person coalitions, the alternate single costs can be seen as a upper bound for any stable cost allocation, i.e.

$$y_i \leq c(\{i\}) \quad \text{for all } i \in N \quad \text{whenever } y \in CORE(c). \tag{16}$$

The figure $c(\{i\})$ is said to be a sharp upper bound for the core if there exists a stable cost allocation y with $y_i = c(\{i\})$. In the first example of the previous section, the alternate single costs are sharp upper bounds for the core, but in the second example the alternate single cost $c(\{3\})$ is not sharp since $y_3 \le 48 < 55 = c(\{3\})$ for any stable cost allocation $y = (y_1, y_2, y_3)$. However, in this second example the figures $SC_i(c) + \lambda_i(c)$ are sharp upper bounds for the core. These figures turn out to be upper bounds for the core in general. So,

$$y_i \le SC_i(c) + \lambda_i(c) \quad \text{for all } i \in N \quad \text{whenever } y \in CORE(c). \qquad (17)$$

A simple proof of this result is given in the appendix.

In view of (9), (14)-(17), a geometrical characterization of both the NSCG- and the SCRB-method for cost games with a nonempty core can now be stated as follows. The cost allocation for a cost game with a nonempty core by the NSCG-method (respectively SCRB-method) is equal to that unique cost allocation, that lies on the straight line segment with end points the lower bound of the core determined by the separable costs and the upper bound of the core determined by the figures $SC_i(c) + \lambda_i(c)$, $i = 1,2,\dots,n$ (respectively the alternate single costs $c(\{i\})$, $i = 1,2,\dots,n$). It follows that the cost allocations by both methods coincide whenever $\lambda_i(c) = c(\{i\}) - SC_i(c) = g^c(\{i\})$ for all $i \in N$. These cost games will be studied in the next section.

Heaney and Dickinson [1982] propose the so-called minimum costs, remaining savings (MCRS) method, which is based on lower and upper bounds for the core that are as sharp as possible. For games $(N;c)$ with a nonempty core those sharp lower and upper bounds can be found by solving for any $i \in N$ the following linear program:

$$\text{minimize or maximize } y_i \qquad (18)$$

subject to y satisfying (2) and (4).

Note that the constraint set of any of these 2n linear programs is identical. Let the solutions be given by y_i^{min} and y_i^{max}, $i = 1,\dots,n$.

Notice that these solutions will depend on the involved cost game. Then the cost allocated to player i by the MCRS-method is given by

$$MCRS_i(c) = y_i^{min} + u_i(c)[c(N) - \sum_{j=1}^{n} y_j^{min}]$$

where $$u_i(c) = [y_i^{max} - y_i^{min}][\sum_{j=1}^{n} (y_j^{max} - y_j^{min})]^{-1}.$$

Let's compare the several cost allocation methods by means of the next example.

Consider the 4-person cost game (N;c) with

c(1) = 7	c(12) = 11	c(23) = 11	c(123) = 15	c(1234) = 20
c(2) = 7	c(13) = 10	c(24) = 11	c(124) = 16	
c(3) = 7	c(14) = 11	c(34) = 11	c(134) = 17	
c(4) = 9			c(234) = 18	

The results are collected in table 3. Note that NSC(c) = 20 - 14 = 6 and that the several cost allocations are different, while only the cost allocation by the MCRS-method is stable. In general it is not guaranteed that the cost allocation by the MCRS-method (or any other method) is stable although the bounds of the core are as sharp as possible. This point will be considered in the next section.

[About here table 3]

5. CONVEX, SEMICONVEX AND 1-CONVEX COST GAMES

In the previous section it was already noted that for cost games with a nonempty core, the cost allocations by the NSCG- and SCRB-method coincide whenever $\lambda_i(c) = g^c(\{i\})$ for all $i \in N$. (19)

In view of (12), this condition is equivalent to

$g^c(\{i\}) \le g^c(T)$ for all $i \in N$ and $T \subset N$ with $i \in T$. (20)

A cost game (N;c) which satisfies (20) is called semiconvex. Semiconvex games were introduced in *Driessen and Tijs* [1982]. The subadditivity implies that a semiconvex cost game satisfies also (11) and (13), so its cost allocation by the NSCG-method is well-defined. As a matter of fact, for semiconvex cost games the cost allocations by the NSCG- and SCRB-method coincide. This is easily seen from (14) and (9) since $\lambda_j(c) = g^c(\{j\}) = c(\{j\}) - SC_j(c)$ for all $j \in N$ by using the semiconvexity condition (19).

