# Cooperative games with countably many players

van Damme, E.E.C.

*Published in:*
Methods of Operations Research

*Publication date:*
1980

# COOPERATIVE GAMES WITH COUNTABLY MANY PLAYERS

E.E.C. van Damme, Eindhoven

A cooperative game with countably many players (in characteristic function form) is a function $v : 2^N \to \mathbb{R}^+$, where $2^N$ is the powerset of N ($N=\{1,2,3,\ldots\}$) and $\mathbb{R}^+=\{x \in \mathbb{R} ; x \geq 0\}$. Mostly we will write "game" instead of "cooperative game with countably many players".
We investigate which results from the theory of cooperative games with finitely many players are still valid for games with countably many players.

We will consider three solution concepts :

i) stable sets (von Neumann Morgenstern solutions)
ii) the core (the core of the game v will be denoted by $C(v)$)
iii) a Shapley value

The definitions of a stable set and of the core are direct adaptations of the definitions of these concepts in the finite case. A Shapley value on a subclass G of the class of all games is a function $\varphi : G \to \ell^1$ that satisfies the axioms 1-5 of [1]. Artstein [1] defined a Shapley value on a certain subclass $B_0$. We will define a value in a slightly different way, later on.

Let us first turn to the core and the stable sets. In the following example we see that not all results from the finite case can be generalized.

<u>EXAMPLE 1</u> Let the game v be defined by :

$v(S) = 1$, if N-S is finite
$v(S) = 0$, otherwise.

Then v is a convex game ($v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S,T \in 2^N$). By a theorem of Shapley [4] we know that any finite convex game has a nonempty core.
Since, $C(v) = \emptyset$, this result is not valid in the countable case.

Moreover, Shapley [4] proved that for any finite convex game the core is a stable set. Clearly, for the game above, the core is not stable. However, even if we have a convex game with nonempty core, then this core need not be stable, as we see in example 2.

EXAMPLE 2 Let the game v be defined by :

$v(S) = 1$, if $1 \in S$ and $N - S$ is finite
$v(S) = 0$, otherwise.

Then v is convex, $C(v) = \{(1,0,0,\ldots)\}$, but $C(v)$ is not a stable set.

We will indicate the fact that we cannot obtain many positive results for games with countably many players by giving another basic theorem from the theory of finite cooperative games, which cannot be generalized. Therefore, let us introduce the notion of balancedness. A game v is called balanced if for all functions $\lambda : 2^N \to \mathbb{R}^+$, with $\sum_S \lambda_S \chi_S \leq \chi_N$ we have $\sum_S \lambda_S v(S) \leq v(N)$.
($\chi_S$ is the function from N to $\ell^\infty$ with $\chi_S(i) = 1$, if $i \in S$; $\chi_S(i)=0$ otherwise)

We know that the core of a finite game is nonempty if and only if this game is balanced ([3]). Example 1 shows that this result is not true for countable player games. The game of example 1 is balanced, but its core is empty.
So, we see that we can only obtain positive results on a subset of the set of all games.

We will now introduce a class of games on which positive results can indeed be obtained, namely the class of all continuous games.
A game v is called continuous if for all sequences $S, S_1, S_2, \ldots$ of elements of $2^N$ with $S_n \uparrow S (n \to \infty)$ we have

$$\lim_{n \to \infty} v(S_n) = v(S)$$

($S_n \uparrow S(n \to \infty)$ means $S_1 \subset S_2 \subset \ldots$ and $\bigcup_{n=1}^{\infty} S_n = S$)

We can prove the following propositions :

PROPOSITION 1 Let v be a continuous game. Then $C(v) \neq \emptyset$ if and only if v is balanced.

PROPOSITION 2 Let v be a convex and continuous game. Then $C(v) \neq \emptyset$. Moreover, $C(v)$ is the unique stable set.

We will now define a Shapley value on a subset of the set of all continuous games.

Let $i \in N$. Let $\{S_n\}_{n \in N}$ be a sequence of elements of $2^N$ such that

$$\text{i) } S_1 = \{i\}, \qquad \text{ii) } S_n \subsetneq S_{n+1} \quad (n \in N)$$

$$\text{iii) } S_n \text{ is finite,} \qquad \text{iv) } S_n \uparrow N \quad (n \to \infty)$$

Let $s_n$ be the number of elements of $S_n$ and let the game $v_{S_n}$ be defined by : $v_{S_n}(T) = v(T) \quad (T \in 2^{S_n})$.

So, $v_{S_n}$ is a game with $s_n$ players. Let $\varphi_{S_n}$ be the Shapley value on the set of all games with $s_n$ players ([2]).

Since $i \in S_n$ for all n, we have that $\varphi_{S_n}(v_{S_n})(i)$ is defined. If $\lim_{n \to \infty} \varphi_{S_n}(v_{S_n})(i)$ exists and if this limit is independent of the sequence $\{S_n\}_{n \in N}$ that was chosen, then we denote this limit by $\varphi_i(v)$.

If we can carry out the procedure described above for all $i \in N$, then we define

$$\varphi(v) := (\varphi_1(v), \varphi_2(v), \ldots)$$

We can prove the following proposition :

PROPOSITION 3 If we define $\varphi$ as above, then $\varphi$ is a Shapley value on a subset $G_0$ of the set of all games. $G_0$ contains all games with finite carrier and all games v that can be written as $v = u-w$ with u and w convex and continuous.

We close by mentioning a property of this Shapley value analogous to a property of the Shapley value in the finite case.

PROPOSITION 4 If v is convex and continuous, then $\varphi(v) \in C(v)$.

REFERENCES

[1] Artstein, Z., Values of games with denumerably many players. Int. J. Game Theory 1 (1971) 27-37.

[2] Shapley, L.S., A value for n person games.
Annals of Mathematics Study 28, Princeton University Press, Princeton, 1953, 307-317.

[3] Shapley, L.S., On balanced sets and cores.
Naval Res. Logistics Quart. 14 (1967) 435-460.

[4] Shapley, L.S., Cores of convex games.
Int. J. Game Theory 1 (1971) 11-26.

Address:
Eindhoven University of Technology
Department of Mathematics
P.O. Box 513
NL-5600 MB Eindhoven