# The t-value as a feasible compromise between utopia and disagreement

Tijs, S.H.; Driessen, T.S.H.

*Published in:*
Axiomatics and pragmatics of conflict analysis

*Publication date:*
1987

# THE $\tau$-VALUE AS A FEASIBLE COMPROMISE BETWEEN UTOPIA AND DISAGREEMENT

by

S.H. TIJS AND T.S.H. DRIESSEN

*Report 8312*
*March 1983*

DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands

The $\tau$-value as a feasible compromise
between utopia and disagreement

S.H. Tijs and T.S.H. Driessen

presented at

the International Symposium on Research on
Social Conflict and Harmony

Erasmus University Rotterdam,

January 17-18, 1983

# The τ-value as a feasible compromise between utopia and disagreement

S.H. Tijs and T.S.H. Driessen

Department of Mathematics
Catholic University
Nijmegen, The Netherlands

Abstract:

First a description is given of some economic and political situations which give rise to cooperative n-person games. For such a game, the earnings of the grand coalition have to be divided among the players. A new division scheme, the τ-value, is discussed. In the definition of the τ-value a utopia vector and a disagreement vector play a role. The disagreement vector is constructed with the aid of the utopia vector and a concession vector, based on the gap function of the game. For the games describing the discussed economic and political situations, the τ-value is calculated, which appears to be in those cases a reasonable feasible pay-off vector. Also some nice properties of the τ-value are given.

## 1. Introduction and examples

Let us begin by describing an economic situation, which gives rise to a cooperative game, a notion, which will play an important role in this paper. We explain this notion after the first example. Then we treat several other economic and political situations which give rise to cooperative games.

Example 1, a production game.

There are three persons, who are the owners of four raw materials $M_1$, $M_2$, $M_3$, $M_4$. When someone owns one unit of each raw material, he is able to produce one unit of a product P, i.e.

$$M_1 + M_2 + M_3 + M_4 \rightarrow P.$$

By producing one unit of product P, the earnings are \$ 100. In the beginning each person owns a certain amount of units of each raw material. Assume these amounts are given by table 1.

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| person 1 | 5 | 5 | 0 | 0 |
| person 2 | 5 | 0 | 10 | 0 |
| person 3 | 0 | 5 | 0 | 10 |

TABLE 1

This situation corresponds to a three-person game v. The player set N consists of the three owners, i.e. $N = \{1,2,3\}$. When there is no cooperation, none of the three persons can produce the product P because none of them owns all raw materials. So, a person on one's own earns nothing. We denote it by $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. But, when the persons 2 and 3 cooperate, they together possess at least five units of each of the raw materials. Hence, they can produce five units of the product P and they will earn \$ 500. We denote it by $v(\{2,3\}) = 500$. Similarly, we also obtain $v(\{1,2\}) = v(\{1,3\}) = 0$ and $v(\{1,2,3\}) = 1000$.

Before introducing the notion of a cooperative n-person game, we first settle some notation. Let n be an integer such that $n \geq 2$. Let N be the set $\{1,2,\ldots,n\}$. The family of all subsets of N is denoted by $2^N$, i.e. $2^N := \{T;\ T \subset N\}$.
Elements of the set N are called <u>players</u> and elements of the set $2^N$ are called <u>coalitions</u>.

<u>Definition 1</u>. A <u>cooperative</u> n-<u>person game</u> (in characteristic function form) is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$.

Thus, to each group S of players, called a coalition, the game v assigns a real number, denoted by v(S). This number v(S) represents the worth of the coalition S in the game v. We assume that the worth of the empty set is zero, i.e. $v(\emptyset) = 0$.

We give some other examples of n-person games.

Example 2, a glove game (cf. Rosenmüller, 1971, page 13).
Now we look at the glove game v for which the player set N is divided in two disjoint non-empty subsets L and R, where the players in L possess a left hand glove, and those in R a right hand glove. The worth of a single glove is zero, while each pair, consisting of a left and a right hand glove, has worth \$ 1. The n-person game v, which corresponds to this situation, is defined by

$$v(S) := \min\{|L \cap S|, |R \cap S|\} \text{ for any } S \subset N$$

i.e. for any coalition S we first determine the number of players in S, who possess a left (respectively right) hand glove and secondly the number of pairs of gloves that can be formed within that coalition S.