Note that the convex cost game of the first example in section 3 is semiconvex because it follows from table 1 that (19) is satisfied. In general, any convex cost game turns out to be semiconvex, which will be shown in the appendix. Further, for convex cost games the separable costs (respectively the alternate single costs) are sharp lower (upper) bounds for the core [*Shapley*, 1971], which implies that the solutions y_i^{min} and y_i^{max}, $i = 1,2,\ldots,n$, of the linear program (18) are given by

$$y_i^{min} = SC_i(c) \quad \text{and} \quad y_i^{max} = c(\{i\}).$$

Therefore, the cost allocations for any convex cost game by the NSCG-, SCRB- and MCRS-method coincide. It is guaranteed that this single cost allocation is stable whenever there are not more than four players, but if there are at least five players, then this cost allocation may fall outside the core of the convex cost game [*Driessen and Tijs*, 1982].

The semiconvexity condition (20) rewritten in terms of the cost game itself is given by

$$c(\{i\}) + \sum_{j \in T-\{i\}} SC_j(c) \le c(T) \quad \text{whenever } i \in T \subset N. \tag{21}$$

An interpretation of (21) can be given as follows. The alternate cost of any coalition exceeds the sum of the minimal charges (i.e. separable costs) to all its members, except one whose minimal charge is replaced by his maximal charge (i.e. his alternate single cost).

The definition of the concession amount in (12) gives also rise to consider the cost games, whose gap functions are minimal for the grand coalition, i.e.

$$g^{c}(N) \leq g^{c}(T) \quad \text{for all} \quad \emptyset \neq T \subset N. \tag{22}$$

A cost game (N;c) which satisfies (22) and $g^{c}(N) \geq 0$ is called 1-convex. 1-Convex games are introduced in *Driessen and Tijs* [1983] and are treated extensively in *Driessen* [1984]. The 1-convexity condition can also be rewritten in terms of the cost game itself as follows:

$$NSC(c) \geq 0 \quad \text{and} \quad c(N) \leq c(T) + \sum_{j\in N-T} SC_j(c) \quad \text{for all} \quad \emptyset \neq T \subset N.$$

By subadditivity we have

$$c(N) \leq c(T) + \sum_{j\in N-T} c(\{j\}) \text{ for all } \quad \emptyset \neq T \subset N \tag{23}$$

and hence, for 1-convex cost games the inequalities (23) hold even when the maximal charges (i.e. alternate single costs) are replaced by the minimal charges (i.e. separable costs).

Note that the 1-convexity implies that the concession amount is the same for any player, namely $\lambda_j(c) = g^{c}(N)$ for all $j \in N$. Hence, 1-convex cost games satisfy (11) and (13), while the NSCG-method allocates the non-separable cost equally. So, for 1-convex cost games the cost allocations by the NSCG- and ENSC-method coincide.

Let's consider an example of an 1-convex cost game (N;c) given by $N = \{1,2,3\}$ and

$c(1) = 9 \qquad c(12) = 11 \qquad c(123) = 15$

$c(2) = 9 \qquad c(13) = 12$

$c(3) = 10 \qquad c(23) = 11$

The results are collected in table 4. Note that $NSC(c) = 15 - 11 = 4$.

[About here table 4 and figure 3]

The core of the cost game, which is drawn in figure 3, turns out to be the convex hull of the cost allocations

(8,3,4) , (4,7,4) and (4,3,8)

and hence, the cost allocations by the NSCG-, ENSC- and MCRS-method coincide with the centre of gravity of the core. This last observatio holds always for 1-convex cost games. The cornerstone of the theory o 1-convex cost games is the next statement [*Driessen*, 1984]:

a cost game (N;c) is 1-convex if and only if its core is the con hull of the cost allocations

$$(SC_1(c)+NSC(c), \quad SC_2(c), \quad \ldots, \quad SC_{n-1}(c), \quad SC_n(c)),$$
$$(SC_1(c), \quad SC_2(c)+NSC(c), \quad \ldots, \quad SC_{n-1}(c), \quad SC_n(c)),$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

and $(SC_1(c), \quad SC_2(c), \quad \ldots, \quad SC_{n-1}(c), \quad SC_n(c)+NSC(c))$.