Example 3, a market game (cf. von Neumann and Morgenstern, 1944, page 564).
Consider the three-person market with one seller of a commodity and two potential buyers. The value of the commodity is \$ 100 for the seller, \$ 200 for the weak buyer and \$ 300 for the strong buyer. This market corresponds to the 3-person game v defined by
$v(\{1\}) = 100$, $v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0$, $v(\{1,2\}) = 200$ and
$v(\{1,3\}) = v(\{1,2,3\}) = 300$,
where player 1 is the seller, player 2 (player 3) is the weak (strong) buyer.

So, the worth of a coalition S is zero if player 1 is not a member of S. Otherwise the worth equals the maximum of the values, which the players in S assign to the commodity.

Example 4, a flow game (cf. Kalai and Zemel, 1982).
Consider the directed network of figure 1, in which the arcs are owned by three different persons.

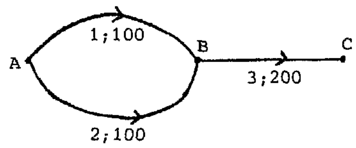


FIGURE 1

Each arc is owned by a certain person (indicated by the first number) and has a certain capacity (indicated by the second number). E.g. the arc from B to C is owned by person 3 and has capacity 200. The transport of one unit of a flow from the source A to the sink C is worth \$ 1. This situation corresponds to the three-person game v defined by

$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 0$, $v(\{1,3\}) = v(\{2,3\}) = 100$ and $v(\{1,2,3\}) = 200$.

The worth of a coalition S is equal to a maximum flow through the network obtained from figure 1 by retaining only those arcs which are owned by the members of S.

Example 5, a voting game.
Consider the Security Council of the United Nations, consisting of fifteen members, from which five members have a veto right. In order to pass a bill, nine votes are needed including those of the five veto players. The 15-person game v, which corresponds to this voting situation, is defined by

$$v(S) = 1 \text{ if } |S| \geq 9 \text{ and } \{1,2,3,4,5\} \subset S$$
$$= 0 \text{ otherwise,}$$

where $\{1,2,3,4,5\}$ is the set of the five veto players. The number $v(S)$ represents the power of the coalition $S$ to pass a bill: 1 (0) corresponds to a powerful (powerless) coalition.

Example 6, a game with a landlord and landless workers (cf. Chetty, Dasgupta and Raghavan, 1976, or Shapley and Shubik, 1967, or Driessen and Tijs, 1982a, example 3.5, respectively 1982b, page 20.)
We consider a situation in which one landlord and n workers are involved. The total gain is denoted by $f(s)$, if s workers are hired by the landlord. This situation corresponds to the $(n+1)$-person game v, defined by

$$v(S) = 0 \quad \text{if } 1 \notin S$$
$$= f(|S|-1) \text{ if } 1 \in S,$$

where player 1 is the landlord. The worth of a coalition S depends on the fact whether or not the landlord is in that coalition and on the number of workers in S.

## 2. Game-theoretical concepts

Let v be a cooperative n-person game. We allow cooperation between the players. Furthermore, we assume that the grand coalition N will be formed and that its worth $v(N)$ will be distributed among all players. The main problem in cooperative game theory is: How to divide the worth $v(N)$ among the players? Many concepts have been introduced, which describe a specific distribution of $v(N)$. We mention some of those solution concepts: the Shapley value (cf. Shapley, 1953), the nucleolus (cf. Schmeidler, 1969), the core and the τ-value (cf. Tijs, 1981 or Driessen and Tijs, 1982a, 1982b). The last two concepts will be studied in this paper. Therefore we introduce another notion.

Definition 2. A vector $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ is called a (feasible) pay-off vector for the n-person game v if $\sum_{i=1}^{n} x_i = v(N)$.

This means that a pay-off vector $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ corresponds to a distribution of the worth v(N) among the players in such a way that player i gets the amount $x_i$. The core of a game v, denoted by C(v), consists of those pay-off vectors for the game v, which can not be improved by any coalition S, i.e.

Definition 3. Let v be an n-person game. Then

$$C(v) := \{x \in \mathbb{R}^n;\ \sum_{i=1}^{n} x_i = v(N) \text{ and } \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in 2^N-\{\emptyset\}\}.$$

Core-elements are stable pay-off vectors for a game, because no coalition can object against such a vector. However, in general the core of a game is not a single point. It may even be the empty set. Our purpose is to introduce for a game v a unique pay-off vector. This will be done in section 3. For that purpose we make use of an upper bound for the core.

Let v be an n-person game and $i \in N$. When the player i joins the coalition N-{i} in order to form the grand coalition N, he contributes a certain amount to the worth of the new coalition N with respect to the worth of the old coalition N-{i}. This contribution is just v(N)-v(N-{i}), and we will call it the marginal contribution of player i in the game v. We denote it by $b_i^v$.

Definition 4. $b_i^v := v(N)-v(N-\{i\})$ for each n-person game v and each $i \in N$.

We call the vector $b^v = (b_1^v, \ldots, b_n^v) \in \mathbb{R}^n$ the <u>utopia vector of the game</u> v. This term is explained by our first theorem, which states that this vector $b^v$ is an upper bound for the core of the game v.

<u>Theorem 1</u>. Let v be an n-person game and $x \in \mathbb{R}^n$ a core-element of the game v. Then $x_j \le b_j^v$ for all $j \in N$.

<u>Proof</u>. Because $x \in C(v)$, we have by definition 3 that $\sum_{i=1}^{n} x_i = v(N)$ and $\sum_{i \in N-\{j\}} x_i \ge v(N-\{j\})$. Thus,

$$x_j = \sum_{i=1}^{n} x_i - \sum_{i \in N-\{j\}} x_i = v(N) - \sum_{i \in N-\{j\}} x_i \le v(N) - v(N-\{j\}) = b_j^v. \qquad \square$$

In general, the vector $b^v$ is not a pay-off vector for the game v. But notice, when the game v has a non-empty core, there is a non-negative gap between the sum $\sum_{i=1}^{n} b_i^v$ and the worth v(N) of the grand coalition according to the following proposition.

<u>Proposition 1</u>. Let v be an n-person game with a non-empty core. Then: $\sum_{i \in S} b_i^v \ge v(S)$ for all $S \in 2^N - \{\emptyset\}$.

<u>Proof</u>. We know $C(v) \ne \emptyset$, so take $x \in C(v)$. Let $S \in 2^N - \{\emptyset\}$. Then $\sum_{j \in S} x_j \ge v(S)$ by definition 3. By theorem 1, $x_j \le b_j^v$ all $j \in S$. Thus, it follows that $\sum_{j \in S} b_j^v \ge \sum_{j \in S} x_j \ge v(S)$. $\square$

We are now interested in those gaps.

<u>Definition 5</u>. Let v be an n-person game and $S \in 2^N - \{\emptyset\}$. We call the expression $\sum_{i \in S} b_i^v - v(S)$ the <u>gap of the coalition</u> S <u>in the game</u> v and denote it by $g^v(S)$.

For the remainder of the paper we look mainly at games with a non-negative gap function, i.e. games v such that $g^v(S) \ge 0$ for all $S \in 2^N - \{\emptyset\}$. Then, in particular $g^v(N) \ge 0$, which means $\sum_{i=1}^{n} b_i^v \ge v(N)$. If this is an equality, we choose the utopia vector

$b^v$ as the pay-off vector for the game v. In case $\sum_{i=1}^{n} b_i^v > v(N)$, we can not divide the worth v(N) in such a way that every player i gets his marginal contribution $b_i^v$. In this case we construct a pay-off vector x for the game v such that

$$b_i^v - \lambda_i^v \le x_i \le b_i^v \quad \text{for each } i \in N$$

where the vector $\lambda^v \in \mathbb{R}^n$ is defined by

$$\lambda_i^v := \min_{S;i \in S} g^v(S) \quad \text{for each } i \in N.$$

This means that each player i does not get more than his utopia pay-off $b_i^v$ and that he gets at least $b_i^v - \lambda_i^v$. The explanation for this lower bound is as follows. A player i will object against a pay-off vector y with $y_i < b_i^v - \lambda_i^v$ because he can argue: "For each coalition S, to which I belong, I can promise to each other member j of S his utopia pay-off $b_j^v$, if they form with me the coalition S. Then I will keep the remainder of the worth v(S) for myself, i.e. the amount $v(S) - \sum_{j \in S-\{i\}} b_j^v$. Furthermore, if I take the coalition $\hat{S}$ in such a way that $g^v(\hat{S}) = \min_{S;i \in S} g^v(S) = \lambda_i^v$, then my earnings $v(\hat{S}) - \sum_{j \in \hat{S}-\{i\}} b_j^v$ are equal to $b_i^v - \lambda_i^v$, which is larger than $y_i$". So, a player i considers the amount $\lambda_i^v$ as a maximum concession, which he wants to make. In view of this reasoning, we give the following definition.

<u>Definition 6</u>. Let v be an n-person game. The vector $\lambda^v \in \mathbb{R}^n$, defined by

$$\lambda_i^v := \min_{S;i \in S} g^v(S) \quad \text{for each } i \in N,$$

is called the <u>concession vector of the game</u> v. The vector $b^v - \lambda^v$ is called the <u>disagreement vector of the game</u> v.