This statement implies that for 1-convex cost games the solutions y_i^{mi} and y_i^{max}, $i = 1,2,\ldots,n$, of the linear program (18) are given by

$$y_i^{min} = SC_i(c) \quad \text{and} \quad y_i^{max} = SC_i(c) + NSC(c).$$

Hence, for 1-convex cost games the MCRS-method coincides with the NSC and ENSC-methods and in view of the above statement, the single cost allocation by those methods is the centre of gravity of the core of t 1-convex cost game. Finally, we remark that for 1-convex cost games t nucleolus is also the centre of gravity of the core [*Driessen and Tij* 1983].

6. SUMMARY AND CONCLUSIONS

A new method is introduced for apportioning the joint costs of w resource projects among its participants on the basis of separable an nonseparable costs. This new method, the so-called nonseparable cost (NSCG) method, allocates the nonseparable cost in proportion to the concession amounts of the participants. The concession amount of any participant is based on the minimum of the gaps of all those coalitio which include the involved participant. If the one-person coalitions determine those minimal gaps, then the NSCG-method coincides with the well-known SCRB-method.

The separable costs and the concession amounts are closely related to lower and upper bounds for the core of the involved cost game. Hence, the NSCG-method is compared with the MCRS-method, which is based on sharp lower and upper bounds for the core. It is shown that for convex cost games, which are of particular interest, the NSCG-, SCRB- and MCRS-method coincide since they make use of the same bounds for the core of the convex cost game. However, their cost allocation does not necessarily belong to the core of the convex cost game, although the bounds of the core are as sharp as possible. For these convex cost games the egalitarian nonseparable cost (ENSC) method, which allocates the nonseparable cost equally among the participants, is of less importance.

From the game theoretical viewpoint, 1-convex cost games deserve the same careful consideration as the convex cost games. For cost allocation problems which give rise to 1-convex cost games, the NSCG-, ENSC-, and MCRS-method coincide and their cost allocation is exactly the centre of gravity of the core of the 1-convex cost game. For these 1-convex cost games the SCRB-method is of less importance.

APPENDIX

In this section we give the proof of statement (17) and further we show that convex cost games are semiconvex.

First of all, in order to prove (17), consider a stable cost allocation y in a cost game $(N;c)$ and a player i. By the definition of the concession amount, there exists a coalition S such that $i \in S$ and

$\lambda_i(c) = g^c(S)$. Since y is stable, we have $\sum_{j\in S} y_j \le c(S)$ and also $\sum_{j\in S-\{i\}} SC_j(c) \le \sum_{j\in S-\{i\}} y_j$ where the last inequality follows from (15). Now we can conclude that

$$y_i \le \sum_{j\in S} y_j - \sum_{j\in S-\{i\}} SC_j(c) \le c(S) - \sum_{j\in S-\{i\}} SC_j(c) = g^c(S) + SC_i(c) =$$

$= \lambda_i(c) + SC_i(c)$ which completes the proof of statement (17).

Secondly, we prove that for convex cost games (N;c) the corresponding gap function is monotonic, i.e.

$$g^c(S-\{i\}) \le g^c(S) \quad \text{whenever} \quad i \in S \subset N. \tag{24}$$

Let (N;c) be a convex cost game and consider a player i who does belong to the coalition S. The convexity condition (5) implies

$c(S) + c(N-\{i\}) \ge c(N) + c(S-\{i\})$ or equivalently

$c(S) - c(S-\{i\}) \ge SC_i(c)$. Now it follows that

$$g^c(S-\{i\}) = c(S-\{i\}) - \sum_{j\in S-\{i\}} SC_j(c) \le c(S) - \sum_{j\in S} SC_j(c) = g^c(S),$$

which proves (24). But it is obvious that (24) implies $g^c(\{i\}) \le g^c(T)$ whenever $i \in T$. Hence, any convex cost game is semiconvex.

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TABLE 1.