We have seen that the utopia vector $b^v$ is an upper bound for core-elements of the game v. In the next section, we prove that the disagreement vector $b^v - \lambda^v$ is a lower bound for core-elements. Also in section 3, we shall define the pay-off vector for the game v with the aid of the utopia vector $b^v$ and the disagreement vector $b^v - \lambda^v$.

Now we apply the theory of this section to the examples of section 1.

Example 1. Applying definition 4 we get $b_1^v = v(N) - v(\{2,3\}) = 1000-500 = 500$, $b_2^v = v(N) - v(\{1,3\}) = 1000$ and $b_3^v = v(N) - v(\{1,2\}) = 1000$, so $b^v = (500,1000,1000)$. With the aid of table 2, we calculate the gaps of the coalitions.

| S | {1} | {2} | {3} | {1,2} | {1,3} | {2,3} | {1,2,3} |
|---|---|---|---|---|---|---|---|
| v(S) | 0 | 0 | 0 | 0 | 0 | 500 | 1000 |
| $\sum_{i \in S} b_i^v$ | 500 | 1000 | 1000 | 1500 | 1500 | 2000 | 2500 |
| $g^v(S)$ | 500 | 1000 | 1000 | 1500 | 1500 | 1500 | 1500 |

TABLE 2

For coalitions, which contain player 1, the minimal gap is achieved by the coalition {1}, so $\lambda_1^v = g^v(\{1\}) = 500$. In this way, we get also $\lambda_2^v = g^v(\{2\}) = 1000$, and $\lambda_3^v = g^v(\{3\}) = 1000$, so $\lambda^v = (500,1000,1000)$. Notice that $\lambda^v = b^v$ in this case.

Example 2. First we suppose that $|L| < |R|$. Then $v(N) = |L|$, $v(N-\{i\}) = |L|-1$ for each $i \in L$ and $v(N-\{i\}) = |L|$ for each $i \in R$. Thus $b_i^v = 1$ for each $i \in L$ and $b_i^v = 0$ for each $i \in R$. It follows that $g^v(S) = \sum_{i \in S} b_i^v - v(S) = |L \cap S| - v(S) \geq 0$, where the inequality follows from the definition of the game v. Notice that

$g^v(N) = |L \cap N| - v(N) = |L| - |L| = 0$, so $g^v(S) \geq g^v(N)$ for all $S \in 2^N - \{\emptyset\}$. This implies that $\lambda_i^v = g^v(N) = 0$ for all $i \in N$.
In case $|L| = |R|$, $b_i^v = 1$ for all $i \in N$. Then $g^v(\{i\}) = b_i^v - v(\{i\}) = 1-0 = 1$ for all $i \in N$. Furthermore $g^v(S) = \sum_{i \in S} b_i^v - v(S) = |S| - v(S) \geq 1$ for all $S \in 2^N - \{\emptyset\}$. It follows that $\lambda_i^v = g^v(\{i\}) = 1$ for all $i \in N$.

Example 3. It is obvious that $b^v = (300,0,100)$. So $g^v(\{1\}) = 200$, $g^v(\{2\}) = 0$, $g^v(\{3\}) = 100$ and $g^v(\{1,2\}) = g^v(\{1,3\}) = g^v(\{2,3\}) = g^v(\{1,2,3\}) = 100$. Thus $\lambda^v = (100,0,100)$.

Example 4. $b^v = (100,100,200)$, so $g^v(\{1\}) = g^v(\{2\}) = 100$ and $g^v(\{3\}) = g^v(\{1,2\}) = g^v(\{1,3\}) = g^v(\{2,3\}) = g^v(\{1,2,3\}) = 200$. Thus $\lambda^v = (100,100,200)$, so in this case $\lambda^v = b^v$.

Example 5. $v(N) = 1$, $v(N-\{i\}) = 1$ if $i \notin \{1,2,3,4,5\}$ and $v(N-\{i\}) = 0$ if $i \in \{1,2,3,4,5\}$. Thus $b_i^v = 1$ if $i$ is a veto player, $b_i^v = 0$ otherwise. It is easy to see that $\lambda^v = b^v$.

Example 6 will be discussed in section 3.

## 3. The τ-value

In section 2 we introduced for any game $v$ the utopia vector $b^v$ and the concession vector $\lambda^v$. The vector $b^v$ appeared to be an upper bound for the core of the game $v$. The disagreement vector $b^v - \lambda^v$ appears to be a lower bound for the core according to our second theorem.

Theorem 2. Let $v$ be an n-person game and $x \in \mathbb{R}^n$ a core-element of the game $v$. Then $x_j \geq b_j^v - \lambda_j^v$ for all $j \in N$.
Proof. Let $x \in C(v)$ and $j \in N$. By definition 6 there exists a

coalition S such that $j \in S$ and $\lambda_j^v = g^v(S)$. Because $x \in C(v)$, we have by definition 3, $\sum_{i\in S} x_i \geq v(S)$ and by theorem 1, $\sum_{i\in S-\{j\}} x_i \leq \sum_{i\in S-\{j\}} b_i^v$. It follows that

$$x_j = \sum_{i\in S} x_i - \sum_{i\in S-\{j\}} x_i \geq v(S) - \sum_{i\in S-\{j\}} b_i^v = v(S) - \sum_{i\in S} b_i^v + b_j^v = -g^v(S) + b_j^v =$$
$$= -\lambda_j^v + b_j^v. \qquad \square$$

Let v be a game with a non-negative gap function. Then for each $j \in N$, $\lambda_j^v \geq 0$ by definition 6 and thus $b_j^v - \lambda_j^v \leq b_j^v$. Shortly written: $b^v - \lambda^v \leq b^v$. Note that $\sum_{i=1}^n b_i^v \geq v(N)$ because $g^v(N) \geq 0$. Whenever $\sum_{i=1}^n (b_i^v - \lambda_i^v) \leq v(N)$, (i.e. $\sum_{i=1}^n \lambda_i^v \geq g^v(N)$), we are able to define the $\tau$-value of the game v as the unique pay-off vector on the line segment $[b^v - \lambda^v, b^v]$ with end points $b^v - \lambda^v$ and $b^v$.

<u>Definition 7</u>. The subclass $Q^n$ of n-person games is defined by

$$Q^n := \{v;\ v \text{ is an n-person game such that } g^v(S) \geq 0 \text{ for all } S \in 2^N - \{\emptyset\} \text{ and } \sum_{i=1}^n \lambda_i^v \geq g^v(N)\}.$$

<u>Definition 8</u>. Let v be an n-person game such that $v \in Q^n$. Then the $\tau$-<u>value</u> $\tau^v$ <u>of the game</u> v is defined by $\tau^v := b^v - \alpha\lambda^v$, where $\alpha \in [0,1]$ is chosen such that $\sum_{i=1}^n \tau_i^v = v(N)$ (i.e. $\tau^v$ is the pay-off vector for the game v, which lies on the line segment with end points the disagreement vector and the utopia vector).

The idea behind the $\tau$-value as the pay-off vector of a game $v \in Q^n$ is as follows: Because $g^v(N) \geq 0$, we have $\sum_{i=1}^n b_i^v \geq v(N)$. So, the utopia vector $b^v$ is in general not a pay-off vector for the game v and therefore, the grand coalition N has to make a concession (with respect to the utopia vector $b^v$), which equals the amount $g^v(N)$. But every player i considersthe amount $\lambda_i^v$ as a maximum concession which he wants to contribute to the concession amount $g^v(N)$ of the grand coalition. Because $\sum_{i=1}^{n} \lambda_i^v \geq g^v(N)$, each player needs only to contribute a fraction of his maximum concession $\lambda_i^v$, and this fraction is the same for every player.

In case $g^v(N) = 0$, then $\lambda_i^v = 0$ for all $i \in N$ and by definition 8, $\tau^v = b^v$. In case $g^v(N) > 0$, then the number $\alpha$ in definition 8 must be chosen equal to $g^v(N) \left( \sum_{i=1}^{n} \lambda_i^v \right)^{-1}$. It appears to be convenient to introduce the <u>concession</u> <u>rate</u> <u>vector</u> $u^v \in \mathbb{R}^n$, defined by

$$u_i^v := \left( \sum_{j=1}^{n} \lambda_j^v \right)^{-1} \lambda_i^v \quad \text{for all } i \in N.$$

Note that $\sum_{i=1}^{n} u_i^v = 1$. Now we can formulate a general formula for the $\tau$-value $\tau^v$ of a game $v \in Q^n$.

<u>Theorem 3</u>. Let $v \in Q^n$.

(i) If $g^v(N) = 0$, then $\tau^v = b^v$.

(ii) If $g^v(N) > 0$, then $\tau^v = b^v - g^v(N) u^v$.