Municipality i	1	2	3	Total
Separable cost $SC_i(c)$	5.3	2.6	0.3	8.2
$c(\{i\}) - SC_i(c) = g^c(\{i\})$	1.2	1.6	1.2	4.0
concession amount $\lambda_i(c)$	1.2	1.6	1.2	4.0
$ENSC_i(c)$	6.1	3.4	1.1	10.6
$SCRB_i(c) = NSCG_i(c)$	6.02	3.56	1.02	10.6

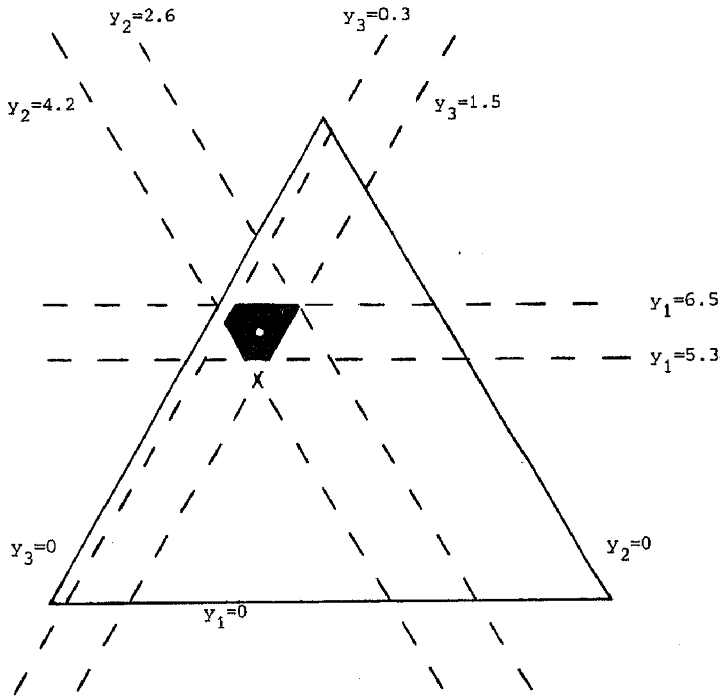

Figure 1. The shaded pentagon represents the core and the indicated point inside it is equal to the cost allocation by the NSCG-method.

TABLE 2.

Player i	1	2	3	Total
Separable cost $SC_i(c)$	13	17	43	73
$c(\{i\}) - SC_i(c) = g^c(\{i\})$	2	3	12	17
concession amount $\lambda_i(c)$	2	3	5	10
$ENSC_i(c)$	$14\frac{2}{3}$	$18\frac{2}{3}$	$44\frac{2}{3}$	78
$SCRB_i(c)$	$13\frac{10}{17}$	$17\frac{15}{17}$	$46\frac{9}{17}$	78
$NSCG_i(c)$	14	$18\frac{1}{2}$	$45\frac{1}{2}$	78