For the games in the examples of section 1 we calculate the $\tau$-value, making use of the calculations of the utopia vector and the concession vector for these games at the end of section 2.

<u>Example 1</u>. $b^v = \lambda^v = (500,1000,1000)$ and $g^v(N) = 1500$. Note that $v \in Q^3$ and $u^v = (\frac{1}{5},\frac{2}{5},\frac{2}{5})$. Using theorem 3 (ii), we get $\tau^v = b^v - g^v(N)u^v =$ $= (200,400,400)$. So according to the $\tau$-value, the \$ 1000 are divided in such a way that the first player obtains \$ 200 and the two other players both \$ 400.

<u>Example 2</u>. In case $|L| < |R|$, $g^v(N) = 0$, so by theorem 3(i) $\tau_j^v = b_j^v = 1$ for each $j \in L$ and $\tau_j^v = b_j^v = 0$ for each $j \in R$. In case $|L| = |R|$, $b^v = \lambda^v = (1,1,\ldots,1)$ and $g^v(N) = |L| = \frac{1}{2}n$, so $\tau^v = b^v - g^v(N)u^v =$ $= (\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$.

<u>Example 3</u>. $b^v = (300,0,100)$, $\lambda^v = (100,0,100)$ and $g^v(N) = 100$. Note

that $v \in Q^3$ and $u^v = (\frac{1}{2},0,\frac{1}{2})$, so $\tau^v = b^v - g^v(N)u^v = (250,0,50)$. According to the $\tau$-value, the strong buyer pays \$ 250 to the seller for the commodity.

Example 4. $b^v = \lambda^v = (100,100,200)$ and $g^v(N) = 200$. Thus $u^v = (\frac{1}{4},\frac{1}{4},\frac{1}{2})$, so $\tau^v = b^v - g^v(N)u^v = (50,50,100)$.

Example 5. $b_i^v = \lambda_i^v = 1$ if i is a veto player, $b_i^v = \lambda_i^v = 0$ otherwise and $g^v(N) = 4$. So $v \in Q^{15}$ and $\tau_i^v = \frac{1}{5}$ if i is a veto player, $\tau_i^v = 0$ otherwise. Thus, the worth v(N) is divided equally among the five veto players if the $\tau$-value is the pay-off vector for the game.

Example 6. The (n+1)-person game v was defined by

$$\begin{aligned} v(S) &= 0 && \text{if } 1 \notin S \\ &= f(|S|-1) && \text{if } 1 \in S. \end{aligned}$$

So $v(N) = f(n)$, $v(N-\{1\}) = 0$ and $v(N-\{j\}) = f(n-1)$ for all $j \in N$, $j \neq 1$. Thus $b_1^v = f(n)$ and $b_j^v = f(n)-f(n-1)$ for all $j \in N$, $j \neq 1$. For the sake of convenience, we write $\Delta$ instead of $f(n)-f(n-1)$. Then $b^v = (f(n),\Delta,\ldots,\Delta)$.

For the gaps we have

(i) if $1 \notin S$, then $g^v(S) = |S|\Delta$.
In particular, $g^v(\{j\}) = \Delta$ for all $j \in N$, $j \neq 1$.

(ii) if $1 \in S$, then $g^v(S) = f(n)+(|S|-1)\Delta-f(|S|-1)$.
In particular, $g^v(N) = n\Delta$ and $g^v(\{1\}) = f(n)-f(0)$.

In the following we assume that $f(s) \geq 0$ for $s \in \{0,1,\ldots,n\}$ (i.e. the total gain is always non-negative) and $f(0) = 0$. In the cases that the function f is convex or concave, we are able to get a nice expression for the $\tau$-value of this game.

Case one. Assume f is concave, i.e. the marginal returns of the function f form a decreasing sequence. That is,

$$0 < f(s+1)-f(s) < f(s)-f(s-1) \text{ for all } s \in \{1,2,\dots,n-1\}.$$

From this, it follows immediately that $g^v(S) = f(n)+(|S|-1)\Delta-f(|S|-1) \geq n\Delta$ whenever $1 \in S$. Hence, $\lambda_j^v = g^v(\{j\}) = \Delta$ for all $j \in N$, $j \neq 1$ and $\lambda_1^v = g^v(N) = n\Delta$. Thus $\lambda^v = (n\Delta,\Delta,\dots,\Delta)$ and we can conclude that $v \in Q^n$. Furthermore, $u^v = (\frac{1}{2},\frac{1}{2n},\dots,\frac{1}{2n})$, so $\tau^v = b^v-g^v(N)u^v =$ $= (f(n)-\frac{1}{2}n\Delta,\frac{1}{2}\Delta,\dots,\frac{1}{2}\Delta)$. This means that half of the marginal contribution $\Delta$ of a worker is for the worker and the other half for the landlord if the $\tau$-value is the pay-off vector for this game.