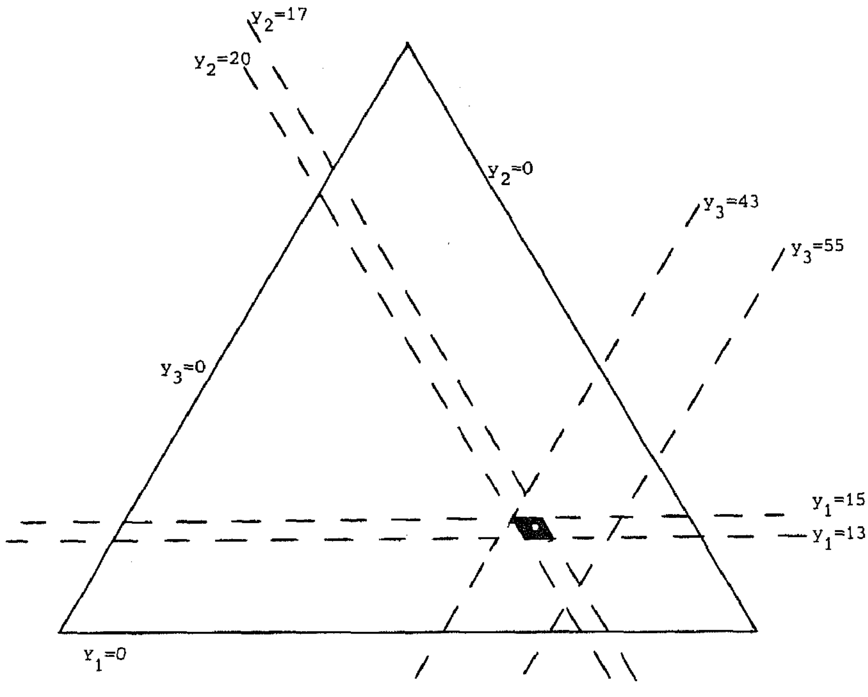

Figure 2. The shaded quadrilateral represents the core and the indicated point inside it is equal to the cost allocation by the NSCG-method.

TABLE 3. Comparative results for the 4-person cost game

Player i	1	2	3	4	Total
Separable cost $SC_i(c)$	2	3	4	5	14
$c(\{i\}) - SC_i(c) = g^c(\{i\})$	5	4	3	4	16
concession amount $\lambda_i(c)$	4	3	2	2	11
$SC_i(c) + \lambda_i(c)$	6	6	6	7	—
y_i^{max}	6	6	6	$6\frac{1}{2}$	$24\frac{1}{2}$
y_i^{min}	$3\frac{1}{2}$	4	4	5	$16\frac{1}{2}$
$ENSC_i(c)$	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$	$6\frac{1}{2}$	20
$SCRB_i(c)$	$3\frac{7}{8}$	$4\frac{1}{2}$	$5\frac{1}{8}$	$6\frac{1}{2}$	20
$NSCG_i(c)$	$4\frac{2}{11}$	$4\frac{7}{11}$	$5\frac{1}{11}$	$6\frac{1}{11}$	20
$MCRS_i(c)$	$4\frac{19}{32}$	$4\frac{7}{8}$	$4\frac{7}{8}$	$5\frac{21}{32}$	20

TABLE 4.

Player i	1	2	3	Total
Separable cost $SC_i(c)$	4	3	4	11
$c(\{i\}) - SC_i(c) = g^c(\{i\})$	5	6	6	17
concession amount $\lambda_i(c)$	4	4	4	12
y_i^{max}	8	7	8	23
y_i^{min}	4	3	4	11
$ENSC_i(c) = NSCG_i(c)$	$5\frac{1}{3}$	$4\frac{1}{3}$	$5\frac{1}{3}$	15
$SCRB_i(c)$	$5\frac{3}{17}$	$4\frac{7}{17}$	$5\frac{7}{17}$	15
$MCRS_i(c)$	$5\frac{1}{3}$	$4\frac{1}{3}$	$5\frac{1}{3}$	15

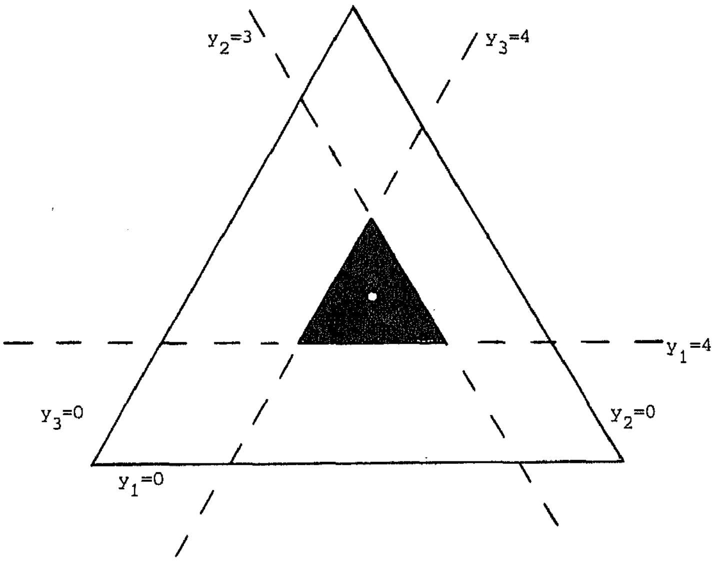

Figure 3. The shaded triangle represents the core and the indicated point inside it is equal to the cost allocation by the NSCG-, ENSC- and MCRS-method.