Case two. Assume f is convex, i.e. the marginal returns of the function f form an increasing sequence. That is,

$$f(s+1)-f(s) > f(s)-f(s-1) \text{ for all } s \in \{1,2,\dots,n-1\}.$$

From this, it follows immediately that

$g^v(S) = f(n)+(|S|-1)\Delta-f(|S|-1) \geq f(n)$ whenever $1 \in S$. Because $f(n) \geq \Delta$, we have $\lambda_i^v = g^v(\{i\})$ for all $i \in N$. Thus, $\lambda^v = (f(n),\Delta,\dots,\Delta)$ and we see that $v \in Q^n$. We get $\tau^v = (n\Delta+f(n))^{-1}f(n)(f(n),\Delta,\dots,\Delta)$. So each worker gets a fraction of his marginal contribution. Notice that this fraction is less than one half because $(f(n)+n\Delta)^{-1}f(n) \leq \frac{1}{2}$ by convexity of f, and that the landlord gets at least as much as a worker, if the $\tau$-value is the pay-off vector.

## 4. Some properties of the $\tau$-value

In this section we list some nice properties of the $\tau$-value.

Property 1. Let $v \in Q^n$. Then

(i) $\tau^v$ is a pay-off vector

(ii) $\tau^v$ is <u>individual rational</u>, i.e.

$\tau_i^v \geq v(\{i\})$ for each $i \in N$.

A player i is called a <u>dummy player</u> in the game v if he contributes nothing to a coalition if he joins that coalition, i.e. $v(S \cup \{i\}) = v(S)$ for all $S \subset N$. The following property states that a dummy player gets no pay-off according to the τ-value.

<u>Property 2</u>. Let $v \in Q^n$ and $i \in N$ a dummy player in the game v. Then $\tau_i^v = 0$.

<u>Proof of the properties 1 and 2</u>.

Let $v \in Q^n$. Then by the definitions 7, 6 and 5 we have

$$0 \leq \lambda_i^v \leq g^v(\{i\}) = b_i^v - v(\{i\}) \text{ for all } i \in N. \qquad (*)$$

By the definition of the τ-value, $\tau_i^v \geq b_i^v - \lambda_i^v$ for all $i \in N$. From (*) it follows that $\tau^v$ is individual rational. Let $i \in N$ be a dummy player. Then in particular, $v(\{i\}) = v(\emptyset) = 0$ and $v(N) = v(N-\{i\})$, so $b_i^v = v(N)-v(N-\{i\}) = 0$. From (*) follows now that $\lambda_i^v = 0$. Using definition 8, we get $\tau_i^v = 0$. □

We remark that also other well-known game-theoretical properties such as symmetry, S-equivalence and continuity, are valid for the τ-value.

We conclude this section by introducing a subclass $\tilde{Q}^n$ of n-person games, for which many solution concepts can easily be calculated with the aid of the utopia vector $b^v$ and the gap function $g^v$. Thus, if a game v is classified as an element of the class $\tilde{Q}^n$, we only need to apply the formulas of the following theorem in order to calculate some solution concepts, which are in general difficult to calculate.

First we define the subclass $\tilde{Q}^n$ as those n-person games with a non-negative gap function such that the gap function takes its minimum in N, i.e.

$$\tilde{Q}^n := \{v;\ v \text{ is an n-person game such that}$$
$$0 \le g^v(N) \le g^v(S) \text{ for all } S \in 2^N-\{\emptyset\}\}.$$

We can now formulate our last theorem.

Theorem 4. Let $v \in \tilde{Q}^n$. Then

(i) $\tau^v = b^v - \frac{1}{n}(g^v(N), g^v(N), \ldots, g^v(N))$.

(ii) the core $C(v)$ is the convex hull of n points $f^1, f^2, \ldots, f^n$ where

$$f^i := (b_1^v, b_2^v, \ldots, b_{i-1}^v, b_i^v - g^v(N), b_{i+1}^v, \ldots, b_n^v) \text{ for each } i \in N.$$

(iii) the $\tau$-value of v is in the center of gravity of the extreme points of the core, i.e. $\tau^v = \frac{1}{n} \sum_{i=1}^{n} f^i$.

(iv) the $\tau$-value of v is equal to the nucleolus of v.

Proof. We shall only prove (i), (iii) and a part of (ii). The proofs of the remaining statements can be found in Driessen and Tijs (1982a). Let $v \in \tilde{Q}^n$. Then by definition of $\tilde{Q}^n$ and definition 6, we have $\lambda_i^v = g^v(N)$ for all $i \in N$. Thus $v \in Q^n$ and $u_i^v = \frac{1}{n}$ for all $i \in N$. By theorem 3, $\tau^v = b^v - g^v(N)u^v = b^v - g^v(N)(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ and statement (i) is shown. Let $f^i$ be as in the theorem. It is obvious that $\sum_{j=1}^{n} (f^i)_j = \sum_{j=1}^{n} b_j^v - g^v(N) = v(N)$, so $f^i$ is a pay-off vector. Let $S \in 2^N-\{\emptyset\}$. If $i \notin S$, then $\sum_{j\in S} (f^i)_j = \sum_{j\in S} b_j^v \ge v(S)$ because $g^v(S) \ge 0$. If $i \in S$, then $\sum_{j\in S} (f^i)_j = \sum_{j\in S} b_j^v - g^v(N) \ge v(S)$ because $g^v(S) \ge g^v(N)$. We can conclude that $f^i \in C(v)$ for each $i \in N$. Statement (iii) follows immediately from the formula of the $\tau$-value, given in (i), and the definition of the vectors $f^i$, given in (ii). □

Example 7. Let v be the three-person game with $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = 7$, $v(\{1,3\}) = v(\{2,3\}) = 5$ and $v(\{1,2,3\}) = 9$.

Then $b^v = (4,4,2)$, so $g^v(\{1\}) = g^v(\{2\}) = 4$, $g^v(\{3\}) = 2$, and $g^v(\{1,2\}) = g^v(\{1,3\}) = g^v(\{2,3\}) = g^v(\{1,2,3\}) = 1$. Thus $v \in \tilde{Q}^3$ and so $\tau^v = (4,4,2) - \frac{1}{3}(1,1,1) = (\frac{11}{3},\frac{11}{3},\frac{5}{3})$ by theorem 4. Notice that we don't need to calculate the concession rate vector $u^v$. Furthermore, let $f^1 := (3,4,2)$, $f^2 := (4,3,2)$ and $f^3 := (4,4,1)$. Then $\tau^v = \frac{1}{3}(f^1+f^2+f^3)$ and the core $C(v)$ is the convex hull of those three points $f^1$, $f^2$ and $f^3$.

We conclude with the remark that for any n-person game also the Shapley value can be expressed with the aid of the utopia vector and the gap function (cf. Driessen, 1983).

## 5. References

Chetty, V.K., Dasgupta, D. and Raghavan, T.E.S.: Power and distribution of profits. Discussion paper no.139. Indian Statistical Institute, Delhi Centre, 1976.

Driessen, T.S.H. and Tijs, S.H.: The τ-value, the nucleolus and the core for a subclass of games. Methods of Operations Research, 1982a (to appear).

Driessen, T.S.H. and Tijs, S.H.: Semiconvex games and the τ-value. Report 8228, Department of Mathematics, Catholic University, Nijmegen, The Netherlands, 1982b.

Driessen, T.S.H.: An alternative formula for the Shapley value of a cooperative game. Department of Mathematics, Catholic University, Nijmegen, The Netherlands, 1983 (to appear).

Kalai, E. and Zemel, E.: Totally balanced games and games of flow. Mathematics of Operations Research 7, 476-478, 1982.

von Neumann, J. and Morgenstern, O.: Theory of games and economic behavior. Princeton University Press, Princeton, 1944.

Rosenmüller, J.: Kooperatieve Spiele und Märkte. Springer Verlag, Berlin, 1971.

Schmeidler, D.: The nucleolus of a characteristic function game. SIAM Journal of Applied Mathematics 17, 1163-1170, 1969.

Shapley, L.S.: A value for n-person games. Annals of Mathematics Studies 28, 307-317, 1953.

Shapley, L.S. and Shubik, M.: Ownership and the production function. The Quarterly Journal of Economics 81, 88-111, 1967.

Tijs, S.H.: Bounds for the core and the $\tau$-value. In: Game Theory and Mathematical Economics. (Eds. O. Moeschlin and D. Pallaschke), North-Holland Publ. Cie, 123-132, 1981.

Tijs, S.H. and Lipperts, F.A.S.: The hypercube and the core cover of n-person cooperative games. Cahiers du C.E.R.O. 24, 27-37, 1982.